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ON RELAXING THE POSITIVITY CONDITION OF LINEAR OPERATORS IN STATISTICAL KOROVKIN-TYPE APPROXIMATIONS

GEORGE A. ANASTASSIOU AND OKTAY DUMAN*

Abstract. In this paper, we relax the positivity condition of linear operators in the Korovkin-type approximation theory via the concept of statistical convergence. Especially, we obtain various Korovkin-type approximation theorems providing the statistical convergence to derivatives of functions by means of a class of linear operators.

1. Introduction

Classical Korovkin-type theorems are mainly concerned with the approximation of real-valued functions by means of positive linear operators (see, for instance, [1, 2]). As usual, by positive linear operators we mean linear operators mapping all non-negative functions into non-negative functions. In recent years, by relaxing this positivity condition on linear operators, various approximation theorems have also been obtained. For example, in [3], it was considered linear operators acting from positive and convex functions into positive functions, and from positive and concave functions into concave functions, and also from positive and increasing functions into increasing functions. Some related results may also be found in the papers [4, 5, 6]. However, almost all results in the classical theory are based on the validity of the ordinary limit. In this study, by using the notion of statistical convergence, we obtain various Korovkin-type theorems in statistical sense although the classical limit fails. Actually, in classical convergence, almost all elements of the sequence have to belong to arbitrarily small neighborhood of the limit while the main idea of statistical convergence is to relax this condition and to demand validity of the convergence condition only for a majority of elements. When proving our results, we use not only classical techniques from approximation theory but also new methods from summability theory.

We first recall some basic concepts used in the paper.

The (natural) density of a subset K of $\mathbb{N}$, the set of all natural numbers, is defined by

$$\delta(K) := \lim_{j} \frac{\#\{n \leq j : n \in K\}}{j} \tag{1.1}$$

when the limit exists, where $\#\{B\}$ denotes the cardinality of a set B. For example, $\delta(\mathbb{N}) = 1$, a set with finite number of elements and a set of squares have density zero

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while even and odd numbers have density 1/2 (see [7]). In [8], using the density, Fast introduced an alternative approach of the classical convergence, the so-called *statistical convergence*, as follows: A sequence (x_n) of real (or, complex) numbers is statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\delta\left(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}\right) = 0. \tag{1.2}$$

Later, Freedman and Sember [9] generalized the concepts of density and statistical convergence as *A-density* and *A-statistical convergence* by using a non-negative regular summability matrix $A = (a_{jn})$ in the following way. The A-density of a subset K is given by

$$\delta_A(K) := \lim_j \sum_{n \in K} a_{jn}. \tag{1.3}$$

A sequence (x_n) is called A-statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\delta_A\left(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}\right) = 0. \tag{1.4}$$

It is easy to check that (1.4) is equivalent to

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0 \text{ for every } \varepsilon > 0. \tag{1.5}$$

Observe that every convergent sequence is both statistical convergent and A-statistical convergent to the same value. But their converses do not hold true (see [10, 11]). If one takes $A = C_1 = (c_{jn})$, the Cesáro matrix given by

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j \\ 0, & \text{otherwise,} \end{cases}$$

then, (1.3) and (1.5) reduce to (1.1) and (1.2), respectively. Furthermore, if we replace the matrix A by the identity matrix, then A-statistical convergence coincides with ordinary convergence. In [12], Kolk proved that A-statistical convergence is stronger than ordinary convergence provided that $\lim_j \max_n\{a_{jn}\} = 0$.

2. Statistical Korovkin-Type Results

Let k be a non-negative integer. As usual, by $C^k[0,1]$ we denote the space of all k-times continuously differentiable functions on $[0,1]$ endowed with the sup-norm $\|\cdot\|$. Then, throughout the paper we consider the following function spaces:

$$\begin{aligned}
\mathcal{A} &: = \left\{f \in C^2[0,1] : f \geq 0\right\}, \\
\mathcal{B} &: = \left\{f \in C^2[0,1] : f'' \geq 0\right\}, \\
\mathcal{C} &: = \left\{f \in C^2[0,1] : f'' \leq 0\right\}, \\
\mathcal{D} &: = \left\{f \in C^1[0,1] : f \geq 0\right\}, \\
\mathcal{E} &: = \left\{f \in C^1[0,1] : f' \geq 0\right\}, \\
\mathcal{F} &: = \left\{f \in C^1[0,1] : f' \leq 0\right\}. \\
\mathcal{G} &: = \left\{f \in C[0,1] : f \geq 0\right\}.
\end{aligned}$$

We also consider the test functions

$$e_i(y) = y^i, \ i = 0, 1, 2, ...$$

Then we present the following results.

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of linear operators mapping $C^2[0,1]$ onto itself. Assume that*

$$\delta_A\left(\{n \in \mathbb{N} : L_n\left(\mathcal{A} \cap \mathcal{B}\right) \subset \mathcal{A}\}\right) = 1. \tag{2.1}$$

Then

$$st_A - \lim_n \|L_n(e_i) - e_i\| = 0 \quad \text{for} \quad i = 0, 1, 2 \tag{2.2}$$

if and only if

$$st_A - \lim_n \|L_n(f) - f\| = 0 \quad \text{for all} \quad f \in C^2[0,1]. \tag{2.3}$$

Proof. The implication (2.3) $\Rightarrow$ (2.2) is clear. Assume now that (2.2) holds. Let $x \in [0,1]$ be fixed, and let $f \in C^2[0,1]$. By the boundedness and continuity of f, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$-\varepsilon - \frac{2M_1\beta}{\delta^2}\varphi_x(y) \leq f(y) - f(x) \leq \varepsilon + \frac{2M_1\beta}{\delta^2}\varphi_x(y) \tag{2.4}$$

holds for all $y \in [0,1]$ and for any $\beta \geq 1$, where $M_1 = \|f\|$ and $\varphi_x(y) = (y-x)^2$. Then, by (2.4), we obtain that

$$g_\beta(y) := \frac{2M_1\beta}{\delta^2}\varphi_x(y) + \varepsilon + f(y) - f(x) \geq 0$$

and

$$h_\beta(y) := \frac{2M_1\beta}{\delta^2}\varphi_x(y) + \varepsilon - f(y) + f(x) \geq 0$$

hold for all $y \in [0,1]$. So, the functions g_β and h_β belong to $\mathcal{A}$. On the other hand, it is clear that, for all $y \in [0,1]$,

$$g_\beta''(y) = \frac{4M_1\beta}{\delta^2} + f''(y)$$

and

$$h_\beta''(y) = \frac{4M_1\beta}{\delta^2} - f''(y).$$

If we choose a number β such that

$$\beta \geq \max\left\{1, \frac{\|f''\|\,\delta^2}{4M_1}\right\}, \tag{2.5}$$

we observe that (2.4) holds for such β's and also the functions g_β and h_β belong to $\mathcal{B}$ because of $g_\beta''(y) \geq 0$ and $h_\beta''(y) \geq 0$ for all $y \in [0,1]$. So, we have $g_\beta, h_\beta \in \mathcal{A} \cap \mathcal{B}$ under the condition (2.5). Let

$$K_1 := \{n \in \mathbb{N} : L_n\left(\mathcal{A} \cap \mathcal{B}\right) \subset \mathcal{A}\}.$$

By (2.1), it is clear that $\delta_A(K_1) = 1$, and so

$$\delta_A(\mathbb{N} \backslash K_1) = 0. \tag{2.6}$$

Then, we may write that

$$L_n(g_\beta; x) \geq 0 \quad \text{and} \quad L_n(h_\beta; x) \geq 0 \quad \text{for every} \quad n \in K_1.$$

Now using the fact that $\varphi_x \in \mathcal{A} \cap \mathcal{B}$ and considering the linearity of L_n, we obtain that, for every $n \in K_1$,

$$\frac{2M_1\beta}{\delta^2} L_n(\varphi_x; x) + \varepsilon L_n(e_0; x) + L_n(f; x) - f(x) L_n(e_0; x) \geq 0$$

and

$$\frac{2M_1\beta}{\delta^2} L_n(\varphi_x; x) + \varepsilon L_n(e_0; x) - L_n(f; x) + f(x) L_n(e_0; x) \geq 0,$$

or equivalently

$$\begin{aligned} -\frac{2M_1\beta}{\delta^2} L_n(\varphi_x; x) - \varepsilon L_n(e_0; x) + f(x)\left(L_n(e_0; x) - e_0\right) \\ \leq L_n(f; x) - f(x) \\ \leq \frac{2M_1\beta}{\delta^2} L_n(\varphi_x; x) + \varepsilon L_n(e_0; x) + f(x)\left(L_n(e_0; x) - e_0\right). \end{aligned}$$

Then, we have

$$|L_n(f; x) - f(x)| \leq \varepsilon + \frac{2M_1\beta}{\delta^2} L_n(\varphi_x; x) + (\varepsilon + |f(x)|)\,|L_n(e_0; x) - e_0|$$

holds for every $n \in K_1$. The last inequality gives that, for every $\varepsilon > 0$ and $n \in K_1$,

$$\begin{aligned} \|L_n(f) - f\| \leq \varepsilon + (\varepsilon + M_1)\,\|L_n(e_0) - e_0\| \\ + \frac{2M_1\beta}{\delta^2}\,\|L_n(e_2) - e_2\| \\ + \frac{4M_1\beta}{\delta^2}\,\|L_n(e_1) - e_1\| \\ + \frac{2M_1\beta}{\delta^2}\,\|L_n(e_0) - e_0\|. \end{aligned}$$

Hence, we get

$$\|L_n(f) - f\| \leq \varepsilon + C_1 \sum_{k=0}^{2} \|L_n(e_k) - e_k\| \quad \text{for every } n \in K_1, \tag{2.7}$$

where $C_1 := \max\left\{\varepsilon + M_1 + \frac{2M_1\beta}{\delta^2}, \frac{4M_1\beta}{\delta^2}\right\}$. Now, for a given $r > 0$, choose an $\varepsilon > 0$ such that $\varepsilon < r$, and define the following sets:

$$\begin{aligned} F &:= \{n \in \mathbb{N} : \|L_n(f) - f\| \geq r\}, \\ F_k &:= \left\{n \in \mathbb{N} : \|L_n(e_k) - e_k\| \geq \frac{r - \varepsilon}{3C_1}\right\}, \quad k = 0, 1, 2. \end{aligned}$$

Then, it follows from (2.7) that

$$F \cap K_1 \subset \bigcup_{k=0}^{2} (F_k \cap K_1),$$

which yields, for every $j \in \mathbb{N}$, that

$$\sum_{n \in F \cap K_1} a_{jn} \leq \sum_{k=0}^{2} \left(\sum_{n \in F_k \cap K_1} a_{jn} \right) \leq \sum_{k=0}^{2} \left(\sum_{n \in F_k} a_{jn} \right) \tag{2.8}$$

Now, taking limit as $j \to \infty$ in the both-sides of (2.8) and using (2.2), we immediately see that

$$\lim_j \sum_{n \in F \cap K_1} a_{jn} = 0. \tag{2.9}$$

Furthermore, since

$$\begin{aligned} \sum_{n \in F} a_{jn} &= \sum_{n \in F \cap K_1} a_{jn} + \sum_{n \in F \cap (\mathbb{N} \setminus K_1)} a_{jn} \\ &\leq \sum_{n \in F \cap K_1} a_{jn} + \sum_{n \in (\mathbb{N} \setminus K_1)} a_{jn} \end{aligned}$$

holds for every $j \in \mathbb{N}$, letting again limit as $j \to \infty$ in the last inequality and using (2.6), (2.9) we obtain

$$\lim_j \sum_{n \in F} a_{jn} = 0,$$

which means that

$$st_A - \lim_n \|L_n(f) - f\| = 0.$$

The theorem is proved. □

Theorem 2.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of linear operators mapping $C^2[0,1]$ onto itself. Assume that*

$$\delta_A \left(\{ n \in \mathbb{N} : L_n \left(\mathcal{A} \cap \mathcal{C} \right) \subset \mathcal{C} \} \right) = 1. \tag{2.10}$$

Then

$$st_A - \lim_n \|[L_n(e_i)]'' - e_i''\| = 0 \quad \text{for } i = 0, 1, 2, 3, 4 \tag{2.11}$$

if and only if

$$st_A - \lim_n \|[L_n(f)]'' - f''\| = 0 \quad \text{for all} \quad f \in C^2[0,1]. \tag{2.12}$$

Proof. It is enough to prove the implication (2.11) $\Rightarrow$ (2.12). Let $f \in C^2[0,1]$ and $x \in [0,1]$ be fixed. As in the proof of Theorem 2.1, we can write that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$-\varepsilon + \frac{2M_2\beta}{\delta^2}\sigma_x''(y) \leq f''(y) - f''(x) \leq \varepsilon - \frac{2M_2\beta}{\delta^2}\sigma_x''(y) \tag{2.13}$$

holds for all $y \in [0,1]$ and for any $\beta \geq 1$, where $\sigma_x(y) = -\dfrac{(y-x)^4}{12} + 1$ and $M_2 = \|f''\|$. Then, define the following functions on $[0,1]$:

$$u_\beta(y) := \frac{2M_2\beta}{\delta^2}\sigma_x(y) + f(y) - \frac{\varepsilon}{2}y^2 - \frac{f''(x)}{2}y^2,$$

and

$$v_\beta(y) := \frac{2M_2\beta}{\delta^2}\sigma_x(y) - f(y) - \frac{\varepsilon}{2}y^2 + \frac{f''(x)}{2}y^2.$$

It follows from (2.13) that

$$u_\beta''(y) \leq 0 \quad \text{and} \quad v_\beta''(y) \leq 0 \quad \text{for all } y \in [0,1],$$

which implies that the functions u_β and v_β belong to $\mathcal{C}$. Observe that $\sigma_x(y) \geq \frac{11}{12}$ for all $y \in [0,1]$. Then

$$\frac{\left(\pm f(y) + \frac{\varepsilon}{2}y^2 \pm \frac{f''(x)}{2}y^2\right)\delta^2}{2M_2\sigma_x(y)} \leq \frac{(M_1 + M_2 + \varepsilon)\delta^2}{M_2}$$

holds for all $y \in [0,1]$, where $M_1 = \|f\|$ and $M_2 = \|f''\|$ as stated before. Now, if we choose a number β such that

$$\beta \geq \max\left\{1, \frac{(M_1 + M_2 + \varepsilon)\delta^2}{M_2}\right\}, \tag{2.14}$$

then inequality (2.13) holds for such β's and

$$u_\beta(y) \geq 0 \quad \text{and} \quad v_\beta(y) \geq 0 \quad \text{for all } y \in [0,1].$$

Hence, we also get $u_\beta,\, v_\beta \in \mathcal{A}$, which gives that the functions u_β and v_β belong to $\mathcal{A} \cap \mathcal{C}$ under the condition (2.14). Now let

$$K_2 := \{n \in \mathbb{N} : L_n(\mathcal{A} \cap \mathcal{C}) \subset \mathcal{C}\}.$$

Then, by (2.10), we have

$$\delta_A(\mathbb{N} \backslash K_2) = 0. \tag{2.15}$$

Also we get, for every $n \in K_2$,

$$[L_n(u_\beta)]'' \leq 0 \quad \text{and} \quad [L_n(v_\beta)]'' \leq 0.$$

Then, we obtain, for every $n \in K_2$, that

$$\frac{2M_2\beta}{\delta^2}[L_n(\sigma_x)]'' + [L_n(f)]'' - \frac{\varepsilon}{2}[L_n(e_2)]'' - \frac{f''(x)}{2}[L_n(e_2)]'' \leq 0$$

and

$$\frac{2M_2\beta}{\delta^2}[L_n(\sigma_x)]'' - [L_n(f)]'' - \frac{\varepsilon}{2}[L_n(e_2)]'' + \frac{f''(x)}{2}[L_n(e_2)]'' \leq 0.$$

These inequalities yield that

$$\begin{aligned}
&\frac{2M_2\beta}{\delta^2}[L_n(\sigma_x)]''(x) - \frac{\varepsilon}{2}[L_n(e_2)]''(x) + \frac{f''(x)}{2}[L_n(e_2)]''(x) - f''(x) \\
&\quad \leq [L_n(f)]''(x) - f''(x) \\
&\quad \leq -\frac{2M_2\beta}{\delta^2}[L_n(\sigma_x)]''(x) + \frac{\varepsilon}{2}[L_n(e_2)]''(x) + \frac{f''(x)}{2}[L_n(e_2)]''(x) - f''(x).
\end{aligned}$$

Observe now that $[L_n(\sigma_x)]'' \leq 0$ on $[0,1]$ for every $n \in K_2$ because of $\sigma_x \in \mathcal{A} \cap \mathcal{C}$. Using this, the last inequality gives, for every $n \in K_2$, that

$$\begin{aligned}
|[L_n(f)]''(x) - f''(x)| \quad \leq \quad & -\frac{2M_2\beta}{\delta^2}[L_n(\sigma_x)]''(x) + \frac{\varepsilon}{2}\left|[L_n(e_2)]''(x)\right| \\
& + \frac{|f''(x)|}{2}\left|[L_n(e_2)]''(x) - 2\right|,
\end{aligned}$$

and hence

$$\begin{aligned}
|[L_n(f)]''(x) - f''(x)| \quad \leq \varepsilon + \frac{\varepsilon + |f''(x)|}{2}\left|[L_n(e_2)]''(x) - e_2''(x)\right| \\
+ \frac{2M_2\beta}{\delta^2}[L_n(-\sigma_x)]''(x).
\end{aligned} \tag{2.16}$$

Now we compute the quantity $[L_n(-\sigma_x)]''$ in inequality (2.16). Observe that

$$\begin{aligned}
[L_n(-\sigma_x)]''(x) &= \left[L_n\left(\frac{(y-x)^4}{12}-1\right)\right]''(x)\\
&= \frac{1}{12}\left[L_n(e_4)\right]''(x)-\frac{x}{3}\left[L_n(e_3)\right]''(x)+\frac{x^2}{2}\left[L_n(e_2)\right]''(x)\\
&\quad -\frac{x^3}{3}\left[L_n(e_1)\right]''(x)+\left(\frac{x^4}{12}-1\right)\left[L_n(e_0)\right]''(x)\\
&= \frac{1}{12}\left\{\left[L_n(e_4)\right]''(x)-e_4''(x)\right\}-\frac{x}{3}\left\{\left[L_n(e_3)\right]''(x)-e_3''(x)\right\}\\
&\quad +\frac{x^2}{2}\left\{\left[L_n(e_2)\right]''(x)-e_2''(x)\right\}-\frac{x^3}{3}\left\{\left[L_n(e_1)\right]''(x)-e_1''(x)\right\}\\
&\quad +\left(\frac{x^4}{12}-1\right)\left\{\left[L_n(e_0)\right]''(x)-e_0''(x)\right\}.
\end{aligned}$$

Combining this with (2.16), for every $\varepsilon>0$ and $n\in K_2$, we have

$$\begin{aligned}
|[L_n(f)]''(x)-f''(x)| &\le \varepsilon+\left(\frac{\varepsilon+|f''(x)|}{2}+\frac{M_2\beta x^2}{\delta^2}\right)|[L_n(e_2)]''(x)-e_2''(x)|\\
&\quad +\frac{M_2\beta}{6\delta^2}\left|\left[L_n(e_4)\right]''(x)-e_4''(x)\right|\\
&\quad +\frac{2M_2\beta x}{3\delta^2}\left|\left[L_n(e_3)\right]''(x)-e_3''(x)\right|\\
&\quad +\frac{2M_2\beta x^3}{3\delta^2}\left|\left[L_n(e_1)\right]''(x)-e_1''(x)\right|\\
&\quad +\frac{2M_2\beta}{3\delta^2}\left(1-\frac{x^4}{12}\right)\left|\left[L_n(e_0)\right]''(x)-e_0''(x)\right|.
\end{aligned}$$

Therefore, we obtain, for every $\varepsilon>0$ and $n\in K_2$, that

$$\left\|[L_n(f)]''-f''\right\|\le\varepsilon+C_2\sum_{k=0}^{4}\left\|[L_n(e_k)]''-e_k''\right\|, \tag{2.17}$$

where $C_2:=\dfrac{\varepsilon+M_2}{2}+\dfrac{M_2\beta}{\delta^2}$ and $M_2=\|f''\|$ as stated before. Now, for a given $r>0$, choose an ε such that $0<\varepsilon<r$, and consider the following sets:

$$\begin{aligned}
G &:= \left\{n\in\mathbb{N}:\left\|[L_n(f)]''-f''\right\|\ge r\right\},\\
G_k &:= \left\{n\in\mathbb{N}:\left\|[L_n(e_k)]''-e_k''\right\|\ge\frac{r-\varepsilon}{5C_2}\right\},\quad k=0,1,2,3,4.
\end{aligned}$$

In this case, by (2.17),

$$G\cap K_2\subset\bigcup_{k=0}^{4}(G_k\cap K_2),$$

which yields, for every $j\in\mathbb{N}$, that

$$\sum_{n\in G\cap K_2}a_{jn}\le\sum_{k=0}^{4}\left(\sum_{n\in G_k\cap K_2}a_{jn}\right)\le\sum_{k=0}^{4}\left(\sum_{n\in G_k}a_{jn}\right) \tag{2.18}$$

Letting $j \to \infty$ in the both-sides of (2.18) and using (2.11), we immediately see that

$$\lim_j \sum_{n \in G \cap K_2} a_{jn} = 0. \tag{2.19}$$

Furthermore, if we use the inequality

$$\begin{aligned} \sum_{n \in G} a_{jn} &= \sum_{n \in G \cap K_2} a_{jn} + \sum_{n \in G \cap (\mathbb{N} \setminus K_2)} a_{jn} \\ &\leq \sum_{n \in G \cap K_2} a_{jn} + \sum_{n \in (\mathbb{N} \setminus K_2)} a_{jn} \end{aligned}$$

and if we take limit as $j \to \infty$, then it follows from (2.15) and (2.19) that

$$\lim_j \sum_{n \in G} a_{jn} = 0.$$

Thus, we get

$$st_A - \lim_n \left\| [L_n(f)]'' - f'' \right\| = 0.$$

The theorem is proved. □

Theorem 2.3. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of linear operators mapping $C^1[0,1]$ onto itself. Assume that*

$$\delta_A \left(\{ n \in \mathbb{N} : L_n \left(\mathcal{D} \cap \mathcal{E} \right) \subset \mathcal{E} \} \right) = 1. \tag{2.20}$$

Then

$$st_A - \lim_n \| [L_n(e_i)]' - e_i' \| = 0 \quad \text{for } i = 0, 1, 2, 3 \tag{2.21}$$

if and only if

$$st_A - \lim_n \| [L_n(f)]' - f' \| = 0 \quad \text{for all} \quad f \in C^1[0,1]. \tag{2.22}$$

Proof. It is enough to prove the implication (2.21) $\Rightarrow$ (2.22). Let $f \in C^1[0,1]$ and $x \in [0,1]$ be fixed. Then, for every $\varepsilon > 0$, there exists a positive number δ such that

$$-\varepsilon - \frac{2M_3\beta}{\delta^2} w_x'(y) \leq f'(y) - f'(x) \leq \varepsilon + \frac{2M_3\beta}{\delta^2} w_x'(y) \tag{2.23}$$

holds for all $y \in [0,1]$ and for any $\beta \geq 1$, where $w_x(y) := \dfrac{(y-x)^3}{3} + 1$ and $M_3 := \|f'\|$. Now considering the functions defined by

$$\theta_\beta(y) := \frac{2M_3\beta}{\delta^2} w_x(y) - f(y) + \varepsilon y + y f'(x)$$

and

$$\lambda_\beta(y) := \frac{2M_3\beta}{\delta^2} w_x(y) + f(y) + \varepsilon y - y f'(x),$$

we can easily check that θ_β and λ_β belong to $\mathcal{E}$ for any $\beta \geq 1$, i.e. $\theta_\beta'(y) \geq 0$, $\lambda_\beta'(y) \geq 0$. Also, observe that $w_x(y) \geq \dfrac{2}{3}$ for all $y \in [0,1]$. Then

$$\frac{(\pm f(y) - \varepsilon y \pm f'(x) y)\, \delta^2}{2M_3 w_x(y)} \leq \frac{(M_1 + M_3 + \varepsilon)\delta^2}{M_3}$$

holds for all $y \in [0,1]$, where $M_1 = \|f\|$. Now, if we choose a number β such that

$$\beta \geq \max\left\{1, \frac{(M_1 + M_3 + \varepsilon)\delta^2}{M_3}\right\}, \tag{2.24}$$

then inequality (2.23) holds for such β's and

$$\theta_\beta(y) \geq 0 \quad \text{and} \quad \lambda_\beta(y) \geq 0 \quad \text{for all } y \in [0,1],$$

which yields that $\theta_\beta, \lambda_\beta \in \mathcal{D}$. Thus, we get $\theta_\beta, \lambda_\beta \in \mathcal{D} \cap \mathcal{E}$ for any β satisfying (2.24). Let

$$K_3 := \{n \in \mathbb{N} : L_n(\mathcal{D} \cap \mathcal{E}) \subset \mathcal{E}\}.$$

Then, by (2.20), we have

$$\delta_A(\mathbb{N} \backslash K_3) = 0. \tag{2.25}$$

Also we get, for every $n \in K_3$,

$$[L_n(\theta_\beta)]' \geq 0 \quad \text{and} \quad [L_n(\lambda_\beta)]' \geq 0.$$

Hence, we obtain, for every $n \in K_3$, that

$$\frac{2M_3\beta}{\delta^2}[L_n(w_x)]' - [L_n(f)]' + \varepsilon[L_n(e_1)]' + f'(x)[L_n(e_1)]' \geq 0$$

and

$$\frac{2M_3\beta}{\delta^2}[L_n(w_x)]' + [L_n(f)]' + \varepsilon[L_n(e_1)]' - f'(x)[L_n(e_1)]' \geq 0.$$

Then, we get, for any $n \in K_3$, that

$$\begin{aligned} &-\frac{2M_3\beta}{\delta^2}[L_n(w_x)]'(x) - \varepsilon[L_n(e_1)]'(x) + f'(x)[L_n(e_1)]'(x) - f'(x) \\ &\leq [L_n(f)]'(x) - f'(x) \\ &\leq \frac{2M_3\beta}{\delta^2}[L_n(w_x)]'(x) + \varepsilon[L_n(e_1)]'(x) + f'(x)[L_n(e_1)]'(x) - f'(x), \end{aligned}$$

and hence

$$\begin{aligned} |[L_n(f)]'(x) - f'(x)| \quad &\leq \varepsilon + (\varepsilon + |f'(x)|)\,|[L_n(e_1)]'(x) - e_1'(x)| \\ &\quad + \frac{2M_3\beta}{\delta^2}[L_n(w_x)]'(x) \end{aligned} \tag{2.26}$$

holds for every $n \in K_3$ because of the fact that the function w_x belongs to $\mathcal{D} \cap \mathcal{E}$. Since

$$\begin{aligned} [L_n(w_x)]'(x) &= \left[L_n\left(\frac{(y-x)^3}{3} + 1\right)\right]'(x) \\ &= \frac{1}{3}\left[L_n(e_3)\right]'(x) - x\left[L_n(e_2)\right]'(x) \\ &\quad + x^2\left[L_n(e_1)\right]'(x) + \left(1 - \frac{x^3}{3}\right)\left[L_n(e_0)\right]'(x) \\ &= \frac{1}{3}\left\{\left[L_n(e_3)\right]'(x) - e_3'(x)\right\} - x\left\{\left[L_n(e_2)\right]'(x) - e_2'(x)\right\} \\ &\quad + x^2\left\{\left[L_n(e_1)\right]'(x) - e_1'(x)\right\} + \left(1 - \frac{x^3}{3}\right)\left\{\left[L_n(e_0)\right]'(x) - e_0'(x)\right\}, \end{aligned}$$

it follows from (2.26) that

$$\begin{aligned}|[L_n(f)]'(x)-f'(x)| &\leq \varepsilon+\left(\varepsilon+|f'(x)|+\frac{2M_3\beta x^2}{\delta^2}\right)|[L_n(e_1)]'(x)-e_1'(x)| \\ &+\frac{2M_3\beta}{3\delta^2}\left|[L_n(e_3)]'(x)-e_3'(x)\right| \\ &+\frac{2M_3\beta x}{\delta^2}\left|[L_n(e_2)]'(x)-e_2'(x)\right| \\ &+\frac{2M_3\beta}{\delta^2}\left(1-\frac{x^3}{3}\right)\left|[L_n(e_0)]'(x)-e_0'(x)\right|.\end{aligned}$$

Thus, we deduce from the last inequality that

$$\left\|[L_n(f)]'-f'\right\| \leq \varepsilon+C_3\sum_{k=0}^{3}\left\|[L_n(e_k)]'-e_k'\right\| \tag{2.27}$$

holds for any $n\in K_3$, where $C_3:=\varepsilon+M_3+\dfrac{2M_3\beta}{\delta^2}$. Now, for a given $r>0$, choose an ε such that $0<\varepsilon<r$, and consider the following sets:

$$\begin{aligned}H &: =\left\{n\in\mathbb{N}:\left\|[L_n(f)]'-f'\right\|\geq r\right\}, \\ H_k &: =\left\{n\in\mathbb{N}:\left\|[L_n(e_k)]'-e_k'\right\|\geq\frac{r-\varepsilon}{4C_3}\right\},\quad k=0,1,2,3.\end{aligned}$$

In this case, by (2.27),

$$H\cap K_3\subset\bigcup_{k=0}^{3}(H_k\cap K_3),$$

which yields, for every $j\in\mathbb{N}$, that

$$\sum_{n\in H\cap K_3}a_{jn}\leq\sum_{k=0}^{3}\left(\sum_{n\in H_k\cap K_3}a_{jn}\right)\leq\sum_{k=0}^{3}\left(\sum_{n\in H_k}a_{jn}\right) \tag{2.28}$$

Letting $j\to\infty$ in the both-sides of (2.28) and also using (2.21), we immediately see that

$$\lim_j\sum_{n\in H\cap K_3}a_{jn}=0. \tag{2.29}$$

Now, using the fact that

$$\begin{aligned}\sum_{n\in H}a_{jn} &= \sum_{n\in H\cap K_3}a_{jn}+\sum_{n\in H\cap(\mathbb{N}\setminus K_3)}a_{jn} \\ &\leq \sum_{n\in H\cap K_3}a_{jn}+\sum_{n\in(\mathbb{N}\setminus K_3)}a_{jn}\end{aligned}$$

and taking limit as $j\to\infty$, then it follows from (2.25) and (2.29) that

$$\lim_j\sum_{n\in H}a_{jn}=0.$$

Thus, we get

$$st_A-\lim_n\left\|[L_n(f)]'-f'\right\|=0.$$

The theorem is proved. □

Theorem 2.4. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of linear operators mapping $C[0,1]$ onto itself. Assume that*

$$\delta_A\left(\{n \in \mathbb{N} : L_n(\mathcal{G}) \subset \mathcal{G}\}\right) = 1. \tag{2.30}$$

Then

$$st_A - \lim_n \|L_n(e_i) - e_i\| = 0 \quad \text{for } i = 0,1,2 \tag{2.31}$$

if and only if

$$st_A - \lim_n \|L_n(f) - f\| = 0 \quad \text{for all} \quad f \in C[0,1]. \tag{2.32}$$

Proof. See the remarks in the next section. □

3. **Concluding Remarks**

In this section we summarize our results and give some applications in order to show the importance of using the statistical approximation in this study.

1. In Theorem 2.4, if we take the condition

$$\{n \in \mathbb{N} : L_n(\mathcal{G}) \subset \mathcal{G}\} = \mathbb{N} \tag{3.1}$$

instead of (2.30), then we observe that the linear operators L_n are positive for each $n \in \mathbb{N}$. In this case, Theorem 2.4 is an A-statistical version of Theorem 1 of [13], and the proof follows immediately. Actually, as in the previous proofs, we can show that

$$(2.31) \Leftrightarrow (2.32)$$

although the weaker condition (2.30) holds. Because of similarity, we omit the proof of Theorem 2.4. Here, condition (2.30) enables us that L_n does not need to be positive for each $n \in \mathbb{N}$, but it is enough to be positive for each $n \in K$ with $\delta_A(K) = 1$. Observe that condition (2.30), which is weaker than (3.1), can be applied to many well-known results regarding statistical approximation of positive linear operators, such as Theorem 3 of [14], Theorems 2.1 and 2.2 of [15], Theorem 2.1 of [16] and Theorem 1 of [17].

2. We can easily see that all of our theorems in this paper are also valid for any compact subset of $\mathbb{R}$ instead of the unit interval $[0,1]$.

3. In Theorems 2.1-2.3, if we replace the matrix A by the identity matrix and also if we consider the conditions

$$\{n \in \mathbb{N} : L_n(\mathcal{A} \cap \mathcal{B}) \subset \mathcal{A}\} = \mathbb{N}, \tag{3.2}$$

$$\{n \in \mathbb{N} : L_n(\mathcal{A} \cap \mathcal{C}) \subset \mathcal{C}\} = \mathbb{N}, \tag{3.3}$$

$$\{n \in \mathbb{N} : L_n(\mathcal{D} \cap \mathcal{E}) \subset \mathcal{E}\} = \mathbb{N} \tag{3.4}$$

instead of the conditions (2.1), (2.10) and (2.20), respectively, then we obtain Propositions 1-3 of [3]. Indeed, for example, assume that A is the identity matrix and (3.2) holds. In this case, since A-statistical convergence coincides with the ordinary convergence, the conditions (2.2) and (2.3) hold with respect to the classical limit operator. Also, according to (3.2), for each $n \in \mathbb{N}$, the linear operators L_n in Theorem 2.1 map positive and convex functions onto positive functions. Hence, we get Proposition 1 of [3].

4. Theorem 2.3 is valid if we replace the condition (2.20) by

$$\delta_A\left(\{n\in\mathbb{N}: L_n\left(\mathcal{D}\cap\mathcal{F}\right)\subset\mathcal{F}\}\right)=1.$$

To prove this, it is enough to consider the function $\psi_x(y)=-\frac{(y-x)^3}{3}+1$ instead of $w_x(y)$ defined in the proof of Theorem 2.3.

5. The following example clearly shows that our statistical approximation results are stronger than the classical ones.

Example. Take $A=C_1$ and define the linear operators L_n on $C^2[0,1]$ as follows:

$$L_n(f;x)=\begin{cases} -x^2 & \text{if } n=m^2 \ (m\in\mathbb{N}) \\ B_n(f;x); & \text{otherwise,}\end{cases} \tag{3.5}$$

where the operators $B_n(f;x)$ denote the Bernstein polynomials. Then, observe that

$$\begin{aligned}\delta_{C_1}\left(\{n\in\mathbb{N}: L_n\left(\mathcal{A}\cap\mathcal{B}\right)\subset\mathcal{A}\}\right) &= \delta\left(\{n\in\mathbb{N}: L_n\left(\mathcal{A}\cap\mathcal{B}\right)\subset\mathcal{A}\}\right)\\ &= \delta\left(\left\{n\neq m^2: m\in\mathbb{N}\right\}\right)\\ &= 1.\end{aligned}$$

Also we have, for each $i=0,1,2$,

$$st_{C_1}-\lim_n\|L_n(e_i)-e_i\|=st-\lim_n\|L_n(e_i)-e_i\|=0.$$

Then, it follows from Theorem 2.1 that, for all $f\in C^2[0,1]$,

$$st_{C_1}-\lim_n\|L_n(f)-f\|=0.$$

However, for the function $e_0=1$, since

$$L_n(e_0;x):=\begin{cases} -x^2 & \text{if } n=m^2 \ (m\in\mathbb{N}) \\ 1 & \text{otherwise,}\end{cases}$$

we get, for all $x\in[0,1]$, that the sequence $\{L_n(e_0;x)\}$ is non-convergent. This shows that Proposition 1 of [3] does not work while Theorem 2.1 still works for the operators L_n defined by (3.5).

6. Finally, we would like to say that, for a future study, it must be interesting to investigate all results of this paper for linear operators acting on various subclasses of the space of all continuous and 2π-periodic functions on the whole real line.

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STATISTICAL WEIGHTED APPROXIMATION TO DERIVATIVES OF FUNCTIONS BY LINEAR OPERATORS

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Abstract. In this paper, we obtain various Korovkin-type approximation theorems providing the statistical weighted convergence to derivatives of functions by means of a class of linear operators acting on weighted spaces. We also discuss the contribution of our results to the approximation theory.

1. Introduction

Statistical convergence, while introduced over nearly fifty years ago (see [1]), has only recently become an area of active research. Different mathematicians studied properties of statistical convergence and applied this concept in various areas such as measure theory [2], trigonometric series [3], approximation theory [4, 6, 5, 7], locally convex spaces [8], summability theory and the limit points of sequences [9, 10, 11, 12], densities of natural numbers [13], in the study of subsets of the Stone-Čhech compactification of the set of natural numbers [14], and Banach spaces [15]. This is because it is quite effective, especially, when the classical limit of a sequence fails. In classical convergence, almost all elements of the sequence have to belong to arbitrarily small neighborhood of the limit; but the main idea of statistical convergence is to relax this condition and to demand validity of the convergence condition only for a majority of elements. Statistical convergence which is a regular non-matrix summability method is also effective in summing non-convergent sequences. Recent studies demonstrate that the notion of statistical convergence provides an important contribution to improvement of the classical analysis. In this paper, by using statistical convergence, we study the statistical Korovkin theory, which deals with the problem of approximating a function by means of a sequence of linear operators acting on weighted spaces. We should note that the classical Korovkin theory and its applications may be found in [16, 17].

This paper is organized as follows: The first section is devoted to basic definitions and notations used in the paper. In the second section, we obtain some statistical approximations to derivatives of functions by means of a class of linear operators defined on various weighted spaces. Our primary motivation of this section is due to [18]. In the final section, we demonstrate that our theorems generalize and improve many well-known results in the approximation theory settings.

Key words and phrases. Statistical convergence, positive linear operators, weight function, weighted spaces, Korovkin theorem.

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Let (x_n) be a sequence of numbers. Then, (x_n) is called statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\lim_j \frac{\#\{n \leq j : |x_n - L| \geq \varepsilon\}}{j} = 0,$$

where $\#\{B\}$ denotes the cardinality of the subset B (see Fast [1]). We denote this statistical limit by $st - \lim_n x_n = L$. Now, let $A = (a_{jn})$ be an infinite summability matrix. Then, the A-transform of x, denoted $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$, provided the series converges for each j. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$ [19]. Assume now that A is a nonnegative regular summability matrix. Then, a sequence (x_n) is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:\, |x_n - L| \geq \varepsilon} a_{jn} = 0 \tag{1.1}$$

holds (see [20]). It is denoted by $st_A - \lim_n x_n = L$. It is not hard to see that if we take $A = C_1$, the Cesáro matrix defined by

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j \\ 0, & \text{otherwise,} \end{cases}$$

then C_1-statistical convergence coincides with statistical convergence. Also, if A is replaced by the identity matrix, then we get the ordinary convergence of number sequences. Observe that every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A. This follows from the definition (1.1) and the well-known regularity conditions of A introduced by Silverman and Toeplitz (see, for instance, Hardy [21, pp. 43-45]); but its converse is not always true. Actually, if $A = (a_{jn})$ is any nonnegative regular summability matrix for which $\lim_j \max_n \{a_{jn}\} = 0$, then A-statistical convergence is stronger than convergence (see [11]).

Throughout the paper we use the following weighted spaces introduced by Efendiev [18]. Let k be a non-negative integer. By $C^{(k)}(\mathbb{R})$ we denote the space of all functions having k-th continuous derivatives on $\mathbb{R}$. Now, let $M^{(k)}(\mathbb{R})$ denote the class of linear operators mapping the set of functions f that are convex of order $(k-1)$ on $\mathbb{R}$, i.e., $f^{(k)}(x) \geq 0$ holds for all $x \in \mathbb{R}$, into the set of all positive functions on $\mathbb{R}$. More precisely, for a fixed non-negative integer k and a linear operator L,

$$L \in M^{(k)}(\mathbb{R}) \Leftrightarrow L(f) \geq 0 \text{ for every function } f \text{ satisfying } f^{(k)} \geq 0. \tag{1.2}$$

If $k = 0$, then $M^{(0)}(\mathbb{R})$ stands for the class of all positive linear operators. Assume that $\rho : \mathbb{R} \to \mathbb{R}^+ = (0, +\infty)$ is a function such that $\rho(0) = 1$; ρ is increasing on $\mathbb{R}^+$ and decreasing on $\mathbb{R}^-$; and $\lim_{x \to \pm\infty} \rho(x) = +\infty$. In this case, we consider the following weighted spaces:

$$\begin{aligned}
C_\rho^{(k)}(\mathbb{R}) &= \left\{ f \in C^{(k)}(\mathbb{R}) : \text{for some positive } m_f,\ \left| f^{(k)}(x) \right| \leq m_f \rho(x),\ x \in \mathbb{R} \right\}, \\
\widetilde{C}_\rho^{(k)}(\mathbb{R}) &= \left\{ f \in C_\rho^{(k)}(\mathbb{R}) : \text{for some } k_f,\ \lim_{x \to \pm\infty} \frac{f^{(k)}(x)}{\rho(x)} = k_f \right\}, \\
\widehat{C}_\rho^{(k)}(\mathbb{R}) &= \left\{ f \in \widetilde{C}_\rho^{(k)}(\mathbb{R}) : \lim_{x \to \pm\infty} \frac{f^{(k)}(x)}{\rho(x)} = 0 \right\}, \\
B_\rho(\mathbb{R}) &= \{ g : \mathbb{R} \to \mathbb{R} : \text{for some positive } m_g,\ |g(x)| \leq m_g \rho(x),\ x \in \mathbb{R} \}.
\end{aligned}$$

As usual, the weighted space $B_\rho(\mathbb{R})$ is endowed with the norm

$$\|g\|_\rho := \sup_{x\in\mathbb{R}} \frac{|g(x)|}{\rho(x)} \quad \text{for } g \in B_\rho(\mathbb{R}).$$

If $k = 0$, then we write $M(\mathbb{R})$, $C_\rho(\mathbb{R})$, $\widetilde{C}_\rho(\mathbb{R})$ and $\widehat{C}_\rho(\mathbb{R})$ instead of $M^{(0)}(\mathbb{R})$, $C_\rho^{(0)}(\mathbb{R})$, $\widetilde{C}_\rho^{(0)}(\mathbb{R})$ and $\widehat{C}_\rho^{(0)}(\mathbb{R})$, respectively.

2. Statistical Approximation Theorems

We first recall that the system of functions $f_0, f_1, ..., f_n$ continuous on an interval $[a, b]$ is called a Tschebyshev system of order n, or $T-$system, if any polynomial

$$P(x) = a_0 f_0(x) + a_1 f_1(x) + ... + a_n f_n(x)$$

has not more than n zeros in this interval with the condition that the numbers $a_0, a_1, ..., a_n$ are not all equal to zero.

Now, following Theorem 3.5 of Duman and Orhan [6] (see also [4, 7]), we obtain the following statistical approximation result at once.

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, and let $\{f_0, f_1, f_2\}$ be $T-$system on an interval $[a, b]$. Assume that $\{L_n\}$ is a sequence of positive linear operators from $C[a, b]$ into itself. If*

$$st_A - \lim_n \|L_n(f_i) - f_i\|_{C[a,b]} = 0, \quad i = 0, 1, 2,$$

then, for all $f \in C[a, b]$, we have

$$st_A - \lim_n \|L_n(f) - f\|_{C[a,b]} = 0,$$

where the symbol $\|\cdot\|_{C[a,b]}$ denotes the usual supremum norm on $C[a, b]$.

We first consider the case of $k = 0$.

Theorem 2.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that the operators $L_n : C_\rho(\mathbb{R}) \to B_\rho(\mathbb{R})$ belong to the class $M(\mathbb{R})$, i.e., they are positive linear operators. Assume further that the following conditions hold:*

(i) $\{f_0, f_1\}$ *and* $\{f_0, f_1, f_2\}$ *are $T-$systems on $\mathbb{R}$,*

(ii) $\lim_{x\to\pm\infty} \dfrac{f_i(x)}{1 + |f_2(x)|} = 0$ *for each* $i = 0, 1$,

(iii) $\lim_{x\to\pm\infty} \dfrac{f_2(x)}{\rho(x)} = m_{f_2} > 0$,

(iv) $st_A - \lim_n \|L_n(f_i) - f_i\|_\rho = 0$ *for each* $i = 0, 1, 2$.

Then, for all $f \in \widetilde{C}_\rho(\mathbb{R})$, we have

$$st_A - \lim \|L_n(f) - f\|_\rho = 0.$$

Proof. Let $f \in \widetilde{C}_\rho(\mathbb{R})$ and define a function g on $\mathbb{R}$ as follows

$$g(y) = m_{f_2}\ f(y) - k_f\ f_2(y), \tag{2.1}$$

where m_{f_2} and k_f are certain constants as in the definitions of the weighted spaces. Then, we easily observe that $g \in \widehat{C}_\rho(\mathbb{R})$. Now we first prove that

$$st_A - \lim_n \|L_n(g) - g\|_\rho = 0. \tag{2.2}$$

Since $\{f_0, f_1\}$ is $T-$system on $\mathbb{R}$, we know from Lemma 2 of [18] that, for each $a \in \mathbb{R}$ satisfying $f_i(a) \neq 0$, $i = 0, 1$, there exists a function $\Phi_a(y)$ such that

$$\Phi_a(a) = 0 \text{ and } \Phi_a(y) > 0 \text{ for } y < a,$$

and the function Φ_a has the following form

$$\Phi_a(y) = \gamma_0(a) f_0(y) + \gamma_1(a) f_1(y), \tag{2.3}$$

where $|\gamma_0(a)| = \left| \dfrac{f_1(a)}{f_0(a)} \right|$, and $|\gamma_1(a)| = 1$. In fact here we define

$$\Phi_a(y) = \begin{cases} F(y), & \text{if } F(y) > 0 \text{ for } y < a \\ -F(y), & \text{if } F(y) < 0 \text{ for } y < a, \end{cases}$$

where

$$F(y) = \frac{f_1(a)}{f_0(a)} f_0(y) - f_1(y).$$

Clearly here $F(a) = 0$, and F has no other root by $\{f_0, f_1\}$ being a $T-$system. On the other hand, by (ii) and (iii), we see, for each $i = 0, 1$, that

$$\frac{f_i(y)}{\rho(y)} = \frac{f_i(y)}{1 + |f_2(y)|} \left(\frac{1}{\rho(y)} + \frac{|f_2(y)|}{\rho(y)} \right) \to 0 \quad \text{as } y \to \pm\infty. \tag{2.4}$$

Now using the fact that $g \in \widehat{C}_\rho(\mathbb{R})$ and also considering (2.4) and (iii), for every $\varepsilon > 0$, there exists a positive number u_0 such that the conditions

$$|g(y)| < \varepsilon \rho(y), \tag{2.5}$$

$$|f_i(y)| < \varepsilon \rho(y), \quad i = 0, 1, \tag{2.6}$$

$$\rho(y) < s_0 f_2(y), \quad \text{(for a certain positive constant } s_0), \tag{2.7}$$

hold for all y with $|y| > u_0$. By (2.5)-(2.7), we can write that

$$|g(y)| < s_0 \varepsilon f_2(y) \quad \text{whenever} \quad |y| > u_0 \tag{2.8}$$

and, for a fixed $a > u_0$ such that $f_i(a) \neq 0, i = 0, 1$,

$$|g(y)| \leq \frac{M}{m_a} \Phi_a(y) \quad \text{whenever} \quad |y| \leq u_0 \tag{2.9}$$

where

$$M := \max_{|y| \leq u_0} |g(y)| \quad \text{and} \quad m_a := \min_{|y| \leq u_0} \Phi_a(y). \tag{2.10}$$

So, combining (2.8) with (2.9), we get

$$|g(y)| < \frac{M}{m_a} \Phi_a(y) + s_0 \varepsilon f_2(y) \quad \text{for all } y \in \mathbb{R}. \tag{2.11}$$

Now, using linearity and monotonicity of the operators L_n, also considering (2.11) and $|\gamma_1(a)| = 1$, we obtain

$$\begin{aligned}
|L_n(g;x)| &\leq L_n\left(|g(y)|;x\right)\\
&\leq \frac{M}{m_a}L_n\left(\Phi_a(y);x\right)+\varepsilon s_0 L_n(f_2(y);x)\\
&= \frac{M}{m_a}\left\{\gamma_0(a)L_n(f_0(y);x)+\gamma_1(a)L_n(f_1(y);x)\right\}+s_0\varepsilon L_n(f_2(y);x)\\
&\leq \frac{M}{m_a}\{|\gamma_0(a)|\,|L_n(f_0(y);x)-f_0(x)|+|L_n(f_1(y);x)-f_1(x)|\}\\
&\quad+\frac{M}{m_a}\left\{\gamma_0(a)f_0(x)+\gamma_1(a)f_1(x)\right\}+\varepsilon s_0\,|L_n(f_2(y);x)-f_2(x)|\\
&\quad+\varepsilon s_0 f_2(x).
\end{aligned}$$

So we have

$$\begin{aligned}
\sup_{|x|>u_0}\frac{|L_n(g(y);x)|}{\rho(x)} &\leq \frac{M}{m_a}\left\{|\gamma_0(a)|\sup_{|x|>u_0}\frac{|L_n(f_0(y);x)-f_0(x)|}{\rho(x)}\right.\\
&\quad\left.+\sup_{|x|>u_0}\frac{|L_n(f_1(y);x)-f_1(x)|}{\rho(x)}\right\}\\
&\quad+\frac{M}{m_a}\left\{|\gamma_0(a)|\sup_{|x|>u_0}\frac{|f_0(x)|}{\rho(x)}+\sup_{|x|>u_0}\frac{|f_1(x)|}{\rho(x)}\right\}\\
&\quad+\varepsilon s_0\sup_{|x|>u_0}\frac{|L_n(f_2(y);x)-f_2(x)|}{\rho(x)}+\varepsilon s_0\sup_{|x|>u_0}\frac{|f_2(x)|}{\rho(x)}.
\end{aligned}$$

But by (2.4) and (iii), we get that

$$A(\varepsilon):=\frac{M}{m_a}\left\{|\gamma_0(a)|\sup_{|x|>u_0}\frac{|f_0(x)|}{\rho(x)}+\sup_{|x|>u_0}\frac{|f_1(x)|}{\rho(x)}\right\}+\varepsilon s_0\sup_{|x|>u_0}\frac{|f_2(x)|}{\rho(x)}$$

is finite for every $\varepsilon>0$. Call now

$$B(\varepsilon):=\max\left\{\frac{M\,|\gamma_0(a)|}{m_a},\frac{M}{m_a},s_0\varepsilon\right\},$$

which is also finite for every $\varepsilon>0$. Then we obtain

$$\sup_{|x|>u_0}\frac{|L_n(g(y);x)|}{\rho(x)}\leq A(\varepsilon)+B(\varepsilon)\sum_{i=0}^{2}\sup_{|x|>u_0}\frac{|L_n(f_i(y);x)-f_i(x)|}{\rho(x)},$$

which implies that

$$\sup_{|x|>u_0}\frac{|L_n(g(y);x)|}{\rho(x)}\leq A(\varepsilon)+B(\varepsilon)\sum_{i=0}^{2}\left\|L_n(f_i)-f_i\right\|_\rho. \tag{2.12}$$

On the other hand, since

$$\|L_n(g)-g\|_\rho\leq\sup_{|x|\leq u_0}\frac{|L_n(g(y);x)-g(x)|}{\rho(x)}+\sup_{|x|>u_0}\frac{|L_n(g(y);x)|}{\rho(x)}+\sup_{|x|>u_0}\frac{|g(x)|}{\rho(x)},$$

it follows from (2.5) and (2.12) that

$$\begin{aligned}\|L_n(g)-g\|_\rho &\leq \varepsilon + A(\varepsilon) + B_1 \|L_n(g)-g\|_{C[-u_0,u_0]} \\ &\quad + B(\varepsilon)\sum_{i=0}^{2}\|L_n(f_i)-f_i\|_\rho\end{aligned} \tag{2.13}$$

holds for every $\varepsilon>0$ and all $n\in\mathbb{N}$, where $B_1=\max_{x\in[-u_0,u_0]}\dfrac{1}{\rho(x)}$. By (iv), we write immediately that

$$st_A-\lim_n \|L_n(f_i)-f_i\|_{C[-u_0,u_0]}=0,\quad i=0,1,2. \tag{2.14}$$

Since $\{f_0,f_1,f_2\}$ is $T-$system and $g\in C[-u_0,u_0]$, we get from (2.14) and Theorem 2.1 that

$$st_A-\lim_n \|L_n(g)-g\|_{C[-u_0,u_0]}=0. \tag{2.15}$$

Now, for a given $r>0$, choose $\varepsilon>0$ such that $0<\varepsilon+A(\varepsilon)<r$. Then, consider the following sets:

$$\begin{aligned}D &:= \left\{n\in\mathbb{N}:\|L_n(g)-g\|_\rho \geq r\right\},\\ D_1 &:= \left\{n\in\mathbb{N}:\|L_n(g)-g\|_{C[-u_0,u_0]} \geq \frac{r-\varepsilon-A(\varepsilon)}{4B_1}\right\},\\ D_2 &:= \left\{n\in\mathbb{N}:\|L_n(f_0)-f_0\|_\rho \geq \frac{r-\varepsilon-A(\varepsilon)}{4B(\varepsilon)}\right\},\\ D_3 &:= \left\{n\in\mathbb{N}:\|L_n(f_1)-f_1\|_\rho \geq \frac{r-\varepsilon-A(\varepsilon)}{4B(\varepsilon)}\right\},\\ D_4 &:= \left\{n\in\mathbb{N}:\|L_n(f_2)-f_2\|_\rho \geq \frac{r-\varepsilon-A(\varepsilon)}{4B(\varepsilon)}\right\}.\end{aligned}$$

From (2.13), we easily observe that

$$D\subseteq D_1\cup D_2\cup D_3\cup D_4,$$

which guarantees

$$\sum_{n\in D}a_{jn}\leq\sum_{n\in D_1}a_{jn}+\sum_{n\in D_2}a_{jn}+\sum_{n\in D_3}a_{jn}+\sum_{n\in D_4}a_{jn}. \tag{2.16}$$

Letting $j\to\infty$ in both sides of the inequality (2.16) and also considering (iv) and (2.15) we get

$$\lim_j\sum_{n\in D}a_{jn}=0.$$

Therefore, we prove (2.2). Now, by (2.1), since $f(y)=\dfrac{1}{m_{f_2}}g(y)+\dfrac{k_f}{m_{f_2}}\ f_2(y)$, we may write, for all $n\in\mathbb{N}$, that

$$\begin{aligned}\|L_n(f)-f\|_\rho &= \left\|L_n\left(\frac{1}{m_{f_2}}g+\frac{k_f}{m_{f_2}}f_2\right)-\left(\frac{1}{m_{f_2}}g+\frac{k_f}{m_{f_2}}f_2\right)\right\|_\rho\\ &\leq \frac{1}{m_{f_2}}\|L_n(g)-g\|_\rho+\frac{k_f}{m_{f_2}}\|L_n(f_2)-f_2\|_\rho.\end{aligned} \tag{2.17}$$

Now for a given $r' > 0$, define the sets

$$\begin{aligned} E &: = \left\{ n \in \mathbb{N} : \|L_n(f) - f\|_\rho \geq r' \right\}, \\ E_1 &: = \left\{ n \in \mathbb{N} : \|L_n(g) - g\|_\rho \geq \frac{m_{f_2}\ r'}{2} \right\}, \\ E_2 &: = \left\{ n \in \mathbb{N} : \|L_n(f_2) - f_2\|_\rho \geq \frac{m_{f_2}\ r'}{2k_f} \right\}. \end{aligned}$$

Then, (2.17) implies that

$$E \subseteq E_1 \cup E_2.$$

So, we get, for all $j \in \mathbb{N}$, that

$$\sum_{n \in E} a_{jn} \leq \sum_{n \in E_1} a_{jn} + \sum_{n \in E_2} a_{jn}. \tag{2.18}$$

Taking limit as $j \to \infty$ in the inequality (2.18), and applying (iv) and (2.2), we immediately obtain

$$\lim_j \sum_{n \in E} a_{jn} = 0,$$

which gives

$$st_A - \lim_n \|L_n(f) - f\|_\rho = 0.$$

The theorem is proved. □

Now, we consider the case of $k \geq 1$.

Theorem 2.3. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that the operators $L_n : C_\rho^{(k)}(\mathbb{R}) \to B_\rho(\mathbb{R})$ belong to the class $M^{(k)}(\mathbb{R})$. Let f_0, f_1, f_2 be functions having k-th continuous derivatives on $\mathbb{R}$. Assume further that the following conditions hold:*

(a) $\{f_0^{(k)}, f_1^{(k)}\}$ *and* $\{f_0^{(k)}, f_1^{(k)}, f_2^{(k)}\}$ *are $T-$systems on $\mathbb{R}$,*

(b) $\lim\limits_{x \to \pm\infty} \dfrac{f_i^{(k)}(x)}{1 + \left| f_2^{(k)}(x) \right|} = 0$ *for each $i = 0, 1$,*

(c) $\lim\limits_{x \to \pm\infty} \dfrac{f_2^{(k)}(x)}{\rho(x)} = m_{f_2}^{(k)} > 0$,

(d) $st_A - \lim_n \left\| L_n(f_i) - f_i^{(k)} \right\|_\rho = 0$ *for each $i = 0, 1, 2$.*

Then, for all $f \in \widetilde{C}_\rho^{(k)}(\mathbb{R})$, we have

$$st_A - \lim_n \left\| L_n(f) - f^{(k)} \right\|_\rho = 0.$$

Proof. We say that $f,\ g \in \widetilde{C}_\rho^{(k)}(\mathbb{R})$ are equivalent provided that $f^{(k)}(x) = g^{(k)}(x)$ for all $x \in \mathbb{R}$. We denote the equivalent classes of $f \in \widetilde{C}_\rho^{(k)}(\mathbb{R})$ by $[f]$. This means that

$$[f] = d^{-k} d^k f,$$

where d^k denotes the k-th derivative operator, and d^{-k} denotes the k-th inverse derivative operator. Thus, by $\left[\widetilde{C}_\rho^{(k)}(\mathbb{R})\right]$ we denote the equivalent weighted spaces

of $\widetilde{C}_{\rho}^{(k)}(\mathbb{R})$. Then, for $f \in \widetilde{C}_{\rho}^{(k)}(\mathbb{R})$, consider

$$L_n([f]) = L_n\left(d^{-k}d^k f\right) =: L_n^*(\psi), \tag{2.19}$$

where $f^{(k)} = \psi \in \widetilde{C}_{\rho}(\mathbb{R})$; and L_n^* is an operator such that $L_n^* = L_n d^{-k}$. Then, we can show that each L_n^* is a positive linear operator from $\widetilde{C}_{\rho}(\mathbb{R})$ into $B_{\rho}(\mathbb{R})$. Indeed, if $\psi \geq 0$, i.e., $f^{(k)} \geq 0$, then since each L_n belongs to the class $M^{(k)}(\mathbb{R})$, it follows from (1.2) that $L_n([f]) \geq 0$, i.e., $L_n^*(\psi) \geq 0$ (see also [18]). Now, for every $x \in \mathbb{R}$, defining

$$\psi_i(x) := f_i^{(k)}(x), \quad i = 0, 1, 2,$$

it follows from $(a) - (d)$ that

$$\begin{aligned}
&\{\psi_0, \psi_1\} \text{ and } \{\psi_0, \psi_1, \psi_2\} \text{ are } T - \text{systems on } \mathbb{R},\\
&\lim_{x\to\pm\infty} \frac{\psi_i(x)}{1 + |\psi_2(x)|} = 0 \ \text{ for each } i = 0, 1,\\
&\lim_{x\to\pm\infty} \frac{\psi_2(x)}{\rho(x)} = m_{\psi_2} > 0\\
&st_A - \lim_n \left\| L_n([f_i]) - f_i^{(k)} \right\|_{\rho} = st_A - \lim_n \left\| L_n^*(\psi_i) - \psi_i \right\|_{\rho} = 0, \ i = 0, 1, 2.
\end{aligned}$$

So, all conditions of Theorem 2.2 are satisfied for the functions ψ_0, ψ_1, ψ_2 and the positive linear operators L_n^* given by (2.19). Therefore, we immediately get

$$st_A - \lim_n \left\| L_n^*(\psi) - \psi \right\|_{\rho} = 0,$$

or equivalently,

$$st_A - \lim_n \left\| L_n(f) - f^{(k)} \right\|_{\rho} = 0.$$

The theorem is proved. □

Finally, we have the following result.

Theorem 2.4. *Assume that conditions* (a), (b) *and* (d) *of Theorem* 2.3 *hold. Let* $\rho_1 : \mathbb{R} \to \mathbb{R}^+ = (0, +\infty)$ *be a function such that* $\rho_1(0) = 1$; ρ_1 *is increasing on* $\mathbb{R}^+$ *and decreasing on* $\mathbb{R}^-$; *and* $\lim_{x\to\pm\infty} \rho_1(x) = +\infty$.*If*

$$\lim_{x\to\pm\infty} \frac{\rho(x)}{\rho_1(x)} = 0, \tag{2.20}$$

and

$$\lim_{x\to\pm\infty} \frac{f_2^{(k)}(x)}{\rho_1(x)} = m_{f_2}^{(k)} > 0 \tag{2.21}$$

then, for all $f \in C_{\rho}^{(k)}(\mathbb{R})$, *we have*

$$st_A - \lim_n \left\| L_n(f) - f^{(k)} \right\|_{\rho_1} = 0.$$

Proof. Let $f \in C_{\rho}^{(k)}(\mathbb{R})$. Since $\dfrac{\left|f^{(k)}(x)\right|}{\rho(x)} \leq m_f$ for every $x \in \mathbb{R}$, we get

$$\lim_{x\to\pm\infty} \frac{\left|f^{(k)}(x)\right|}{\rho_1(x)} \leq \lim_{x\to\pm\infty} \frac{\left|f^{(k)}(x)\right|}{\rho(x)} \frac{\rho(x)}{\rho_1(x)} \leq m_f \lim_{x\to\pm\infty} \frac{\rho(x)}{\rho_1(x)}.$$

Then, by (2.20), we easily obtain that

$$\lim_{x\to\pm\infty}\frac{f^{(k)}(x)}{\rho_1(x)}=0,$$

which yields

$$f\in\widehat{C}_{\rho_1}^{(k)}(\mathbb{R})\subset\widetilde{C}_{\rho_1}^{(k)}(\mathbb{R}).$$

Also observe that, by (2.20), condition (d) of Theorem 2.3 are satisfied for the weight function ρ_1. So, the proof follows from Theorem 2.3 and condition (2.21) at once. □

3. Concluding Remarks

If we replace the matrix $A=(a_{jn})$ in Theorems 2.3 and 2.4 with the identity matrix, then one can immediately get the following results in [18], respectively.

Corollary 3.1 ([18]). *Let f_0,f_1,f_2 be functions having k-th continuous derivatives on $\mathbb{R}$ such that $\{f_0^{(k)},f_1^{(k)}\}$ and $\{f_0^{(k)},f_1^{(k)},f_2^{(k)}\}$ are $T-$systems on $\mathbb{R}$. Assume that the operators $L_n:C_\rho^{(k)}(\mathbb{R})\to B_\rho(\mathbb{R})$ belong to the class $M^{(k)}(\mathbb{R})$. Assume further that the following conditions hold:*

(i) $\lim\limits_{t\to\pm\infty}\dfrac{f_i^{(k)}(x)}{1+\left|f_2^{(k)}(x)\right|}=0\quad(i=0,1)$,

(ii) $\lim\limits_{t\to\pm\infty}\dfrac{f_2^{(k)}(x)}{\rho(x)}=m_{f_2}^{(k)}>0$,

$(iii)\lim_n\left\|L_n(f_i)-f_i^{(k)}\right\|_\rho=0\quad(i=0,1,2)$.

Then, for all $f\in\widetilde{C}_\rho^{(k)}(\mathbb{R})$, $\lim_n\left\|L_n(f)-f^{(k)}\right\|_\rho=0$.

Corollary 3.2 ([18]). *Assume that conditions (i) and (iii) of Corollary 3.1 are satisfied. If (2.20) and (2.21) hold, then, for all $C_\rho^{(k)}(\mathbb{R})$,*

$$\lim_n\left\|L_n(f)-f^{(k)}\right\|_{\rho_1}=0.$$

Assume now that $\{L_n\}$ is a sequence of linear operators satisfying all conditions of Corollary 3.1. Let $A=(a_{nk})$ be a non-negative regular matrix such that $\lim_j\max_n\{a_{jn}\}=0$. In this case, we know [11] that $A-$statistical convergence is stronger than the ordinary convergence. So, we can choose a sequence (u_n) that is $A-$statistically null but non-convergent (in the usual sense). Without loss of generality we may assume that (u_n) is a non-negative; otherwise we would replace (u_n) by $(|u_n|)$. Now define

$$T_n(f;x):=(1+u_n)L_n(f;x). \tag{3.1}$$

By Corollary 3.1, we get, for all $f\in\widetilde{C}_\rho^{(k)}(\mathbb{R})$, that

$$\lim_n\left\|L_n(f)-f^{(k)}\right\|_\rho=0. \tag{3.2}$$

Since $st_A-\lim u_n=0$, it follows from (3.1) and (3.2) that

$$st_A-\lim_n\left\|T_n(f)-f^{(k)}\right\|_\rho=0.$$

However, since (u_n) is non-convergent, the sequence $\left\{\left\|T_n(f)-f^{(k)}\right\|_\rho\right\}$ does not converge to zero. So, Corollary 3.1 does not work for the operators T_n given by (3.1) while Theorem 2.3 still works. It clearly shows that our results are non-trivial generalizations of that of Efendiev [18]. Observe that if one takes $A=C_1$, the Cesàro matrix of order one, then Theorem 1 of [7] is an immediate consequence of our Theorem 2.3.

Now, in Theorem 2.4, take $k=0$ and choose

$$f_i(x)=\frac{x^i\rho(x)}{1+x^2},\quad i=0,1,2. \tag{3.3}$$

Then, it is easy to check that $\{f_0,f_1\}$ and $\{f_0,f_1,f_2\}$ are $T-$systems on $\mathbb{R}$. We also observe that the test functions f_i given by (3.3) satisfy the following conditions.

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{f_0(x)}{1+|f_2(x)|} &= \lim_{x\to\pm\infty}\frac{\rho(x)}{1+x^2+x^2\rho(x)}=0,\\ \lim_{x\to\pm\infty}\frac{f_1(x)}{1+|f_2(x)|} &= \lim_{x\to\pm\infty}\frac{x\rho(x)}{1+x^2+x^2\rho(x)}=0,\\ \lim_{x\to\pm\infty}\frac{f_2(x)}{\rho(x)} &= \lim_{x\to\pm\infty}\frac{x^2}{1+x^2}=1.\end{aligned}$$

Therefore, with these choices, Theorem 3 of [5] is an immediate consequence of Theorem 2.4 for $k=0$ as follows:

Corollary 3.3 ([5])**.** *Let $A=(a_{jn})$ be a non-negative regular summability matrix, and let $\{L_n\}$ be a sequence of positive linear operators from $C_\rho(\mathbb{R})$ into $B_\rho(\mathbb{R})$. Assume that the weight functions ρ and ρ_1 satisfy* (2.20)*. If*

$$st_A-\lim_n\|L_n(f_i)-f_i\|_\rho=0\ ,\quad i=0,1,2,$$

where the functions f_i is given by (3.3)*, then, for all $f\in C_\rho(\mathbb{R})$, we have*

$$st_A-\lim_n\|L_n(f)-f\|_{\rho_1}=0.$$

Finally, if we replace the matrix $A=(a_{jn})$ in Corollary 3.3 with the identity matrix, then we obtain the following classical weighted approximation result for a sequence of positive linear operators (see [22, 23]).

Corollary 3.4. *Let $\{L_n\}$ be a sequence of positive linear operators from $C_\rho(\mathbb{R})$ into $B_\rho(\mathbb{R})$. Assume that the weight functions ρ and ρ_1 satisfy* (2.20)*. If*

$$\lim_n\|L_n(f_i)-f_i\|_\rho=0\ ,\quad i=0,1,2,$$

where the functions f_i is given by (3.3)*, then, for all $f\in C_\rho(\mathbb{R})$, we have*

$$\lim_n\|L_n(f)-f\|_{\rho_1}=0.$$

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GAMMA BETA PRODUCTS

by

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ABSTRACT: The important problem of the product of gamma and beta distributed random variables is considered. More than ten motivating applications are discussed from diverse areas such as expected cost modeling, fading channels in wireless communication, bioinformatics, income modeling, intracell interference, insurance, reliability, mortality, video teleconferencing, production economics, industrial hygiene, time series modeling, probability theory, hydrology and building energy consumption. Exact expressions are derived for the probability density function, cumulative distribution function and moment properties of the product. A program is provided for computing the associated percentage points.

KEYWORDS AND PHRASES: Beta distribution; Gamma distribution; Products of random variables.

1 INTRODUCTION

For given random variables X and Y, the distribution of the product $Z = XY$ is of interest in many areas of the sciences, engineering and medicine. In this paper, we study the distribution of $Z = XY$ when X and Y are independent random variables with X having the gamma distribution given by the probability density function (pdf):

$$f_X(x) = \frac{\lambda^\beta x^{\beta-1} \exp(-\lambda x)}{\Gamma(\beta)} \tag{1}$$

(for $x > 0$, $\beta > 0$ and $\lambda > 0$) and Y having the beta distribution given by the pdf:

$$f_Y(y) = \frac{y^{a-1}(1-y)^{b-1}}{B(a,b)} \tag{2}$$

(for $0 < y < 1$, $a > 0$ and $b > 0$). The study of this particular product is of importance in many applied areas. More than ten motivating examples are discussed in Section 2. The exact expressions for both the pdf and the cumulative distribution function (cdf) of $Z = XY$ are derived in Section 3. The moment properties of $Z = XY$ including characteristic function, moments, factorial moments, skewness and kurtosis are considered in Section 4. The proofs are given for the main results but are omitted when particular cases are considered. The detailed proofs of all the results can be obtained from the author. In Section 5, the percentage points associated with $Z = XY$ are considered and a computer program provided.

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The calculations of this paper involve several special functions, including the Gauss hypergeometric function defined by

$$ {}_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\frac{x^k}{k!}, $$

the ${}_2F_2$ hypergeometric function defined by

$$ {}_2F_2\left(a,b;c,d;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,(d)_k}\frac{x^k}{k!}, $$

the incomplete gamma function defined by

$$ \gamma(a,x) = \int_0^x \exp(-t)t^{a-1}dt, $$

the complementary incomplete gamma function defined by

$$ \Gamma(a,x) = \int_x^{\infty} \exp(-t)t^{a-1}dt $$

and, the Kummer function defined by

$$ \Psi(a,b;x) = \frac{1}{\Gamma(a)}\int_0^{\infty} \exp\left(-xt\right)t^{a-1}(1+t)^{b-a-1}dt, $$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 MOTIVATING APPLICATIONS

2.1 UNDER REPORTED INCOME

In the economic literature, the under reported income is commonly expressed by the multiplicative relationship $Z = XY$, where Y is a multiplicative error and X denotes the true income. It is well known that if Y has the power function distribution (a particular case of the beta distribution) then X is Pareto distributed if and only if Z is also, see Krishnaji (1970). In practice, the gamma distribution is often preferred as a model for income, see e.g. Grandmont (1987), Milevsky (1997), Sarabia *et al.* (2002), and Silver *et al.* (2002). This raises the important question: what is the distribution of the under reported income $Z = XY$ if X is gamma distributed?

2.2 HYDROLOGY

Let X and Y be independent random variables representing the rainfall intensity and the duration of a storm, respectively. Then $Z = XY$ will represent the amount of rainfall produced by that storm. Since gamma distributions are popular models for rainfall intensity and since Y will have a probable maximum it will be most reasonable to assume that X and Y are distributed according to (1) and (2), respectively, after suitable scaling.

Another example is the distribution of peak runoff considered by Gottschalk and Weingartner (1998). Peak runoff is the product of the rainfall volume scaled with respect to its duration, which is assumed to follow a gamma distribution, and the runoff coefficient, which is assumed to follow a beta distribution.

2.3 INSURANCE MODELING

A recent paper by Frostig (2001) states that "... an individual risk is a product of two random variables: (1) a Bernoulli random variable which is an indicator for the event that claim has occurred; (2) the claim amount, which is a positive random variable" From a statistical perspective, it will be reasonable to rephrase the definition of individual risk as the product $Z = XY$, where X denotes the claim amount and Y denotes the probability that the event that claim has occurred. Since gamma distributions are popular models for claim amounts (see ter Berg (1980/81), Gerber (1992) and dos Reis (1993); to mention just a few), it will be reasonable to assume that X is a gamma random variable. On the other hand, since beta distributions are the only standard models for data on the unit interval, it will be reasonable to assume that Y is a beta random variable.

2.4 MORTALITY

An example similar to the above arises with respect to mortality. In fact, mortality is the product of two random variables: 1) a Bernoulli random variable equal to zero with probability h and equal to one with probability $1 - h$. It determines whether or not any biomass is lost to mortality; 2) L, the amount of biomass lost given a mortality event.

2.5 BIOINFORMATICS

Products of gamma and beta random variables arise also in theoretical aspects of bioinformatics. An example is the model for crossover process involving two non–sister chromatids, i.e. chromatids originating from homologous chromosomes, see Zhao and Speed (2006). Models for the crossover process are important to the understanding of crossing over mechanisms, to the construction of genetic maps, and to the strategy of finding disease genes.

2.6 FADING CHANNELS

Wireless communication systems are subject to fading arising out of multipath propagation. This short–term fading leads to variation in signal strength which may result in the loss of signal. The most commonly known model for the received signal envelope, Z say, is the Nakagami distribution (Nakagami, 1960) given by the pdf:

$$f_Z(z) = \frac{2\alpha^\alpha z^{2\alpha-1}\exp\left(-\alpha z^2/\lambda\right)}{\Gamma(\alpha)\lambda^\alpha},$$

where α is the Nakagami parameter and λ is the average power given by $< Z^2 >$. When the wireless channel is also subject to shadowing, the local mean power becomes random (Simon and Alouni, 2000). This can be taken into account by defining λ to be a random

variable, say Λ. Thus, in the presence of shadowing, the pdf of the envelope in combined fading and shadowing can be expressed as:

$$f_Z(z) = 2\alpha z \int_0^\infty \frac{x^{\alpha-1}\exp(-x/\lambda)}{\Gamma(\alpha)\lambda^\alpha} g(\lambda)d\lambda, \quad (3)$$

where $x = \alpha z^2$ and where $g(\cdot)$ denotes the pdf of Λ. If $g(\cdot)$ is a beta pdf then the integral in (3) amounts to computing the product of gamma and beta random variables.

2.7 VIDEO TELECONFERENCING

A flexible model called the group-of-pictures gammabeta auto-regression (GOP GBAR) model for video teleconferencing is based on products of independent gamma and beta random variables, see Heyman (1997) and Frey and Nguyen–Quang (2000) for details.

2.8 EXPECTED COST

Suppose an event occurs with probability p and results in a cost c. The expected cost will be $c \times p + 0 \times (1-p) = cp$. In reality, both c and p will be subject to some random errors and so will the expected cost. Assume that X and Y are independent random variables representing the values of c and p, respectively. The most natural model for X will be the gamma distribution (it being the most popular model for skewed data) given by (1). The most natural model for Y will be the beta distribution (the only standard model for data on the unit interval) given by (2). Thus, the expected cost can be represented by the random variable $Z = XY$.

2.9 INTRACELL INTERFERENCE

In multi–cellular multiple input–multiple-output (MIMO) systems, there is co–channel interference from users within the same cell as well as from other cell users. Consistent with practical scenarios, the co–channel interference is categorized into two groups: intracell interference from users within the same cell as the desired user and intercell interference from outer cell users. Tokgoz and Rao (2006) have shown that the intracell interference distribution can be approximated by that of a product of independent gamma and beta random variables.

2.10 PRODUCTION ECONOMICS

In production economics, one is often interested in estimating the system failure rate in various levels of criticality in order to incorporate failure data in one level into analyzing failure rates in any other levels. A common model is that the gamma distribution is assumed for the initial failure rate and the beta for the criticality probabilities, see e.g. Jun *et al.* (1999). This naturally gives rise to products of gamma and beta random variables.

2.11 INDUSTRIAL HYGIENE

It is well established in the industrial hygiene literature that the beta distribution is a good descriptor of respirator penetration values experienced by an individual worker from wearing to wearing, and of average respirator penetration values experienced by different workers, see e.g. Nicas (1996). Also, the gamma distribution can reasonably describe the time-varying Mycobacterium tuberculosis aerosol exposure levels experienced by health care workers. Thus, the computation of such quantities as an individual worker's cumulative risk of infection, and the worker population mean cumulative risk of infection will involve products of gamma and beta random variables.

2.12 RELIABILITY

Rekha and Sunder (1997) examined the survival of a component, prone to attacks by a succession of random stresses, under the assumption that the strength of the component is attenuated and these attenuated factors are random variables. Three scenarios were considered: (i) when the stress strength follows exponential distribution and the random attenuation factors follow rectangular distribution; (ii) when the stress strength follows exponential distribution with the attenuation factors following beta distribution; (iii) when the stress following exponential distribution, strength following gamma distribution and the random attenuation factors following beta distribution. Each case clearly involves the product of gamma and beta random variables.

Another reliability problem with products of the above kind has been studied by Sahinoglu and Libby (2003). The Forced Outage Ratio of a hardware or a software component is defined as the failure rate divided by the sum of the failure and the repair rates. The common model for the FOR is a beta distribution. The common models for failure and repair rates are the gamma distributions.

2.13 BEHRENS–FISHER PROBLEM

The generalized Behrens-Fisher distribution is defined as the convolution of two Student's t distributions and is related to the inverted–gamma distribution by means of a representation theorem as a scale mixture of normals where the mixing distribution is a convolution of two inverted–gamma distributions. Thus, results on the convolution of inverted–gamma distributions may result on interesting properties of the Behrens–Fisher distribution (Giron and del Castillo, 2001). If the two inverted–gamma random variables, say X and Y, have the same scale parameter then their convolution is equivalent to the product of gamma and beta random variables. This is easy to see because $X + Y = 1/(1/X) + 1/(1/Y) = (1/X + 1/Y)/((1/X)(1/Y))$.

2.14 BUILDING ENERGY CONSUMPTION

Meng *et al.* (2007) modeled monthly, daily and hourly rainfall patterns in Guangzhou area of China in an effort to cut down on building energy consumption. Meng *et al.* (2007) found that "distribution of daily rainfall month–by–month shows a gamma distribution model agrees well with daily rainfall distribution" and that "distribution patterns of hourly

rainfall percentage, both in the rainy season and non–rainy season, coincide well with the beta distribution". Thus, we have a case of product of gamma and beta random variables.

2.15 TIME SERIES MODELING

Finally, we would like to mention that products of gamma and beta random variables arise with respect to time series modeling of count data, see e.g. Kuk (1999).

3 EXACT DISTRIBUTION OF $Z = XY$

Theorem 1 expresses the pdf and the cdf of $Z = XY$ in terms of the Kummer function and the ${}_2F_2$ hypergeometric function, respectively.

THEOREM 1 *Suppose X and Y are distributed according to (1) and (2), respectively. The cdf of $Z = XY$ can be expressed as:*

$$F_Z(z) = \frac{1}{\Gamma(\beta)B(a,b)}\left[\frac{(\lambda z)^\beta}{\beta}B(b,a-\beta)\ {}_2F_2(1-a-b+\beta,\beta;\beta+1,1-a+\beta;-\lambda z)\right.$$
$$\left.+\frac{(\lambda z)^a}{a}\Gamma(\beta-a)\ {}_2F_2(a,1-b;a-\beta+1,1+a;-\lambda z)\right] \quad (4)$$

for $z > 0$. The corresponding pdf of $Z = XY$ is

$$f_Z(z) = \frac{\lambda^\beta\Gamma(b)}{\Gamma(\beta)B(a,b)}z^{\beta-1}\exp(-\lambda z)\Psi(b,1+\beta-a;\lambda z) \quad (5)$$

for $z > 0$.

PROOF: The cdf corresponding to (1) is $1-\Gamma(\beta,\lambda x)/\Gamma(\beta)$. Thus, one can write the cdf of $Z = XY$ as

$$\begin{aligned}\Pr(XY \le z) &= \int_0^1 F_X(z/y)f_Y(y)dy \\ &= 1-\frac{1}{\Gamma(\beta)B(a,b)}\int_0^1\Gamma\left(\beta,\frac{\lambda z}{y}\right)y^{a-1}(1-y)^{b-1}dy \\ &= 1-\frac{1}{\Gamma(\beta)B(a,b)}\int_1^\infty\Gamma(\beta,\lambda zw)\,w^{-(a+b)}(w-1)^{b-1}dw \\ &= 1-\frac{1}{\Gamma(\beta)B(a,b)}I, \end{aligned} \quad (6)$$

which follows after setting $w = 1/y$. Application of equation (2.10.2.3) in Prudnikov *et al.* (1986, volume 2) shows that the integral I can be calculated as

$$I = \Gamma(\beta)B(a,b)-\frac{(\lambda z)^\beta}{\beta}B(b,a-\beta)\ {}_2F_2(1-a-b+\beta,\beta;\beta+1,1-a+\beta;-\lambda z)$$
$$-\frac{(\lambda z)^a}{a}\Gamma(\beta-a)\ {}_2F_2(a,1-b;a-\beta+1,1+a;-\lambda z). \quad (7)$$

The result in (4) follows by substituting (7) into (6). The pdf in (5) follows by differentiation and using properties of the ${}_2F_2$ hypergeometric function. ■

Springer and Thompson (1970) derived an exact expression involving the MeijerG function for the pdf of the product of m beta distributed random variables with parameters (a_k, b_k) and $n-m$ gamma distributed random variables with shape parameters c_k and scale parameters set to 1 (where all of the n random variables are assumed independent). The result of Theorem 1 is a particular case of this result for $m = 1$ and $n = 2$. The advantage of Theorem 1 over the result of Springer and Thompson (1970) is that the ${}_2F_2$ hypergeometric function is much simpler to compute than the MeijerG function. In fact, there are not many numerical routines for computing the MeijerG function.

If $a = q$, $b = p - q$ (for $0 < q < p$) and $\beta = p$ then it can be shown that (4) and (5) reduce to a gamma distribution. This corresponds to a result given in Stuart (1962). One can derive several other simpler forms of (4) when a, b and β take integer or half integer values. This is illustrated in the corollaries below.

COROLLARY 1 *If $\beta \geq 1$ is an integer then (4) can be reduced to the simpler form*

$$F_Z(z) = 1 - \frac{\Gamma(b)\exp(-\lambda z)}{B(a,b)} \sum_{k=0}^{\beta-1} \frac{(\lambda z)^k}{k!} \Psi\left(b, 1+k-a; \lambda z\right)$$

for $z > 0$.

COROLLARY 2 *If $\beta - 1/2 = n \geq 1$ is an integer then (4) can be reduced to the simpler form*

$$\begin{aligned} F_Z(z) = & -2\sqrt{\frac{\lambda z}{\pi}} \frac{B\left(b, a-1/2\right)}{B\left(a,b\right)} \, {}_2F_2\left(\frac{1}{2}, \frac{3}{2}-a-b; \frac{3}{2}-a, \frac{3}{2}; -\lambda z\right) \\ & - \frac{(\lambda z)^a}{\sqrt{\pi} a} \frac{\Gamma\left(1/2-a\right)}{B\left(a,b\right)} \, {}_2F_2\left(a, 1-b; 1+a, a+\frac{1}{2}; -\lambda z\right) \\ & + \frac{(-1)^n \Gamma(b) \sqrt{\lambda z} \exp(-\lambda z)}{\Gamma(\beta) B(a,b)} \sum_{k=0}^{n-1} (-\lambda z)^k \left(\frac{1}{2}-n\right)_{n-k-1} \Psi\left(b, k+\frac{3}{2}-a; \lambda z\right) \end{aligned}$$

for $z > 0$.

COROLLARY 3 *If $a \geq 1$ is an integer then (4) can be reduced to the simpler form*

$$F_Z(z) = 1 - \frac{b(\lambda z)^\beta \exp(-\lambda z)}{\Gamma(\beta)} \sum_{k=0}^{a} \frac{\Gamma(b+k-1)}{\Gamma(k)} \Psi\left(b+1, \beta-k+2; \lambda z\right)$$

for $z > 0$.

COROLLARY 4 *If $b \geq 1$ is an integer then (4) can be reduced to the simpler form*

$$F_Z(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} + \frac{(\lambda z)^\beta \exp(-\lambda z)}{\Gamma(a)\Gamma(\beta)} \sum_{k=0}^{b} \Gamma(a+k-1) \Psi\left(k, \beta-a+1; \lambda z\right)$$

for $z > 0$.

COROLLARY 5 *If $a = 1/2$ and $b = 1/2$ then (4) can be reduced to the simpler form*

$$F_Z(z) = \frac{\gamma(\beta, z)}{\Gamma(z)} - \frac{2\lambda^\beta K}{\pi \Gamma(\beta)}$$

for $z > 0$, where

$$K = \int_z^\infty x^{\beta-1} \exp(-\lambda x) \arctan\left(\frac{z}{x-z}\right) dx. \quad (8)$$

COROLLARY 6 *If $a + 1/2 = n \geq 1$ is an integer and $b = 1/2$ then (4) can be reduced to the simpler form*

$$F_Z(z) = \frac{\gamma(\beta,z)}{\Gamma(z)} + \frac{2\lambda^\beta K}{\pi\Gamma(\beta)} - \frac{(\lambda z)^\beta \exp(-\lambda z)}{2\Gamma(\beta)} \sum_{k=1}^{n-1} \frac{\Gamma(k)}{\Gamma(k+1/2)} \Psi\left(\frac{3}{2}, \beta - k + \frac{3}{2}; \lambda z\right)$$

for $z > 0$, where K is given by (8).

COROLLARY 7 *If $b + 1/2 = n \geq 1$ is an integer and $a = 1/2$ then (4) can be reduced to the simpler form*

$$F_Z(z) = \frac{\gamma(\beta,z)}{\Gamma(z)} + \frac{2\lambda^\beta K}{\pi\Gamma(\beta)} + \frac{(\lambda z)^\beta \exp(-\lambda z)}{\sqrt{\pi}\Gamma(\beta)} \sum_{k=1}^{n-1} \Gamma(k) \Psi\left(k + \frac{1}{2}, \beta + \frac{1}{2}; \lambda z\right)$$

for $z > 0$, where K is given by (8).

COROLLARY 8 *If $a + 1/2 = m \geq 1$ and $b + 1/2 = n \geq 1$ are integers then (4) can be reduced to the simpler form*

$$\begin{aligned} F_Z(z) = {} & \frac{\gamma(\beta,z)}{\Gamma(z)} + \frac{2\lambda^\beta K}{\pi\Gamma(\beta)} - \frac{(\lambda z)^\beta \exp(-\lambda z)}{2\Gamma(\beta)} \sum_{k=1}^{m-1} \frac{\Gamma(k)}{\Gamma(k+1/2)} \Psi\left(\frac{3}{2}, \beta - k + \frac{3}{2}; \lambda z\right) \\ & + \frac{(\lambda z)^\beta \exp(-\lambda z)}{\Gamma(\beta)} \sum_{k=1}^{n-1} \frac{\Gamma(m+k-1)}{\Gamma(m-1/2)} \Psi\left(k + \frac{1}{2}, \beta - m + \frac{3}{2}; \lambda z\right) \end{aligned}$$

for $z > 0$, where K is given by (8).

The formulas for $F_Z(z)$ in the corollaries above can be used to save computational time since the computation of the hypergeometric term in (4) can be more demanding. They can be computed by using the KummerU $(\cdot, \cdot, \cdot)$ function in MAPLE.

[Figure 1 about here.]

Figure 1 illustrates possible shapes of the pdf (5) for selected values of a, b and β (the pdf computed by using the KummerU $(\cdot, \cdot, \cdot)$ function in MAPLE). The four curves in each plot correspond to selected values of β. As expected, the densities are unimodal and the effect of the parameters is evident. The parameter β controls the skewness of the densities while their general shape is determined by a and b.

4 MOMENT PROPERTIES OF $Z = XY$

The moment properties of $Z = XY$ can be derived by knowing the same for X and Y. It is well known (see, for example, Johnson *et al.* (1994)) that

$$E(X^n) = \frac{\Gamma(\beta+n)}{\lambda^n \Gamma(\beta)}$$

and

$$E\left(Y^n\right) = \frac{B(a+n,b)}{B(a,b)}.$$

Thus, the nth moment of $Z = XY$ is

$$E\left(Z^n\right) = \frac{\Gamma(\beta+n)B(a+n,b)}{\lambda^n\Gamma(\beta)B(a,b)}.$$

In particular,

$$E\left(Z\right) = \frac{\beta a}{\lambda(a+b)},$$

$$E\left(Z^2\right) = \frac{\beta(\beta+1)a(a+1)}{\lambda^2(a+b)(a+b+1)},$$

$$E\left(Z^3\right) = \frac{\beta(\beta+1)(\beta+2)a(a+1)(a+2)}{\lambda^3(a+b)(a+b+1)(a+b+2)}$$

and

$$E\left(Z^4\right) = \frac{\beta(\beta+1)(\beta+2)(\beta+3)a(a+1)(a+2)(a+3)}{\lambda^4(a+b)(a+b+1)(a+b+2)(a+b+3)}.$$

The factorial moments, skewness and the kurtosis can be calculated by using the relationships that

$$E\left[(Z)_n\right] = E\left[Z(Z-1)\cdots(Z-n+1)\right],$$

$$Skewness(Z) = \frac{E\left(Z^3\right) - 3E(Z)E\left(Z^2\right) + 2E^3(Z)}{\left\{E\left(Z^2\right) - E^2\left(Z\right)\right\}^{3/2}},$$

and

$$Kurtosis(Z) = \frac{E\left(Z^4\right) - 4E(Z)E\left(Z^3\right) + 6E\left(Z^2\right)E^2(Z) - 3E^4(Z)}{\left\{E\left(Z^2\right) - E^2\left(Z\right)\right\}^2}.$$

Using the fact that the characteristic function (chf) of X is:

$$E\left[\exp\left(itX\right)\right] = \left(\frac{\lambda}{\lambda - it}\right)^{\beta},$$

the chf of $Z = XY$ can be expressed as

$$\begin{aligned} E\left[\exp\left(itXY\right)\right] &= \int_0^1 \left(\frac{\lambda}{\lambda - ity}\right)^{\beta} f_Y(y)dy \\ &= \frac{\lambda^\beta}{B(a,b)}\int_0^1 \frac{y^{a-1}(1-y)^{b-1}}{(\lambda - ity)^\beta}dy \\ &= \frac{\lambda^\beta}{B(a,b)(-it)^\beta}I. \end{aligned} \tag{9}$$

Application of equation (2.2.6.15) in Prudnikov *et al.* (1986, volume 1) shows that the integral I can be calculated as

$$I = B(a,b)\left(-\frac{\lambda}{it}\right)^{-\beta} {}_2F_1\left(a,\beta;a+b;\frac{it}{\lambda}\right). \tag{10}$$

Substituting (10) into (9), one obtains

$$E\left[\exp\left(itXY\right)\right] = {}_2F_1\left(a,\beta;a+b;\frac{it}{\lambda}\right). \tag{11}$$

Using well-known transformation formulas for the Gauss hypergeometric function, one can obtain the following alternative forms of (11):

$$E\left[\exp\left(itXY\right)\right] = \left(1-\frac{it}{\lambda}\right)^{-a} {}_2F_1\left(a,a+b-\beta;a+b;\frac{it}{it-\lambda}\right),$$

$$E\left[\exp\left(itXY\right)\right] = \left(1-\frac{it}{\lambda}\right)^{-\beta} {}_2F_1\left(\beta,b;a+b;\frac{it}{it-\lambda}\right)$$

and

$$E\left[\exp\left(itXY\right)\right] = \left(1-\frac{it}{\lambda}\right)^{b-\beta} {}_2F_1\left(b,a+b-\beta;a+b;\frac{it}{\lambda}\right).$$

If a, b and β take integer values then, using special properties of the Gauss hypergeometric function, one can obtain the following elementary form of (11):

$$E\left[\exp\left(itXY\right)\right] = \frac{\lambda^a}{t^a B(a,b)}\sum_{k=0}^{b-1}\sum_{l=0}^{\beta}\binom{b-1}{k}\binom{\beta}{l}(-1)^k(-i)^l(\lambda/t)^k P(a+k+l-1),$$

where $P(m)$ satisfies the recurrence relation

$$P(m) = \frac{1}{1+m-2\beta}\frac{(t/\lambda)^{m-1}}{(1+t/\lambda)^{\beta-1}} + \frac{m-1}{2\beta-m-1}P(m-2)$$

with the initial values

$$P(1) = \begin{cases} \frac{1}{2}\log\left(1+\frac{t^2}{\lambda^2}\right), & \text{if } \beta = 1, \\ \frac{1}{2(1-\beta)}\left\{\left(1+\frac{t^2}{\lambda^2}\right)^{1-\beta}-1\right\}, & \text{if } \beta > 1 \end{cases}$$

and

$$P(0) = \frac{t}{(2\beta-1)\lambda}\sum_{k=1}^{\beta-1}\frac{(2\beta-1)(2\beta-3)\cdots(2\beta-2k+1)}{2^k(\beta-1)(\beta-2)\cdots(\beta-k)}\left(1+\frac{t^2}{\lambda^2}\right)^{k-\beta} + \frac{(2\beta-3)!!}{2^{\beta-1}(\beta-1)!}\arctan\left(\frac{t}{\lambda}\right).$$

5 PERCENTILES OF $Z = XY$

In this section, we provide a program for computing the percentage points z_p associated with the cdf of $Z = XY$. The percentage points are obtained numerically by solving the equation

$$\frac{1}{\Gamma(\beta)B(a,b)}\left[\frac{(\lambda z_p)^{\beta}}{\beta}B\left(b,a-\beta\right)\ {}_2F_2\left(1-a-b+\beta,\beta;\beta+1,1-a+\beta;-\lambda z_p\right)\right.$$
$$\left.+\frac{(\lambda z_p)^{a}}{a}\Gamma\left(\beta-a\right)\ {}_2F_2\left(a,1-b;a-\beta+1,1+a;-\lambda z_p\right)\right]=p. \qquad (12)$$

Evidently, this involves computation of the hypergeometric function and routines for this are widely available. We used the function hypergeom ([·, ·],[·, ·], ·) in MAPLE. The following 6–line program in MAPLE solves (12) for given p, λ, β, a and b.

```
c1:=(1/beta)*((lambda*z)**beta)*Beta(b,a-beta):
c2:=(1/a)*((lambda*z)**a)*GAMMA(beta-a):
f1:=hypergeom([1-a-b+beta,beta],[beta+1,1-a+beta],-lambda*z):
f2:=hypergeom([a,1-b],[a-beta+1,1+a],-lambda*z):
ff:=(c1*f1+c2*f2)/(Beta(a,b)*GAMMA(beta)):
fsolve(ff=p,z=0..10000):
```

We expect that this program could be useful for applications of the type described in Section 2. For instance, z_{1-p} will be the under reported income that will be exceeded with probability p, see Example 1 of Section 2. Similarly, in example 2 of Section 2, the extreme percentile points of the amount of rainfall will be useful for purposes such as building of dams.

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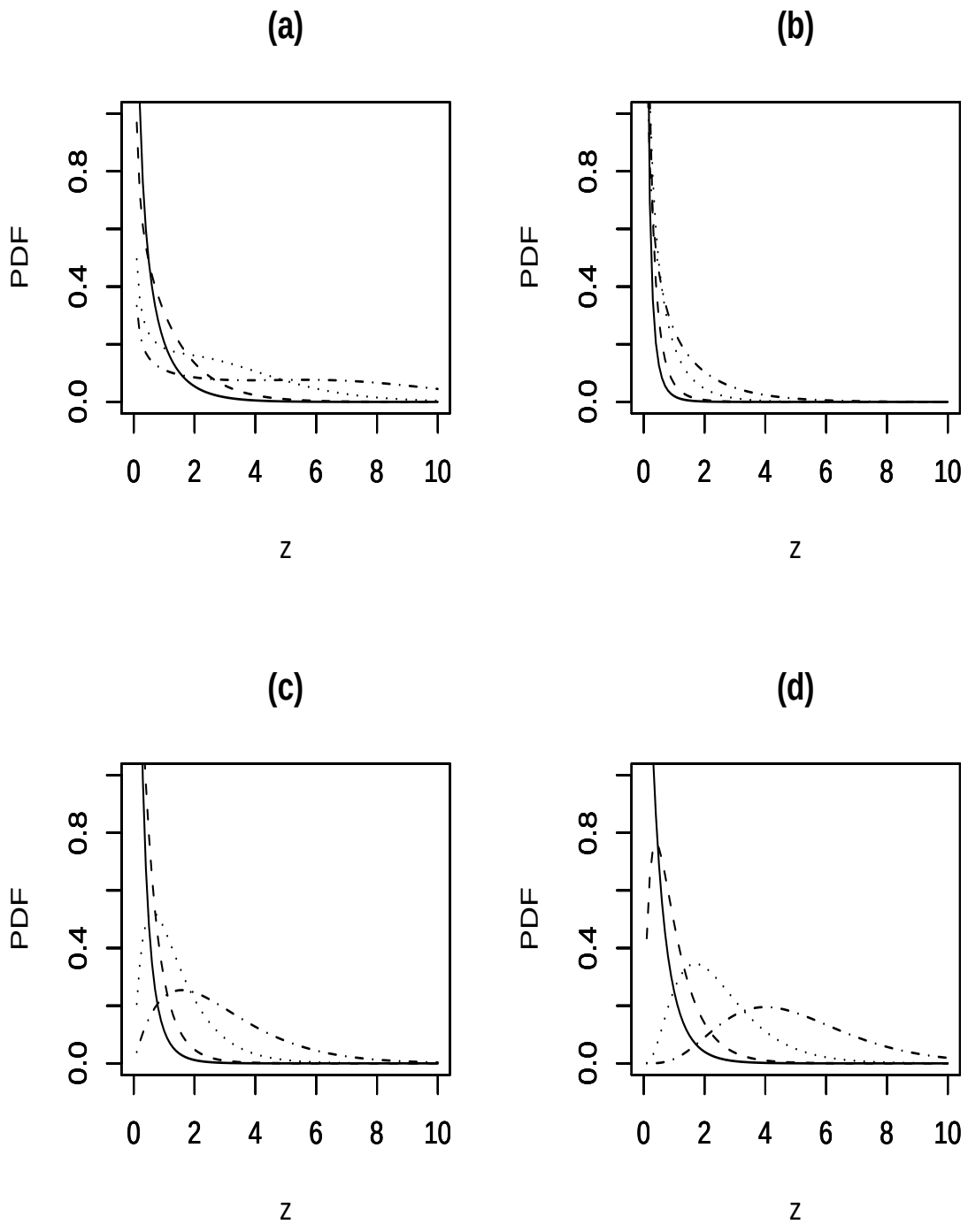


FIGURE 1. Plots of the pdf (5) for $\lambda = 1$, $\beta = 1, 2, 5, 10$ and (a): $a = 0.5$, $b = 0.5$; (b): $a = 0.5$, $b = 5$; (c): $a = 2$, $b = 5$; and, (d): $a = 5$, $b = 5$. The four curves in each plot are: the solid curve ($\beta = 1$), the curve of dots ($\beta = 2$), the curve of lines ($\beta = 5$), and the curve of dots and lines ($\beta = 10$).

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A summability factor theorem by using an almost increasing sequence

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Abstract

In this paper, by applying the concept of an almost increasing sequence, the author presents a generalization of a known result on the $\mid \bar{N}, p_n \mid_k$ summability for the $\mid A, p_n \mid_k$ summability factors.

1 Introduction.

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \ n = 0, 1, ... \tag{1}$$

The series $\sum a_n$ is said to be summable $\mid A \mid_k, k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} n^{k-1} \mid \overline{\Delta} A_n(s) \mid^k < \infty \tag{2}$$

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where

$$\overline{\Delta}A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{3}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{4}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty, \tag{5}$$

and it is said to be summable $\mid A, p_n \mid_k$, $k \geq 1$, if (see [7])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid \overline{\Delta}A_n(s) \mid^k < \infty. \tag{6}$$

In the special case when $p_n = 1$ for all n, $\mid A, p_n \mid_k$ summability is the same as $\mid A \mid_k$ summability. Also if we take $a_{nv} = \frac{p_v}{P_n}$, then $\mid A, p_n \mid_k$ summability is the same as $\mid \bar{N}, p_n \mid_k$ summability.

2 Known results.

Bor [3] has proved the following theorem on $\mid \bar{N}, p_n \mid_k$ summability factors of an infinite series.

Theorem A. Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{7}$$

If (X_n) is a positive monotonic non-decreasing sequence such that

$$\mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty \tag{8}$$

$$\sum_{n=1}^{m} nX_n \mid \Delta^2 \lambda_n \mid = O(1) \tag{9}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \mid t_n \mid^k = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k, k \geq 1$.

If we take $p_n = 1$ for all values of n, then we get a result of Mazhar [5] for $\mid C, 1 \mid_k$ summability factors.

Later on, Mazhar [6] has proved Theorem A under weaker conditions for $\mid \bar{N}, p_n \mid_k$ summability in the following form by using an almost increasing sequence.

Theorem B. Let (X_n) be an almost increasing sequence such that conditions (8)-(10) of Theorem A are satisfied. If

$$\sum_{n=1}^{m} \frac{\mid t_n \mid^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty, \tag{12}$$

then the series $\sum a_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k, k \geq 1$.

It should be remarked that the condition (7) implies the condition (12), but the converse need not be true (see [6] for details.)

3 The main result.

In the present paper, we make use of the concept of an almost increasing sequence in order to generalize Theorem B for the $\mid A, p_n \mid_k$ summability. Before stating the main theorem we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semi-matrices $\overline{A} = (\overline{a}_{nv})$ and $\widehat{A} = (\widehat{a}_{nv})$ as follows:

$$\overline{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \ n, v = 0, 1, ... \tag{13}$$

and

$$\widehat{a}_{00} = \overline{a}_{00} = a_{00}, \ \widehat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}, \ n = 1, 2, ... \tag{14}$$

It may be noted that $\overline{A}$ and $\widehat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) \quad = \quad \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} a_{nv} \sum_{i=0}^{v} a_i$$

$$= \sum_{i=0}^{n} a_i \sum_{v=i}^{n} a_{nv} = \sum_{i=0}^{n} \overline{a}_{ni} a_i$$

and

$$\begin{aligned} \overline{\Delta} A_n(s) &= \sum_{i=0}^{n} \overline{a}_{ni} a_i - \sum_{i=0}^{n-1} \overline{a}_{n-1,i} a_i \\ &= \overline{a}_{nn} a_n + \sum_{i=0}^{n-1} (\overline{a}_{ni} - \overline{a}_{n-1,i}) a_i \\ &= \widehat{a}_{nn} a_n + \sum_{i=0}^{n-1} \widehat{a}_{ni} a_i = \sum_{i=0}^{n} \widehat{a}_{ni} a_i. \end{aligned} \tag{15}$$

Now we shall prove the following theorem.

Theorem. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{no} = 1, \ n = 0, 1, ..., \tag{16}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{17}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{18}$$

$$\sum_{v=1}^{n-1} \frac{\widehat{a}_{n,v+1}}{v} = O(a_{nn}). \tag{19}$$

If (X_n) is an almost increasing sequence such that conditions (8)-(11) are satisfied, then the series $\sum a_n \lambda_n$ is summable $\mid A, p_n \mid_k$, $k \geq 1$.

It should be noted that if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem B. Indeed, in this case condition (19) reduces to condition (12).

We need the following lemma for the proof of our theorem.

Lemma ([6]). If (X_n) is an almost increasing sequence, then under the conditions (8) and (9), we have that

$$\sum_{n=1}^{\infty} X_n \mid \Delta \lambda_n \mid < \infty, \tag{20}$$

$$nX_n \mid \Delta \lambda_n \mid = O(1) \ as \ \ n \to \infty. \tag{21}$$

3. Proof of the Theorem. Let (T_n) denotes A-transform of the series $\sum a_n \lambda_n$. Then we have, by (15),

$$\overline{\Delta} T_n = \sum_{v=0}^{n} \widehat{a}_{nv} a_v \lambda_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\overline{\Delta}T_n &= \sum_{v=1}^{n}\frac{\widehat{a}_{nv}\lambda_v}{v}va_v \\
&= \sum_{v=1}^{n-1}\Delta_v\left(\frac{\widehat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v+\frac{n+1}{n}\widehat{a}_{nn}\lambda_n t_n \\
&= \sum_{v=1}^{n-1}\frac{v+1}{v}\Delta_v(\widehat{a}_{nv})\lambda_v t_v+\sum_{v=1}^{n-1}\frac{v+1}{v}\widehat{a}_{n,v+1}\Delta\lambda_v t_v \\
&+ \sum_{v=1}^{n-1}\frac{\widehat{a}_{n,v+1}\lambda_{v+1}t_v}{v}+\frac{n+1}{n}a_{nn}\lambda_n t_n \\
&= T_n(1)+T_n(2)+T_n(3)+T_n(4),\ say.
\end{aligned}$$

Since

$$|\,T_n(1)+T_n(2)+T_n(3)+T_n(4)\,|^k\leq 4^k\left(|\,T_n(1)\,|^k+|\,T_n(2)\,|^k+|\,T_n(3)\,|^k+|\,T_n(4)\,|^k\right),$$

to complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\,T_n(r)\,|^k<\infty,\quad for\quad r=1,2,3,4.$$

Firstly, since

$$\begin{aligned}
\Delta_v(\widehat{a}_{nv}) &= \widehat{a}_{nv}-\widehat{a}_{n,v+1} \\
&= \overline{a}_{nv}-\overline{a}_{n-1,v}-\overline{a}_{n,v+1}+\overline{a}_{n-1,v+1} \\
&= a_{nv}-a_{n-1,v}, \qquad (22)
\end{aligned}$$

by using (16) and (17)

$$\sum_{v=0}^{n-1}|\,\Delta_v(\widehat{a}_{nv})\,|=\sum_{v=0}^{n-1}(a_{n-1,v}-a_{nv})=1-1+a_{nn}=a_{nn}, \qquad (23)$$

and applying Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|\,T_n(1)\,|^k &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\,\Delta_v(\widehat{a}_{nv})\,||\,\lambda_v\,||\,t_v\,|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\,\Delta_v(\widehat{a}_{nv})\,||\,\lambda_v\,|^k|\,t_v\,|^k\right) \\
&\times \left(\sum_{v=1}^{n-1}|\,\Delta_v(\widehat{a}_{nv})\,|\right)^{k-1}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid\mid t_v \mid^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1} \mid \Delta_v(\widehat{a}_{nv}) \mid \\
&= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid\mid t_v \mid^k \sum_{n=v+1}^{m+1} \mid \Delta_v \widehat{a}_{nv} \mid .
\end{aligned}
$$

By (22), we have

$$
\sum_{n=v+1}^{m+1} \mid \Delta_v \widehat{a}_{nv} \mid = \sum_{n=v+1}^{m+1} (a_{n-1,v} - a_{nv}) = \sum_{n=v}^{m} a_{nv} - \sum_{n=v+1}^{m+1} a_{nv} = a_{vv} - a_{m+1,v} \leq a_{vv}.
$$

Thus, we obtain

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(1) \mid^k &= O(1)\sum_{v=1}^{m} \mid \lambda_v \mid\mid t_v \mid^k a_{vv} \\
&= O(1)\sum_{v=1}^{m} \frac{p_v}{P_v} \mid \lambda_v \mid\mid t_v \mid^k \\
&= O(1)\sum_{v=1}^{m-1} \Delta(\mid \lambda_v \mid) \sum_{r=1}^{v} \frac{p_r}{P_r} \mid t_r \mid^k + O(1) \mid \lambda_m \mid \sum_{v=1}^{m} \frac{p_v}{P_v} \mid t_v \mid^k \\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_m \mid X_m \\
&= O(1),\ as\ m \to \infty,
\end{aligned}
$$

by virtue of the hypothesis of the Theorem and the Lemma.

Again applying Hölder's inequality, we get

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(2) \mid^k &= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} \widehat{a}_{n,v+1} \mid \Delta\lambda_v \mid\mid t_v \mid\right)^k \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} \widehat{a}_{n,v+1} \mid \Delta\lambda_v \mid\mid t_v \mid^k\right) \\
&\times \left(\sum_{v=1}^{n-1} \widehat{a}_{n,v+1} \mid \Delta\lambda_v \mid\right)^{k-1}
\end{aligned}
$$

Taking account of (20), we obtain

$$
\sum_{n=1}^{\infty} \mid \Delta\lambda_n \mid < \infty
$$

and from the conditions due to $\overline{A}$ and $\widehat{A}$ matrices,

$$
\widehat{a}_{n,v+1} = \overline{a}_{n,v+1} - \overline{a}_{n-1,v+1} = \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{n} a_{ni} - \sum_{i=0}^{v} a_{ni} - \sum_{i=0}^{n-1} a_{n-1,i} + \sum_{i=0}^{v} a_{n-1,i} \\
&= \sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) \leq \sum_{i=0}^{n-1} (a_{n-1,i} - a_{ni}) = a_{nn},
\end{aligned}
$$

where

$$
\sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) \geq 0.
$$

Thus,

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(2) \mid^k &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} \widehat{a}_{n,v+1} \mid \Delta\lambda_v \mid\mid t_v \mid^k \right) \\
&= O(1) \sum_{v=1}^{m} \mid \Delta\lambda_v \mid\mid t_v \mid^k \sum_{n=v+1}^{m+1} \widehat{a}_{n,v+1}
\end{aligned}
$$

In view of the definitions of the matrices $\overline{A}$ and $\widehat{A}$, it is clear that,

$$
\begin{aligned}
\sum_{n=v+1}^{m+1} \widehat{a}_{n,v+1} &= \sum_{n=v+1}^{m+1} \sum_{i=0}^{v} (a_{n-1,i} - a_{ni}) = \sum_{i=0}^{v} \sum_{n=v+1}^{m+1} (a_{n-1,i} - a_{ni}) \\
&= \sum_{i=0}^{v} \left(\sum_{n=v+1}^{m+1} a_{n-1,i} - \sum_{n=v+1}^{m+1} a_{ni} \right) \\
&= \sum_{i=0}^{v} (a_{vi} - a_{m+1,i}) \leq \sum_{i=0}^{v} a_{vi} = 1. \qquad (24)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(2) \mid^k &= O(1) \sum_{v=1}^{m} \mid \Delta\lambda_v \mid\mid t_v \mid^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v \mid \Delta\lambda_v \mid) \sum_{r=1}^{v} \frac{\mid t_r \mid^k}{r} \\
&+ O(1)m \mid \Delta\lambda_m \mid \sum_{v=1}^{m} \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta^2\lambda_v \mid X_v \\
&+ O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_{v+1} \mid X_{v+1} + O(1)m \mid \Delta\lambda_m \mid X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypothesis of the Theorem and the Lemma.

Again, using Hölder's inequality, as in $T_n(1)$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_n(3)\mid^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\frac{\widehat{a}_{n,v+1}\mid\lambda_{v+1}\mid\mid t_v\mid}{v}\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}\frac{\widehat{a}_{n,v+1}}{v}\mid\lambda_{v+1}\mid^k\mid t_v\mid^k\\ &\times\left(\sum_{v=1}^{n-1}\frac{\widehat{a}_{n,v+1}}{v}\right)^{k-1}\\ &= O(1)\sum_{v=1}^{m}\frac{\mid\lambda_{v+1}\mid^k\mid t_v\mid^k}{v}\sum_{n=v+1}^{m+1}\widehat{a}_{n,v+1}\\ &= O(1)\sum_{v=1}^{m}\mid\lambda_{v+1}\mid^k\frac{\mid t_v\mid^k}{v}\\ &= O(1)\quad as\qquad m\to\infty,\end{aligned}$$

by virtue of the hypothesis of the Theorem and the Lemma.

Finally, again as in $T_n(1)$, we get

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_n(4)\mid^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^k\mid\lambda_n\mid^k\mid t_n\mid^k\\ &= O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}\mid\lambda_n\mid\mid t_n\mid^k= O(1)\quad as\qquad m\to\infty,\end{aligned}$$

by virtue of the hypothesis of the Theorem and the Lemma.

Therefore, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_n(r)\mid^k= O(1),\ as\ m\to\infty,\ for\ r=1,2,3,4.$$

This completes the proof of the Theorem.

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An Improved Active Set Feasible SQP Algorithm for the Solution of Inequality Constrained Optimization Problems

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Abstract

In this paper, we have proposed an active set FSQP algorithm for nonlinear inequality constraints optimization problems. At each iteration of the proposed algorithm, by solving a reduced quadratic programming subproblem and a reduced system of linear equation without involving multiplier estimate, a feasible direction of descent is generated through a suitable combination of a descent direction and a feasible direction. To overcome the Maratos effect, a higher-order correction direction is obtained by solving another reduced system of linear equation whose coefficient matrix is the same as the previous one. The algorithm is proved to be globally convergent and superlinearly convergent under some mild conditions *without the strict complementarity*. Finally, some numerical results are reported.

Key words. Inequality constrained optimization, quadratic programming, feasible SQP method, global convergence, convergence rate.

MR(2000)Subject Classification: 65K10,90C30

1. Introduction

Consider the following nonlinear inequality constrained optimization:

$$(P) \qquad \begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_j(x) \le 0, \quad j \in I = \{1, 2, ..., m\}, \end{array} \tag{1.1}$$

where $m > 0$ and the functions $f_0, \quad f_j(j \in I) : R^n \to$R are all continuously differentiable.

Up to now, there exist many methods for solving (P) such as gradient projection methods, trust region methods, interior point methods and SQP methods. Among these methods, SQP method is an important one. For an excellent recent survey of SQP algorithms, and the theory behind them, see [2]. Many existing SQP algorithms for handling constrained optimization problems starting with arbitrary initial point (may be infeasible) focus on using penalty functions, see [3]–[8], so the iterative points may be infeasible for the original problem, while a feasible sequence of iterates is

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very important for many practical problems, such as engineering design, real-time applications and that problems whose objective functions are not well defined outside the feasible set. To overcome this shortcoming, in [10], variations on the standard SQP algorithms for solving (P) are proposed which generate iterations lying within the feasible set of (P), which is called as feasible sequential quadratic programming (FSQP) algorithm. It is proved to be globally convergent and superlinearly convergent under some mild assumptions. However, at each iteration, these algorithms require to solve two QP subproblems and a linear least squares problem. Clearly, their computational cost per single iteration is relatively high. In [12], the FSQP is further improved, a feasible descent direction is obtained by a suitable convex combination of a descent direction and a feasible direction, a second-order direction is computed by solving another QP subproblem. In the end, this method is proved to be local two-step superlinearly convergent. Recently, another type of FSQP algorithm [9, 13] is proposed. In this algorithm, they can obtain a feasible descent direction by solving a QP subproblem, then, compute a second-order correction direction by solving another QP subproblem, and perform an arc search. In order to reduce computational cost per iteration, based on the method in [12], Zhu [14] proposed a simple FSQP, a feasible direction of descent is generated through a suitable combination of a descent direction which is generated by solving a quadratic programs and a feasible direction which is obtained by solving a system of linear equation. To overcome the Maratos effect, a higher-order correction direction is obtained by solving another system of linear equation. The algorithm is proved to be globally convergent and superlinearly convergent under some mild conditions. However, in Zhu's method, the systems of linear equation involved multiplier estimate, as [11] pointed out, the algorithm may meet instability problems. In particular, the linear system may become very ill-conditioned if some multiplier corresponding to a nearly active constraint becomes very small. On the other hand, for the above-mentioned algorithms, to obtain locally superlinear convergence, the strict complementary condition is necessary.

In this paper, we have proposed an active set FSQP algorithm for nonlinear inequality constraints optimization problems. At each iteration of the proposed algorithm, by solving a reduced quadratic programming subproblem and a reduced system of linear equation without involving multiplier estimate, a feasible direction of descent is generated through a suitable combination of a descent direction and a feasible direction. To overcome the Maratos effect, a higher-order correction direction is obtained by solving another reduced system of linear equation whose coefficient matrix is the same as the previous one. The algorithm is proved to be globally convergent and superlinearly convergent under some mild conditions *without the strict complementarity*.

The remainder of this paper is organized as follows. The proposed algorithm is stated in Section 2. In Section 3 and Section 4, under some mild assumptions, we show that this algorithm is globally convergent and locally superlinear convergent, respectively. Some numerical results are reported in section 5. Finally, we give concluding remarks about the proposed algorithm.

2. Description of algorithm

We denote the feasible set X of (P) by

$$X = \{x \in R^n : f_i(x) \leq 0, i \in I\},$$

and, for a feasible point $x \in X$, define the active set by

$$I(x) = \{i \in I : f_i(x) = 0\}.$$

In this paper, we suppose that the feasible set X is not empty and the following basic hypothesis holds.

Assumption A_1. *The gradient vectors* $\{\nabla f_j(x), j \in I(x)\}$ *are linearly independent for each feasible point* $x \in X$.

For $x \in X$, we now give an estimate of the active set $I(x)$ ([19]):

$$A(x;\varepsilon) = \{i : f_i(x) + \varepsilon\rho(x,\lambda(x)) \geq 0\},$$

where ε is a nonnegative parameter and $\rho(x,\lambda(x)) = \sqrt{\| \Phi(x,\lambda) \|}$ with

$$\Phi(x,\lambda(x)) = \begin{bmatrix} \nabla_x L(x,\lambda(x)) \\ \min\{-f(x),\lambda(x)\} \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ ... \\ f_m(x) \end{bmatrix}, \quad L(x,\lambda(x)) = f_0(x) + \lambda(x)^T f(x),$$

$$\lambda(x) = -(\nabla f(x)^T \nabla f(x) + diag(f_i(x))^2)^{-1} \nabla f(x)^T \nabla f_0(x) (See[17]), \quad \nabla f(x) = (\nabla f_i(x), i \in I).$$

It is obvious that (x^*,λ^*) is a KKT point of (P) if and only if $\Phi(x^*,\lambda^*) = 0$ or $\rho(x^*,\lambda^*) = 0$. Facchinei et al [19] showed that if the second order sufficient condition and the Mangassarian-Fromovotz Constraint Qualification hold, then for any $\varepsilon > 0$, when x is sufficient close to x^*, $A(x;\varepsilon)$ is an exact identification of $I(x^*)$.

The following algorithm is proposed for solving (P).

ALGORITHM

Parameters $\tau \in (2,3),\ \beta \in (0,1),\ \alpha \in (0,\frac{1}{2}),\ \ \delta > 2,\ \ \rho > 0,\ \ \gamma > 0.$

Data Choose an initial feasible point $x^1 \in X$, a symmetric positive matrix H_1 and $\varepsilon^0 > 0$. Set $k = 1$.

Step 1 Set $\varepsilon = \varepsilon^{k-1}$.

Step 2 Set $A^k(\varepsilon) = A(x^k,\varepsilon)$. If $\nabla f_{A^k(\varepsilon)}(x^k)$ is not of full rank, then set $\varepsilon := \sigma\varepsilon$, go to step 2.(Where $\nabla f_{A^k(\varepsilon)}(x^k) = (\nabla f_j(x^k),\ j \in A^k(\varepsilon)))$

Step 3 Set $\varepsilon^k = \varepsilon,\ A^k = A^k(\varepsilon)$.

Step 4 (Compute the search direction)

1.1 Computation of the descent direction d_0^k:

For the current iteration point x^k, solve

$$(QP) \qquad \begin{array}{ll} \min & \nabla f_0(x^k)^T d + \frac{1}{2} d^T H_k d \\ \text{s.t.} & f_j(x^k) + \nabla f_j(x^k)^T d \leq 0, \quad j \in A^k, \end{array} \tag{2.1}$$

to obtain an optimal solution d_0^k, let $u_{A^k}^k$ be corresponding KKT multipliers. If $d^k = 0$, then x^k is a KKT point for (P) and stop; otherwise go to 1.2.

1.2 Computation of the feasible direction d_1^k:

Solve the following system of linear equation:

$$\begin{bmatrix} H_k & \nabla f_{A^k}(x^k) \\ \nabla f_{A^k}(x^k)^T & 0 \end{bmatrix} \begin{bmatrix} d \\ \lambda \end{bmatrix} = \begin{bmatrix} -\nabla f_0(x^k) \\ -\|d_0^k\|^\delta e_{A^k} \end{bmatrix}, \tag{2.2}$$

where $e_{A^k} = (1, ..., 1)^T \in R^{|A^k|}$. Let d_1^k be the solution.

1.3 Computation of the feasible descent direction d^k:
Establish a suitable combination of d_0^k and d_1^k:

$$d^k = d_0^k + \rho_k d_1^k, \tag{2.3}$$

$$\rho_k = \begin{cases} \rho, & \text{if } \nabla f_0(x^k)^T d_1^k \leq 0; \\ \frac{-\nabla f_0(x^k)^T d_0^k}{\nabla f_0(x^k)^T d_1^k + \gamma}, & otherwise. \end{cases} \tag{2.4}$$

Step 5 Compute the higher-order correction direction $\widetilde{d}^k$ by solving the following system of linear equation:

$$\begin{bmatrix} H_k & \nabla f_{A^k}(x^k) \\ \nabla f_{A^k}(x^k)^T & 0 \end{bmatrix} \begin{bmatrix} d \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ -\|d^k\|^\tau e_{A^k} - f_{A^k}(x^k + d^k) + f_{A^k}(x^k) + \nabla f_{A^k}(x^k)^T d^k \end{bmatrix}, \tag{2.5}$$

If $\|\widetilde{d^k}\| > \|d^k\|$, set $\widetilde{d^k} = 0$.

Step 6 (Do curve search) Compute the step size λ_k, which is the first number λ of the sequence $\{1, \beta, \beta^2, ...\}$ satisfying

$$f_0(x^k + \lambda d^k + \lambda^2 \widetilde{d^k}) \leq f_0(x^k) + \alpha\lambda \nabla f_0(x^k)^T d^k, \tag{2.6}$$

$$f_j(x^k + \lambda d^k + \lambda^2 \widetilde{d^k}) \leq 0, \quad \forall j \in I. \tag{2.7}$$

Step 7 Set a new iteration point by $x^{k+1} = x^k + \lambda_k d^k + \lambda_k^2 \widetilde{d^k}$. Compute a new symmetric positive definite matrix H_{k+1}, set $k := k + 1$, and go back to Step 1.

Remark 1 *Unlike [14], the coefficient matrix for the system of linear equation (2.2) do not involve multiplier estimate.*

3. Global Convergence

In this section, firstly, we show that the proposed algorithm is well defined.

Lemma 3.1 *Let $x^k \in X$ and suppose that Assumption A_1 holds. Then Step 2 in the proposed algorithm can be finished in a finite number of computations.*

The proof is similar to the one of Lemma 1.1 and Lemma 2.8 in [20].

Lemma 3.2 *Suppose that H_k is symmetric positive definite. Then the linear system (2.2) and (2.5) always has a unique optimal solution under Assumption A_1, respectively.*

Proof. The symmetric positive definite property and Assumption A_1 imply that the coefficients matrix $\begin{bmatrix} H_k & \nabla f_{A^k}(x^k) \\ \nabla f_{A^k}(x^k)^T & 0 \end{bmatrix}$ is nonsingular. So the claim holds.

Lemma 3.3 *(1) If $d_0^k = 0$, then x^k is a KKT point of the problem (P).*
(2) If $d_0^k \neq 0$, then

$$\nabla f_0(x^k)^T d_0^k < 0, \quad \nabla f_0(x^k)^T d^k < 0, \quad \nabla f_j(x^k)^T d^k < 0, \quad j \in I(x^k).$$

The proof is similar to the one of Theorem 3.2 in [14].

Lemma 3.4 *The line search in Step 6 of the proposed algorithm yields a stepsize $\lambda_k = \beta^j$ for some finite $j = j(k)$.*

It is not difficult to finish the proof of this lemma.

To analyze the global convergence of the proposed algorithm, the following assumptions are necessary.

Assumption A_2. *The sequence $\{x^k\}$, which is generated by the proposed algorithm, is bounded.*

Assumption A_3. *There exist $a,\ b > 0$ such that $a\|d\|^2 \leq d^T H_k d \leq b\|d\|^2$ for all k and all $d \in R^n$.*

We suppose that x^* is a given accumulation point of $\{x^k\}$. In view of A^k and J_k being a subset of the finite and fixed set I, respectively, there exist an infinite index set K such that

$$\lim_{k\in K} x^k = x^*, \quad A^k \equiv A, \quad J_k \equiv J, \ \ \forall k \in K, \tag{3.1}$$

where

$$J_k = \{j \in A^k : f_j(x^k) + \nabla f_j(x^k)^T d^k = 0\}.$$

Lemma 3.5 *Suppose that Assumptions A_2 and A_3 hold. Then the sequences $\{d_0^k : k \in K\}, \{d_1^k : k \in K\},\ \{\widetilde{d}^k : k \in K\}$ and $\{u_{A^k}^k : K \in K\}$ are all bounded.*

It is easy to finish the proof of this lemma.

Based on Lemma 3.5, we now can present the global convergence theorem of the proposed algorithm as follows.

Theorem 3.1 *Suppose that Assumptions A_1, A_2 and A_3 hold, then the proposed algorithm either stops at a KKT point x^k for problem (P) in a finite number of steps or generates an infinite sequence $\{x^k\}$ of points such that each accumulation point x^* is a KKT point for problem (P).*

Proof. The proof is similar to the one of Theorem 3.4 in [14].

4. Rate of convergence

In this section, we will analyze the convergent rate of the proposed algorithm, for this, the following further hypothesis is necessary.

Assumption A_4 *(i) The functions $f_j(x)(j \in I)$ are all second-order continuously differentiable.*

(ii) The sequence $\{x^k\}$ generated by the algorithm possesses an accumulation point x^* such that KKT pair (x^*, u^*) satisfies the strong second-order sufficiency conditions, i.e.,

$$d^T \nabla_{xx}^2 L(x^*, u^*) d > 0, \quad \forall d \in \Omega \stackrel{def}{=} \{d \in R^n : d \neq 0, \nabla f_{I^+(x^*)}(x^*)^T d = 0\}, \tag{4.1}$$

where

$$L(x, u) = f_0(x) + \sum_{j\in I} u_j f_j(x), \qquad I^+(x^*) = \{j \in I : u_j^* > 0\}. \tag{4.2}$$

Lemma 4.1 *(i) Suppose that Assumptions A_1, A_2 hold. Then* $\lim_{k\to\infty}\|x^{k+1}-x^k\|=0$ *and* $\lim_{k\to\infty}x^k=x^*$.

(ii) If Assumptions A_1,A_2 and A_3 are satisfied, then $\lim_{k\to\infty}d_0^k=0,\quad \lim_{k\to\infty}d_1^k=0,\quad \lim_{k\to\infty}\widetilde{d}^k=0.$

Proof. (i) Similar to the proof in [14]. One has

$$\lim_{k\to\infty}\|x^{k+1}-x^k\|=\lim_{k\to\infty}\|\lambda_k d^k+\lambda_k^2\widetilde{d}^k\|\le\lim_{k\to\infty}2\lambda_k\|d^k\|=0.$$

According to Assumption A_4 (ii), one can conclude that the given limit point x^* is an isolated KKT point of (1.1)(See Theorem 1.2.5 in [16]), therefore x^* is an isolated accumulation point of $\{x^k\}$ from Theorem 3.1, and this together with $\lim_{k\to\infty}\|x^{k+1}-x^k\|=0$ shows that $\lim_{k\to\infty}x^k=x^*$.

(ii) Taking into account $x^k\to x^*,\ k\to\infty$, according to the proof of Theorem 3.1, it is clear to see that $\lim_{k\to\infty}d_0^k=0,\ \lim_{k\to\infty}d_1^k=0,\ \lim_{k\to\infty}\widetilde{d}^k=0$, The proof is finished. □

Lemma 4.2 *Under all the above-mentioned assumptions, when k is sufficiently large, the matrix*

$$M_k\overset{def}{=}\begin{bmatrix}H_k & \nabla f_{A^k}(x^k)\\ \nabla f_{A^k}(x^k)^T & 0\end{bmatrix}.$$

is nonsingular, furthermore, there exists a constant $C>0$ such that $\|M_k^{-1}\|\le C$.

The proof of Lemma 4.2 is similar to that of Lemma 2.2 in [15] or Lemma 2.2.2 in [16], and is omitted.

Lemma 4.3 *Suppose that Assumptions A_1, A_2 and A_3 hold. Then*

$$\|d_1^k\|=o(\|d_0^k\|^2),\quad \|d^k\|=\|d_0^k\|+o(\|d_0^k\|^2),\quad \|\widetilde{d}^k\|=O(\|d^k\|^2),\tag{4.3}$$

$$J_k\subseteq I(x^*)=A^k,\tag{4.4}$$

$$\lim_{k\to\infty}u_{A_k}^k=u^*.\tag{4.5}$$

Proof. Firstly, from (2.2), (2.3) and (2.4), we have

$$\|d_1^k\|=o(\|d_0^k\|^2),\quad \|d^k\|=\|d_0^k\|+o(\|d_0^k\|^2).$$

Secondly, we shall show the third equation of (4.3). In view of (2.5) being equivalent to solve the following system of linear equations:

$$\begin{bmatrix}H_k & \nabla f_{A^k}(x^k)\\ \nabla f_{A^k}(x^k)^T & 0\end{bmatrix}\begin{bmatrix}\widetilde{d}^k\\ \widetilde{\lambda}^k\end{bmatrix}=\begin{bmatrix}0\\ -\|d^k\|^\tau e_{A^k}-f_{A^k}(x^k+d^k)+f_{A^k}(x^k)+\nabla f_{A^k}(x^k)^Td^k\end{bmatrix}.$$

So, it is not difficult to verify that $\|\widetilde{d}^k\|=o(\|d^k\|^2)$.

To show the relationship (4.4), one first gets $J_k\subseteq I(x^*)$ from $\lim_{k\to\infty}(x^k,d^k,z_k,\sigma_k)=(0,0,0,0)$. From Theorem 2.3 and Theorem 3.7 in [19], we know that $I(x^*)=A^k$ under Assumptions $A_1, A_4(ii)$.

Similar to the proof in [14], we get $\lim_{k\to\infty}u_{A_k}^k=u^*$. The proof is complete. □

To ensure the step size $\lambda_k \equiv 1$ for k large enough, an additional assumption as follows is necessary.

Assumption A_5 *Suppose that* $\|(\nabla^2_{xx}L(x^k,u_k)-H_k)d_0^k\| = o(\|d_0^k\|)$, *where*

$$L(x,u_k) = f_0(x) + \sum_{j\in J_k} u_j^k f_j(x).$$

Remark 2 *This assumption is similar to the well-known Dennis-More Assumption [1] that guarantees superlinear convergence for quasi-Newton methods.*

Lemma 4.4 *Suppose that Assumptions* A_1, A_2, A_3, A_4 *and* A_5 *hold. Then the step size of the proposed algorithm always equals one, i.e.,* $\lambda_k \equiv 1$, *if k is sufficiently large.*

Proof. We know that it is sufficient to verify that (2.6) and (2.7) hold for $\lambda = 1$, and the statement "k large enough" will be omitted in the following discussion.

We first prove (2.7) holds for $\lambda = 1$. For $j \notin I(x^*)$, i.e., $f_j(x^*) < 0$, in view of $(x^k, d^k, \widetilde{d^k}) \to (x^*, 0, 0)(k \to \infty)$, we can conclude $f_j(x^k + d^k + \widetilde{d^k}) \le 0$ holds.

For $j \in I(x^*) = A^k(\varepsilon)$, from Taylor expansion and formula (4.3), we have

$$\begin{aligned} f_j(x^k + d^k + \widetilde{d}^k) &= f_j(x^k + d^k) + \nabla f_j(x^k + d^k)^T \widetilde{d}^k + O(\|\widetilde{d}^k\|^2) \\ &= f_j(x^k + d^k) + \nabla f_j(x^k)^T \widetilde{d}^k + O(\|d^k\|\|\widetilde{d}^k\|) + O(\|\widetilde{d}^k\|^2) \\ &= f_j(x^k + d^k) + \nabla f_j(x^k)^T \widetilde{d}^k + O(\|d^k\|^3). \end{aligned} \tag{4.6}$$

Therefore we have from (2.5) and (4.6)

$$f_j(x^k + d^k + \widetilde{d^k}) \le -\|d^k\|^\tau + O(\|d^k\|^3) < 0.$$

This shows that (2.7) holds for $\lambda = 1$.

The next objective is to show (2.6) holds for $\lambda = 1$.

From Taylor expansion and taking into account relationship (4.3), we have

$$\begin{aligned} \omega_k &\overset{def}{=} f_0(x^k + d^k + \widetilde{d^k}) - f_0(x^k) - \alpha\nabla f_0(x^k)^T d^k \\ &= \nabla f_0(x^k)^T(d^k + \widetilde{d^k}) + \tfrac{1}{2}(d^k)^T\nabla^2_{xx} f_0(x^k)d^k - \alpha\nabla f_0(x^k)^T d^k + o(\|d^k\|^2). \end{aligned} \tag{4.7}$$

On the other hand, from the KKT condition of (2.1) and formula (4.3), one has

$$\nabla f_0(x^k)^T(d^k + \widetilde{d^k}) = -(d_0^k)^T H_k d_0^k - \sum_{j\in J_k} u_j^k \nabla f_j(x^k)^T(d^k + \widetilde{d^k}) + o(\|d_0^k\|^2). \tag{4.8}$$

$$\begin{aligned} \nabla f_0(x^k)^T d^k &= -(d_0^k)^T H_k d_0^k - \sum_{j\in J_k} u_j^k \nabla f_j(x^k)^T d_0^k + o(\|d_0^k\|^2) \\ &= -(d_0^k)^T H_k d_0^k + \sum_{j\in J_k} u_j^k f_j(x^k) + o(\|d_0^k\|^2). \end{aligned}$$

Again, from the third equation of (4.6), (2.1) and Taylor expansion, we have

$$f_j(x^k) + \nabla f_j(x^k)^T(d^k + \widetilde{d^k}) + \frac{1}{2}(d^k)^T\nabla^2_{xx} f_j(x^k)d^k = o(\|d^k\|^2), \quad j \in J_k.$$

Thus

$$-\sum_{j\in J_k} u_j^k \nabla f_j(x^k)^T(d^k+\widetilde{d}^k) = \sum_{j\in J_k} u_j^k f_j(x^k) + \tfrac{1}{2}(d_0^k)^T(\sum_{j\in J_k} u_j^k \nabla_{xx}^2 f_j(x^k))d_0^k + o(\|d_0^k\|^2). \tag{4.9}$$

Substituting (4.9) into (4.8), one has

$$\nabla f_0(x^k)^T(d^k+\widetilde{d}^k) = -(d_0^k)^T H_k d_0^k + \sum_{j\in J_k} u_j^k f_j(x^k) + \tfrac{1}{2}(d_0^k)^T(\sum_{j\in J_k} u_j^k \nabla_{xx}^2 f_j(x^k))d_0^k + o(\|d_0^k\|^2). \tag{4.10}$$

Substituting (4.10) and the third equation of (4.8) into (4.7), we obtain

$$\begin{aligned}
\omega_k &= (\alpha-1)(d_0^k)^T H_k d_0^k + \tfrac{1}{2}(d_0^k)^T \nabla_{xx}^2 L(x^k,u^k)d_0^k \\
&+ (1-\alpha)\sum_{j\in J_k} u_j^k f_j(x^k) + o(\|d_0^k\|^2) \\
&\le ((\alpha-1)+\tfrac{1}{2})a\|d_0^k\|^2 + \tfrac{1}{2}(d_0^k)^T(\nabla_{xx}^2 L(x^k,u^k) - B_k)d_0^k \\
&+ (1-\alpha)\sum_{j\in J_k} u_j^k f_j(x^k) + o(\|d_0^k\|^2).
\end{aligned}$$

So, using Assumption A_5 and the given conditions, one has

$$\omega_k \le ((\alpha-1)+\frac{1}{2})a\|d_0^k\|^2 + o(\|d_0^k\|^2).$$

Therefore, according to $\alpha \in (0, \frac{1}{2})$. The whole proof is finished. □

Theorem 4.1 *Under all above-mentioned assumptions, the algorithm is superlinearly convergent. i.e., the sequence* $\{x^k\}$ *generated by the algorithm satisfies* $\|x^{k+1}-x^*\| = o(\|x^k-x^*\|)$.

The proof is similar to the one of Theorem 5.2 in [18], and is omitted.

5. Numerical experiments

In this section, we test some practical problems based on the proposed algorithm. The numerical experiments are implemented on MATLAB 6.5, under Windows XP and 1000MHZ CPU. The (2.1), (2.2) and (2.3) are solved by the Optimization Toolbox. The BFGS formula, which is proposed in [21], is adopted in the algorithm.

During the numerical experiments, we set

$$\tau = 2.8, \quad \rho = 2, \quad \beta = 0.6, \quad \alpha = 0.2, \quad \gamma = 1, \quad \delta = 3.$$

The test problem in Table 5.1 are selected from [22] and [23]. Besides the test problem hs001, the initial points for the selected problems are as same as the ones in [22] and [23]. The columns of Table 5.1 have the following meanings: The **prob** column lists the test problem taken from [22] and [23] in order. The columns labelled **Ni**, **Nf0**, **Ng0**, **Nf** and **Ng** give the number of iterations required to solve the problem, objective function evaluations, objective function gradient evaluations, constraint function evaluations(including linear and nonlinear constraints) and constraint function gradient evaluations, respectively. The columns labelled **objective**, **dnorm** and **eps** denote the final objective value, the norm of d^k and the step criterion threshold ϵ, respectively.

The detailed information of the solutions to the test problems is listed in the following Table 5.1.

Table 5.1 Numerical results

Prob	Ni	Nf0	Ng0	Nf	Ng	objective	dnorm	eps
hs001	6	27	6	62	11	-0.0100e+02	5.2373e-014	0.1e-05
hs12	11	44	11	51	8	-0.3000e+02	2.5691e-008	0.1e-05
hs29	16	77	16	87	11	-0.226274e+02	1.2937e-008	0.1e-05
hs31	9	55	9	434	14	-0.0600e+02	5.4155e-007	0.1e-05
hs34	18	106	18	984	53	-0.0008340e+03	7.2411e-007	0.1e-05
hs35	8	34	8	164	14	0.00011+03	1.0122e-007	0.1e-05
hs43	21	130	21	444	36	0.4400+02	1.2956e-007	0.1e-05
hs100	61	347	61	612	76	6.806301+02	4.6149e-007	0.1e-05
s225	8	20	8	125	14	0.2000+01	0.000e-007	0.1e-05
s264	20	128	20	435	27	-0.441134+02	2.8061e-007	0.1e-05

6. Concluding remarks

In this paper, a simple active set FSQP algorithm for nonlinear inequality constraints optimization problems is presented. At each iteration of the proposed algorithm, through a suitable combination of a descent direction which is generated by solving a reduced quadratic programs and a feasible direction which is obtained by solving a reduced system of linear equation without involving multiplier estimate, a feasible direction of descent is generated. To overcome the Maratos effect, a higher-order correction direction is obtained by solving another reduced system of linear equation whose coefficient matrix is the same as the previous one. The algorithm is proved to be globally convergent and superlinearly convergent under some mild conditions *without the strict complementarity*.

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REMARK ON DOUBLE LACUNARY STATISTICAL CONVERGENCE OF FUZZY NUMBERS

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Abstract. Y. Altin proved the inclusion relations between the sets of the double statistically convergent and double lacunary statistically convergent sequences of fuzzy numbers. In this paper we show that Altin's condition is sufficient as well as necessary.

1. Introduction and Background

Before we enter the motivation for this paper and presentation of the main results we give some preliminaries. A fuzzy number is a function X from R^n to $[0,1]$, which is normal, fuzzy convex and upper- semi continuous and the closure of $\{x \in R^n\colon \mathrm{X}(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the α-level set

$$X^{\alpha} = \{x \in R^n : \mathrm{X}(x) \geq \alpha\}$$

is a nonempty compact convex, subset of R^n, as is the support X^0. Let $L(R^n)$ denote the set of all fuzzy numbers.
Define for each $1 \leq q < \infty$

$$d_q(X,Y) = \left\{ \int_0^1 \delta_{\infty}(X^{\alpha}, Y^{\alpha})^q \, d\alpha \right\}^{\frac{1}{q}}$$

and $d_{\infty} = \sup_{0 \leq \alpha \leq 1} \delta_{\infty}(\mathrm{X}^{\alpha}, \mathrm{Y}^{\alpha})$ where d_{∞} is Hausdorf metric. Clearly $d_{\infty}(X,Y) = \lim_{q \to \infty} d_q(\mathrm{X}, \mathrm{Y})$ with $d_q \leq d_r$ if $q \leq r$. Moreover d_q is a complete, separable and locally compact metric space [2].

Throughout the paper, d will denote d^q with $1 \leq q \leq \infty$.

The concept of statistical convergence of fuzzy numbers was introduced by Kwon [4] in 2000. A sequence $X = (X_k)$ is said to be statistically convergent to the number X_0 if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \leq n : d(X_k, X_0) \geq \epsilon\}| = 0,$$

where by $k \leq n$, we mean that $k = 0, 1, 2, ..., n$ and the vertical bars indicate the number of elements in the enclosed set.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals

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determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

The following concept is due to F. Nuray . Let θ be a lacunary sequence; the sequence X is $S_\theta-$ convergent to X_0 provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r : d(X_k, X_0) \geq \epsilon\}| = 0.$$

We will need the following definitions (see, [8]).

Definition 1.1. *A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or P- convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ there exists $N \in \mathcal{N}$ such that*

$$d(X_{kl}, X_0) < \epsilon \quad for \quad k, l > N,$$

and we denote by $P - limX = X_0$. The number X_0is called the Pringsheim limit of X_{kl}.

More exactly we say that a double sequence (X_{kl}) converges to a finite number X_0 if X_{kl} tend to X_0 as both k and l tends to ∞ independently of one another.

Let $c^2(F)$ denote the set of all double convergent sequences of fuzzy numbers.

Definition 1.2. *A double sequence $X = (X_{kl})$ of fuzzy numbers is bounded if there exists a positive number M such that $d(X_{kl}, X_0) < M$ for all k and l. We will denote the set of all bounded double sequences by $l^2_\infty(F)$.*

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\{\frac{K_{m,n}}{mn}\}^{\infty,\infty}_{m,n=1,1}$ has a limit then we say that K has a natural density and is defined

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, Let $K = \{(i^2, j^2) : i, j \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero). Quite recently, Savas and Mursaleen [8], defined the statistical analogue for double sequences $X = \{X_{k,l}\}$ as follows: A double sequences $X = \{X_{k,l}\}$ of fuzzy numbers is said to be P-statistically convergent to X_0 provided that for every $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \{\text{number of } (j, k) : j \leq m \text{ and } k \leq n, d(X_{j,k}, X_0) \geq \epsilon\} = 0.$$

In this case we write $st_2 - \lim_{m,n} X_{m,n} = X_0$ and we denote the set of all statistically convergent double sequences by $st_2(F)$.
In [1] Altin proved the following theorems:

Theorem 1.1. *For any double lacunary sequence $\theta_{r,s}$, $st_2 - \lim X = L$ implies $S_{\theta_{r,s}} - \lim X = L$ if $\liminf q_r > 1$ and $\liminf \bar{q}_s > 1$.*

Theorem 1.2. *For any double lacunary sequence $\theta_{r,s}$, $S_{\theta_{r,s}} \lim X = X_0$ implies $st_2 - \lim X = X_0$ if $\sup_r q_r < \infty$ and $\sup_s \bar{q}_s < \infty$.*

Theorem 1.3. *Let $\theta_{r,s}$ be a lacunary double sequence; Then $st_2(F) = S_{\theta_{r,s}}(F)$ if*

$$1 < P - liminf_r q_r \leq P - limsup_r q_r < \infty$$

and

$$1 < P - liminf_s \bar{q}_s \leq P - limsup_s \bar{q}_s < \infty.$$

In this paper we will prove that the converses of theorems 1.1, 1.2 and hence 1.3 are also valid.

2. Definitions and Results

We begin with some definitions.

Definition 2.1. *The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called* **double lacunary** *if there exist two increasing of integers such that*

$$k_0 = 0, h_r = k_r - k_{k-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determine by $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \& l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$.

We now have the following definition,

Definition 2.2. *Let $\theta_{r,s}$ be a double lacunary sequence; the double sequence X of fuzzy numbers is $S_{\theta_{r,s}}$-convergent to X_0 provided that for every $\epsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \epsilon\}| = 0.$$

In this case we write $S_{\theta_{r,s}} - \lim X = X_0$ and we denote the set of all lacunary statistically convergent double sequences by $S_{\theta_{r,s}}(F)$.

Definition 2.3. *Let θ be a double lacunary sequence; the double sequence $X = \{X_{kl}\}$ of fuzzy numbers is said to be an $S_{\theta_{r,s}}$-Cauchy double sequence if there exists a double subsequence $\{X_{\bar{k}_r, \bar{l}_s}\}$ of X such that $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ for each (r,s) P-$\lim_{r,s} X_{k_r,l_s} = X_0$ and for every $\epsilon > 0$*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : d(X_{r,s}, X_{\bar{k}_r,\bar{l}_s}) \geq \epsilon\right\}\right| = 0.$$

Proof of the converse of theorem 1.1. For the contrapositive method we suppose that $\liminf_r q_r = 1$ and $\liminf_s \bar{q}_s = 1$. Then we will prove that there is a bounded st_2-convergent sequence that is not $S_{\theta_{r,s}}$. Since $\theta_{r,s}$ is lacunary sequence there are subsequences $\left\{k_{r_j}, l_{s_i}\right\}$ of $\theta_{r,s}$ satisfying

$$\frac{k_{r_j - 1}}{k_{r_{j-1}}} > j, \quad \frac{k_{r_j}}{k_{r_j - 1}} < 1 + \frac{1}{j} \text{ and } \frac{l_{s_i - 1}}{l_{s_{i-1}}} > i, \quad \frac{l_{s_i}}{l_{s_i - 1}} < 1 + \frac{1}{i}$$

where $r_j \geq r_{j-1} + 2$ and $s_i \geq s_{i-1} + 2$. Let us define X as follows

$$X_{k,l} := \begin{cases} \bar{1}, & \text{if} \quad k,l \in I_{r_j s_i}, \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Then , for any fuzzy numbers F and for any $\varepsilon > 0$ with $\varepsilon \leq min\{d(\bar{1}, F), (\bar{0}, F)\}$,

$$P-lim_{j,i}\frac{1}{h_{r_j,s_i}}|\{(k,l)\in I_{r_j,s_i}:d(X_{k,l},F)\geq\varepsilon|=\begin{cases}1, & \text{if} \quad F\neq\bar{1},\\ 0, & \text{if} \quad F=\bar{1},\end{cases}$$

and, for $r\neq r_j$ and $s\neq s_i$.

$$P-lim_{r,s}\frac{1}{h_{r,s}}|\{(k,l)\in I_{r,s}:d(X_{k,l},F)\geq\varepsilon|=\begin{cases}0, & \text{if} \quad F=\bar{0},\\ 1, & \text{if} \quad F\neq\bar{0}.\end{cases}$$

Therefore X is not in $S_{\theta_{r,s}}(F)$.

If m and n are any sufficiently large integers, we can find the unique j and i with

$$k_{r_j-1}<m\leq k_{r_j} \text{ and } l_{s_i-1}<n\leq l_{s_i}.$$

Then, since $m,n\to\infty$, implies $j,i\to\infty$, we have

$$P\text{–}lim_{m,n}\frac{1}{m,n}\left|\{k\leq m \text{ and } l\leq n:d(X_{k,l},\bar{0})\geq\epsilon\}\right|\leq\frac{k_{r_{j-1}}l_{s_{i-1}}+h_{r_js_i}}{k_{r_j-1}l_{s_i-1}}<lim_{ji}\left(\frac{2}{ji}\right)=0.$$

Hence $X\in st_2(F)$.

Proof of the converse of theorem 1.2. We suppose that $\limsup_r q_r=\infty$ or $\limsup_s\bar{q}_s=\infty$. Then we will prove that there is a bounded $S_{\theta_{r,s}}$ -convergent sequence that is not st_2. Now $\theta_{r,s}$ is lacunary and there is subsequences $\{k_{r_j},l_{s_i}\}$ of $\theta_{r,s}$ satisfying $q_{r_j}>j$, $\bar{q}_{s_i}>i$. Define $X=(X_{kl})$ by

$X_{kl}=\bar{1}$, if $k_{r_j-1}<k\leq 2k_{r_j-1}$ and $l_{s_i-1}<l\leq 2l_{s_i-1}$; and $X_{kl}=\bar{0}$, otherwise.

Let $\varepsilon>0$ be given,

$$\frac{1}{h_{r_j,s_i}}|\{(k,l)\in I_{r_j,s_i}:d(X_{k,l},\bar{0})\geq\varepsilon|=\frac{k_{r_j-1}l_{s_i-1}}{h_{r_j,s_i}}\leq\left(\frac{1}{j-1}\right)\left(\frac{1}{i-1}\right).$$

and if $r\neq r_j$ and $s\neq s_i$, then $|\{(k,l)\in I_{r_j,s_i}:d(X_{k,l},\bar{0})|=0$.
Hence

$$\frac{1}{h_{r_j,s_i}}|\{(k,l)\in I_{r_j,s_i}:d(X_{k,l},\bar{0})\geq\varepsilon|=0,$$

from which we have $\{X_{kl}\}\in S_{\theta_{r,s}}(F)$. On the other hand, for the sequence $\{X_{kl}\}$ above we will show that any fuzzy number F can not be a statistical limit of $\{X_{kl}\}$.
If $F=\bar{0}$ and $\varepsilon>0$, then

$$\begin{aligned}\frac{1}{2k_{r_j-1}2l_{s_i-1}}\left|\left\{k\leq 2k_{r_j-1}\text{ and }l\leq 2l_{s_i-1}:d(X_{k,l},\bar{0})\geq\epsilon\right\}\right| &\geq \left(\frac{k_{r_j}-1}{2k_{r_j-1}}\right)\left(\frac{l_{s_i}-1}{2l_{s_i}-1}\right)\\ &= \frac{1}{4}.\end{aligned}$$

If $F=\bar{1}$ and $\varepsilon>0$, then

$$\begin{aligned}\frac{1}{k_{r_j}l_{s_i}}\left|\left\{k\leq k_{r_j-1}\text{ and }l\leq l_{s_i-1}:d(X_{k,l},\bar{1})\geq\epsilon\right\}\right| &\geq \left(\frac{k_{r_j}-2k_{r_j-1}}{k_{r_j}}\right)\left(\frac{l_{s_i}-2l_{s_i-1}}{l_{s_i}}\right)\\ &\geq \left(1-\frac{2}{j}\right)\left(1-\frac{2}{i}\right).\end{aligned}$$

Finally, if $F\neq\bar{0},\bar{1}$ and $min\{d(F,\bar{0}),d(F,\bar{1})\}>\varepsilon>0$, then

$$\frac{1}{k_{nm}}\left|\{k\leq k_n \text{ and } l\leq l_m:d(X_{k,l},F)>\epsilon\}\right|=1.$$

Hence from the above three cases, we conclude that $\{X_{kl}\}$ is not a double statistically convergent sequences.

Proof of the converse of theorem 1.3. It follows combining Theorems 1.1 and 1.2.

We conclude this paper with the following theorem.

Theorem 2.1. *The double sequence X is $S_{\theta_{r,s}}$ - convergent if and only if X is an $S_{\theta_{r,s}}$- Cauchy double sequence.*

Proof. Let $X_{k,l} \to X_0(S_{\theta_{r,s}})(F)$ and $K^{i,j} = \{(k,l) \in \mathcal{N} \times \mathcal{N} : d(X_{k,l}, X_0) < \frac{1}{ij}\}$ for each $(i,j) \in \mathcal{N} \times \mathcal{N}$ we obtain the following $K^{i+1,j+1} \subseteq K^{i,j}$ and

$$\frac{|K^{i,j} \cap I_{r,s}|}{h_{rs}} \to 1 \text{ as } r,s \to \infty.$$

This implies that there exists m_1 and n_1 such that $r \geq m_1$ and $s \geq n_1$ and

$$\frac{|K^{1,1} \cap I_{r,s}|}{h_{rs}} > 0$$

that is $K^{1,1} \cap I_{r,s} \neq \emptyset$. We next choose $m_2 > m_1$ and $n_2 > n_1$ such that $r > m_2$ and $s > n_2$ implies that $K^{2,2} \cap I_{r,s} \neq \emptyset$. Thus for each pairs (r,s) such that $m_1 \leq r < m_2$ and $n_1 \leq s < n_2$ we choose $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ such that $(\bar{k}_r, \bar{l}_s) \in K^{r,s} \cap I_{r,s}$ that is $d(X_{\bar{k}_r,\bar{l}_s}, X_0) < 1$. In general we choose $m_{i+1} > m_i$ and $n_{j+1} > n_j$ such that $r > m_{i+1}$ and $s > n_{j+1}$ this implies $I_{r,s} \cap K^{i+1,j+1} \neq \emptyset$. Thus for all (r,s) such that for $m_i \leq r < m_{i+1}$ and $n_j \leq s < n_{j+1}$ choose $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ i.e. $d(X_{\bar{k}_r,\bar{l}_s}, X_0) < \frac{1}{ij}$. Thus $(\bar{k}_r, \bar{l}_s) \in I_{r,s}$ for each pair (r,s) and $d(X_{\bar{k}_r,\bar{l}_s}, X_0)) < \frac{1}{ij}$ implies $P - \lim_{r,s} X_{\bar{k}_r,\bar{l}_s} = X_0)$. Also, for every $\epsilon > 0$

$$\begin{aligned} & \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{r,s} : d(X_{k,l}, X_{\bar{k}_r,\bar{l}_s}) \geq \epsilon\right\}\right| \\ \leq & \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \frac{\epsilon}{2}\right\}\right| \\ + & \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{r,s} : d(X_{\bar{k}_r,\bar{l}_s}, X_0) \geq \frac{\epsilon}{2}\right\}\right|. \end{aligned}$$

Since $X_{k,l} \to X_0(S_{\theta_{r,s}})(F)$ and $P - \lim_{r,s} X_{\bar{k}_r,\bar{l}_s} = X_0$ it follows that X is an $S_{\theta_{r,s}}$-Cauchy double sequence. Now suppose that X is an $S_{\theta_{r,s}}$-Cauchy double sequence then

$$\begin{aligned} \frac{1}{h_{rs}} |\{(k,l) \in I_{r,s} : d(X_{k,l}, X_0) \geq \epsilon\}| \leq & \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{r,s} : d(X_{k,l}, X_{\bar{k}_r,\bar{l}_s}) \geq \frac{\epsilon}{2}\right\}\right| \\ + & \frac{1}{h_{rs}} \left|\left\{(k,l) \in I_{r,s} : d(X_{\bar{k}_r,\bar{l}_s}, X_0) \geq \frac{\epsilon}{2}\right\}\right|. \end{aligned}$$

Therefore $X_{k,l} \to X_0(S_{\theta_{r,s}})(F)$. This completes the proof.

□

Remark: This paper was completed in 2006 and accepted in 2008 for publication. In the previous version of this paper I proved theorems 1.1 ,1.2 and 1.3 with necessary and sufficient conditions. During the time this paper was waiting for publication Atin [1] proved that the conditions of the Theorems 1.1. , 1.2 and 1.3. are sufficient but his paper does not discuss the necessity of the conditions.

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Weighted Composition Operators from Mixed Norm Spaces into Weighted Bloch Spaces

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Abstract

Assume u is an analytic function on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and φ is an analytic self-map of $\mathbb{D}$. Weighted composition operators $uC_\varphi f = u \cdot (f \circ \varphi)$ from mixed norm spaces into weighted Bloch spaces and little weighted Bloch spaces are characterized by function theoretic properties of the functions u and φ.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all functions analytic on $\mathbb{D}$. Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$, then the linear operator $uC_\varphi f = u \cdot (f \circ \varphi)$ on $H(\mathbb{D})$ is called weighted composition operator. This operator is a generalization of a multiplication operator and a composition operator ([3]). Function theoretic characterizations of when φ induces a bounded or compact composition operator on spaces of analytic functions are problems of some interest.

A positive continuous function ϕ on $[0,1)$ is called normal ([17]) if there is $\delta \in [0,1)$ and a and b, $0 < a < b$ such that

$$\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0;$$
$$\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\phi(r)}{(1-r)^b} = \infty.$$

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For $p, q \in (0, \infty)$ and a normal function ϕ, let $H(p, q, \phi)$ denote the mixed norm space, that is, the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(r, f) \frac{\phi^p(r)}{1-r} r dr \right)^{1/p} < \infty,$$

where the integral means $M_q(f, r)$ are defined by

$$M_q(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q d\theta \right)^{1/q}, \qquad 0 \le r < 1.$$

For $1 \le p < \infty$, $H(p, q, \phi)$, equipped with the norm $\|\cdot\|_{H(p,q,\phi)}$ is a Banach space, while for $0 < p < 1$, $\|\cdot\|_{H(p,q,\phi)}$ is a quasinorm on $H(p, q, \phi)$ and $H(p, q, \phi)$ is a Frechet space. If $p = q$, then $H(p, p, \phi)$ becomes a Bergman-type space, which for $\phi(r) = (1-r)^{1/p}$ is equivalent with the Bergman space $A^p = A^p(\mathbb{D})$.

Let $w(z) = w(|z|)$, $z \in \mathbb{D}$, and w is normal on the interval $[0, 1)$. An analytic function f on $\mathbb{D}$ is said to belong to the weighted Bloch space $\mathcal{B}^w = \mathcal{B}^w(\mathbb{D})$ if

$$B_w(f) = \sup_{z \in \mathbb{D}} w(z)|f'(z)| < \infty.$$

The expression $B_w(f)$ defines a seminorm on $\mathcal{B}^w$, while the natural norm is given by $\|f\|_{\mathcal{B}^w} = |f(0)| + B_w(f)$. With this norm $\mathcal{B}^w$ is a Banach space. Let $\mathcal{B}_0^w$ denote the subspace of $\mathcal{B}^w$ consisting of those $f \in \mathcal{B}^w$ for which

$$\lim_{|z| \to 1} w(z)|f'(z)| = 0.$$

This space is called the little weighted Bloch space. If $w(z) = (1-|z|^2)^\alpha (\ln(1-|z|^2)^{-1})^\beta$, where $\alpha > 0$ and $\beta \ge 0$ we obtain the logarithm-type Bloch space $\mathcal{B}^{\alpha,\beta}$. If $\beta = 0$, $\mathcal{B}^{\alpha,\beta}$ becomes so called α-Bloch space. For $\alpha = \beta = 1$ the space $\mathcal{B}^{\alpha,\beta}$ appears in [1] where it is shown that f is a multiplier for $\mathcal{B}^1(\mathbb{D})$ if and only if $f \in H^\infty(\mathbb{D}) \cap \mathcal{B}^{1,1}$. For more information on Bloch-type spaces see, for example, [2, 3, 5, 15, 18], and the references therein.

In this paper we study the weighted composition operators from the mixed norm space $H(p, q, \phi)$ into the weighted Bloch space $\mathcal{B}^w$ and the little weighted Bloch space $\mathcal{B}_0^w$. For some other closely related papers see, for example, [2], [3], [6]-[14], [16], [19]-[25], and the related references therein. Our main results are motivated by the results in [7]. Here we improve them, namely, we show that the proofs in [7] can be modified so that the results appearing there are naturally extended to the case of mixed norm spaces and weighted Bloch spaces.

Throughout this paper, positive constants are denoted by C and may differ from one occurrence to the next. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2 Auxiliary results

In this section we formulate and prove several lemmas which are used in the proofs of the main results of the paper.

Lemma 1. *Assume $0 < p, q < \infty$, ϕ is normal and $f \in H(p, q, \phi)$. Then the following statements are true.*

(a) There is a positive constant C independent of f such that

$$|f(z)| \leq C \frac{\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{1/q}}, \qquad z \in \mathbb{D}. \tag{1}$$

(b) There is a positive constant C independent of f such that

$$|f'(z)| \leq C \frac{\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{1+1/q}}. \tag{2}$$

Proof. We prove only the statement in *(b)*, the proof of *(a)* is simpler and is omitted. By the monotonicity of the integral means, the asymptotic formula

$$\int_0^1 M_q^p(f, r) \frac{\phi(r)^p}{1-r} r dr \asymp |f(0)|^p + \int_0^1 M_q^p(f', r) \frac{\phi(r)^p}{1-r} (1-r)^p dr,$$

(which can be obtained by a slight modification of Theorem 2 in [4]), Theorem 7.2.5 in [15], and the assumption that ϕ is normal, we have

$$\begin{aligned} \|f\|^p_{H(p,q,\phi)} &\geq C \int_{(1+|z|)/2}^{(3+|z|)/4} M_q^p(f', r) \frac{\phi(r)^p}{1-r} (1-r)^p dr \\ &\geq C M_q^p(f', (1+|z|)/2) \phi(|z|)^p (1-|z|^2)^p \\ &\geq C \phi(|z|)^p (1-|z|^2)^{p+\frac{p}{q}} |f'(z)|^p, \end{aligned}$$

from which the result follows. □

Lemma 2. ([17]) *For $\beta > -1$ and $\gamma > 1 + \beta$ we have*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^\gamma} dr \leq C(1-\rho)^{1+\beta-\gamma}, \qquad 0 < \rho < 1.$$

The following criterion for compactness can be proved in a standard way.

Lemma 3. *The operator $uC_\varphi : H(p, q, \phi) \to \mathcal{B}^w$ is compact if and only if $uC_\varphi : H(p, q, \phi) \to \mathcal{B}^w$ is bounded and for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $H(p, q, \phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|uC_\varphi f_n\|_{\mathcal{B}^w} \to 0$ as $n \to \infty$.*

The next lema can be proved similar to Lemma 1 in [11]. We omit its proof.

Lemma 4. *Assume w is normal. A closed set K in $\mathcal{B}_0^w$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} w(z)|f'(z)| = 0. \tag{3}$$

3 The boundedness and compactnes of the operator $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$

The boundedness and compactness of the weighted composition operator $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ are characterized in this section.

Theorem 1. *Suppose φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, $0 < p,q < \infty$ and ϕ and w are normal on $[0,1)$. Then, $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded if and only if the following conditions are satisfied:*

$$\sup_{z\in\mathbb{D}} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} < \infty, \tag{4}$$

and

$$\sup_{z\in\mathbb{D}} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} < \infty. \tag{5}$$

Proof. First, suppose that $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded. Then, by taking the functions given by $f(z) = 1$ and $f(z) = z$ we obtain that $u \in \mathcal{B}^w$ and

$$\sup_{z\in\mathbb{D}} w(z)|u(z)\varphi'(z) + u'(z)\varphi(z)| < \infty.$$

Using these facts and the boundedness of the function $\varphi(z)$, we have

$$M := \sup_{z\in\mathbb{D}} w(z)|u(z)\varphi'(z)| < \infty. \tag{6}$$

For fixed $w \in \mathbb{D}$ and $t > -1$, set

$$f_w(z) = \frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-\bar{w}z)^{1/q+t+1}}. \tag{7}$$

By [15, Lemma 1.4.10], it follows that

$$M_q(f_w, r) \le C\frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-r|w|)^{t+1}}.$$

Employing Lemma 2 and the assumption that ϕ is normal, we obtain

$$\begin{aligned}
\|f_w\|^p_{H(p,q,\phi)} &= \int_0^1 M_q^p(f_w,r)\frac{\phi^p(r)}{1-r}rdr \le C\int_0^1 \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr \\
&\le C\left(\int_0^{|w|} \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr + \int_{|w|}^1 \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr\right) \\
&\le C\frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)}\frac{\phi^p(|w|)}{(1-|w|^2)^{pt}}\int_0^{|w|}\frac{(1-r)^{pt-1}}{(1-r|w|)^{p(t+1)}}dr \\
&\quad + C\frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)}\frac{\phi^p(|w|)}{(1-|w|^2)^{ps}}\int_{|w|}^1\frac{(1-r)^{ps-1}}{(1-r|w|)^{p(t+1)}}d \le C,
\end{aligned}$$

which implies $\sup_{w\in\mathbb{D}}\|f_w\|_{H(p,q,\phi)}\leq C$. Hence

$$\begin{aligned} & C\|uC_\varphi\|_{H(p,q,\phi)\to\mathcal{B}^w}\geq\|f_{\varphi(\lambda)}\|_{H(p,q,\phi)}\|uC_\varphi\|_{H(p,q,\phi)\to\mathcal{B}^w}\geq\|uC_\varphi f_{\varphi(\lambda)}\|_{\mathcal{B}^w} \\ \geq & \left|(1/q+t+1)\frac{w(\lambda)|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}}-\frac{w(\lambda)|u'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1/q}}\right|, \end{aligned}$$

for every $\lambda\in\mathbb{D}$, and consequently

$$\frac{w(\lambda)|u'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1/q}}\leq C\|uC_\varphi\|_{H(p,q,\phi)\to\mathcal{B}^w}+\frac{Cw(\lambda)|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}}. \quad (8)$$

Now, for $\lambda\in\mathbb{D}$, set

$$g_\lambda(z)=\frac{(1-|\varphi(\lambda)|^2)^{t+2}}{\phi(|\varphi(\lambda)|)(1-\overline{\varphi(\lambda)}z)^{1/q+t+2}}-\frac{(1-|\varphi(\lambda)|^2)^{t+1}}{\phi(|\varphi(\lambda)|)(1-\overline{\varphi(\lambda)}z)^{1/q+t+1}}. \quad (9)$$

Similarly it is proved that $\sup_{\lambda\in\mathbb{D}}\|g_\lambda\|_{H(p,q,\phi)}\leq C$, moreover $g_\lambda(\varphi(\lambda))=0$ and $g'_\lambda(\varphi(\lambda))=\overline{\varphi(\lambda)}/[\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}]$. Therefore

$$C\|uC_\varphi\|_{H(p,q,\phi)\to\mathcal{B}^w}\geq\|uC_\varphi g_\lambda\|_{\mathcal{B}^w}\geq\frac{w(\lambda)|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}}, \quad (10)$$

and consequently

$$\sup_{\lambda\in\mathbb{D}}\frac{w(\lambda)|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}}<\infty. \quad (11)$$

From (11), for a fixed $\delta\in(0,1)$ we obtain

$$\sup_{|\varphi(\lambda)|>\delta}\frac{w(\lambda)|u(\lambda)||\varphi'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}}<\infty. \quad (12)$$

On the other hand, since ϕ is normal, for $\lambda\in\mathbb{D}$ such that $|\varphi(\lambda)|\leq\delta$, we have

$$\frac{w(\lambda)|u(\lambda)\varphi'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}}\leq C\frac{w(\lambda)|u(\lambda)\varphi'(\lambda)|}{(1-\delta^2)^{1+1/q}\phi(\delta)}. \quad (13)$$

Hence, from (6) and (13), we obtain

$$\sup_{|\varphi(\lambda)|\leq\delta}\frac{w(\lambda)|u(\lambda)\varphi'(\lambda)|}{\phi(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{1+1/q}}<\infty. \quad (14)$$

From (12) and (14), (5) follows. Taking the supremum in (8) over $\lambda\in\mathbb{D}$ and using (5), (4) follows, finishing the proof of the implication.

Now, suppose that conditions (4) and (5) hold. Then for each $z\in\mathbb{D}$ and $f\in H(p,q,\phi)$, by Lemma 1 we have

$$\begin{aligned} w(z)|(uC_\varphi f)'(z)| &\leq w(z)|u'(z)||f(\varphi(z))|+w(z)|f'(\varphi(z))||u(z)\varphi'(z)| \\ &\leq Cw(z)|u'(z)|\frac{\|f\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} \\ &\quad +Cw(z)|u(z)\varphi'(z)|\frac{\|f\|_{H(p,q,\phi)}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}}. \quad (15) \end{aligned}$$

Taking the supremum in (15) over $\mathbb{D}$ and then using conditions (4) and (5) we obtain that the operator $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded. $\square$

Theorem 2. *Suppose φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, $0 < p, q < \infty$, ϕ and w are normal on $[0,1)$ and $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded. Then, $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is compact if and only if the following conditions are satisfied:*

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0, \tag{16}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} = 0. \tag{17}$$

Proof. First assume that conditions in (16) and (17) hold, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $H(p,q,\phi)$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_{H(p,q,\phi)} \le L$ and f_n converges to zero uniformly on compacts of $\mathbb{D}$ as $n \to \infty$.

By the assumptions of the theorem we have that for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} < \varepsilon/L$$

and

$$\frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} < \varepsilon/L.$$

Let $\delta\mathbb{D} = \{w \in \mathbb{D} : |w| \le \delta\}$. From this, since ϕ is normal and using the estimates from Lemma 1, it follows that

$$\begin{aligned}
\|uC_\varphi f_n\|_{\mathcal{B}^w} &\le \sup_{z\in\mathbb{D}} w(\lambda)|u'(z)f_n(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}} w(\lambda)|u(z)f_n'(\varphi(z))\varphi'(z)| + |u(0)f_n(\varphi(0))| \\
&\le \sup_{\{z\in\mathbb{D}:\ \varphi(z)\in\delta\mathbb{D}\}} w(\lambda)|u'(z)f_n(\varphi(z))| \\
&\quad + \sup_{\{z\in\mathbb{D}:\ \delta\le|\varphi(z)|<1\}} w(\lambda)|u'(z)f_n(\varphi(z))| \\
&\quad + \sup_{\{z\in\mathbb{D}:\ \varphi(z)\in\delta\mathbb{D}\}} w(\lambda)|u(z)\varphi'(z)||f_n'(\varphi(z))| \\
&\quad + \sup_{\{z\in\mathbb{D}:\ \delta\le|\varphi(z)|<1\}} w(\lambda)|u(z)\varphi'(z)||f_n'(\varphi(z))| + |u(0)f_n(\varphi(0))| \\
&\le \|u\|_{\mathcal{B}^w} \sup_{w\in\delta\mathbb{D}} |f_n(w)| + M \sup_{w\in\delta\mathbb{D}} |f_n'(w)| + |u(0)f_n(\varphi(0))| \\
&\quad + C \sup_{\{z\in\mathbb{D}:\ \delta\le|\varphi(z)|<1\}} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} \|f_n\|_{H(p,q,\phi)} \\
&\quad + C \sup_{\{z\in\mathbb{D}:\ \delta\le|\varphi(z)|<1\}} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} \|f_n\|_{H(p,q,\phi)} \\
&\le \|u\|_{\mathcal{B}^w} \sup_{w\in\delta\mathbb{D}} |f_n(w)| + M \sup_{w\in\delta\mathbb{D}} |f_n'(w)| + |u(0)f_n(\varphi(0))| + 2C\varepsilon,
\end{aligned}$$

where we have used the fact that $u \in \mathcal{B}^w$ and (6) (note that $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded).

Since the sets $\delta\mathbb{D}$ and $\{\varphi(0)\}$ are compact we have, $\lim_{n\to\infty} \sup_{w\in\delta\mathbb{D}} |f_n(w)| = 0$ and $\lim_{n\to\infty} |u(0)f_n(\varphi(0))| = 0$. By Cauchy's estimate, if f_n is a sequence which converges on compacts of $\mathbb{D}$ to zero, then the sequence f_n' also converges on compacts of $\mathbb{D}$ to zero as $n \to \infty$. Employing these facts and letting $n \to \infty$ in the last inequality, we obtain that

$$\limsup_{n\to\infty} \|uC_\varphi f_n\|_{\mathcal{B}^w} \le 2C\varepsilon.$$

Since ε is an arbitrary positive number it follows that the last limit is equal to zero, which implies the compactness of $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$.

Conversely, suppose $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is compact. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. If such a sequence does not exist conditions (16) and (17) are vacously satisfied. Now choose the functions $(f_{\varphi(z_n)})_{n\in\mathbb{N}}$ in (7) which we denote by simplicity by $(f_n)_{n\in\mathbb{N}}$. Then, we know that $\sup_{n\in\mathbb{N}} \|f_n\|_{H(p,q,\phi)} \le C$ and f_n converges to 0 uniformly on compacts of $\mathbb{D}$ as $n \to \infty$. Since uC_φ is compact, we have $\|uC_\varphi f_n\|_{\mathcal{B}^w} \to 0$ as $n \to \infty$.

We also have

$$\begin{aligned} \|uC_\varphi f_n\|_{\mathcal{B}^w} &\ge \sup_{z\in\mathbb{D}} w(z)|(uC_\varphi f_n)'(z)| \\ &\ge \left| (1/q+t+1)\frac{w(z_n)|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1+1/q}} - \frac{w(z_n)|u'(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/q}} \right|. \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{|\varphi(z_n)|\to 1} \frac{(1/q+t+1)w(z_n)|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1+1/q}} \\ &= \lim_{|\varphi(z_n)|\to 1} \frac{w(z_n)|u'(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/q}}, \end{aligned} \tag{18}$$

if one of these two limits exists.

Further, by using the functions $(g_{\varphi(z_n)})_{n\in\mathbb{N}} := (g_n)_{n\in\mathbb{N}}$ defined in (9), we have that $(g_n)_{n\in\mathbb{N}}$ is a bounded sequence in $H(p,q,\phi)$, $g_n \to 0$ uniformly on compacts of $\mathbb{D}$ as $n \to \infty$, $g_n(\varphi(z_n)) = 0$ and

$$g_n'(\varphi(z_n)) = \frac{\overline{\varphi(z_n)}}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1+1/q}}.$$

Hence by the compactness of $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ we have $\|uC_\varphi g_n\|_{\mathcal{B}^w} \to 0$ as $n \to \infty$.

On the other hand by, (10) the following inequality holds

$$\frac{w(z_n)|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1+1/q}} \le \|uC_\varphi g_n\|_{\mathcal{B}^w}. \tag{19}$$

Letting $n \to \infty$ in (19), it follows that (17) holds. From this and (18), (16) follows, as desired. □

4 The boundedness and compactness of the operator $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$

In this section we characterize the boundedness and compactness of the weighted composition operator $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$. Before we formulate and prove the main results of this section we prove two lemmas.

Lemma 5. *Suppose φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, $0 < p, q < \infty$ and ϕ and w are normal on $[0,1)$. Then,*

$$\lim_{|z|\to 1} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0 \tag{20}$$

if and only if $u \in \mathcal{B}_0^w$ and

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0. \tag{21}$$

Proof. First, we assume that (20) holds. If $\|\varphi\|_\infty < 1$ then condition (21) is vacuously satisfied. If $|\varphi(z)| \to 1$, then $|z| \to 1$, from which it follows that

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} = 0,$$

hence (21) holds.

Now, assume to the contrary that $u \notin \mathcal{B}_0^w$. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $|z_n| \in [1/2, 1)$, $|z_n| \to 1$ and $\liminf_{n\to\infty} w(z_n)|u'(z_n)| > 0$. From this and since ϕ is normal, we have

$$\liminf_{n\to\infty} \frac{w(z_n)|u'(z_n)|}{\phi(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{1/q}} > 0,$$

which contradicts to (20). Hence $u \in \mathcal{B}_0^w$.

Conversely, suppose that $u \in \mathcal{B}_0^w$ and (21) holds. From (21) it follows that for every $\varepsilon > 0$, there exists $r \in (0,1)$ such that

$$\frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} < \varepsilon$$

when $r < |\varphi(z)| < 1$. Since $u \in \mathcal{B}_0^w$, there exists a $\sigma \in (0,1)$ such that

$$w(z)|u'(z)| \le \varepsilon(1-r^2)^{1/q}\phi(r).$$

when $\sigma < |z| < 1$.

Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{w(z)|u'(z)|}{\phi(|\varphi(z)|))(1-|\varphi(z)|^2)^{1/q}} < \varepsilon. \tag{22}$$

If $|\varphi(z)| \leq r$ and $\sigma < |z| < 1$, then since ϕ is normal, we obtain

$$\frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} < \frac{(1-r)^s w(z)|u'(z)|}{\phi(r)(1-|\varphi(z)|^2)^{1/q+s}} < \varepsilon. \tag{23}$$

From (22) with (23), condition (20) follows. □

The next lemma can be proved similar to Lemma 5.

Lemma 6. *Suppose φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, $0 < p, q < \infty$ and ϕ and w are normal on $[0,1)$. Then,*

$$\lim_{|z|\to 1} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} = 0 \tag{24}$$

if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} = 0 \tag{25}$$

and

$$\lim_{|z|\to 1} w(z)|u(z)\varphi'(z)| = 0. \tag{26}$$

Theorem 3. *Suppose φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, $0 < p, q < \infty$ and ϕ and w are normal on $[0,1)$. Then, $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$ is bounded if and only if $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded, $u \in \mathcal{B}_0^w$ and*

$$\lim_{|z|\to 1} w(z)|u(z)\varphi'(z)| = 0. \tag{27}$$

Proof. First assume that $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$ is bounded. Then, it is clear that $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded. Taking the functions $f(z) = 1$ and $f(z) = z$, we obtain that $u \in \mathcal{B}_0^w$ and (27) holds.

Conversely, assume that $uC_\varphi : H(p,q,\phi) \to \mathcal{B}^w$ is bounded, $u \in \mathcal{B}_0^w$ and that (27) holds. Then, for each polynomial p, we have that

$$\begin{aligned} w(z)|(uC_\varphi p)'(z)| &\leq w(z)|u'(z)||p(\varphi(z))| + w(z)|u(z)\varphi'(z)p'(\varphi(z))| \\ &\leq w(z)|u'(z)|\|p\|_\infty + w(z)|u(z)\varphi'(z)|\|p'\|_\infty, \end{aligned}$$

from which it follows that $uC_\varphi p \in \mathcal{B}_0^w$. Since the set of all polynomials is dense in $H(p,q,\phi)$, we have that for every $f \in H(p,q,\phi)$ there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that $\|f - p_n\|_{H(p,q,\phi)} \to 0$, as $n \to \infty$. From this and since the operator $uC_\varphi : H(p,q,\phi) \to \mathcal{B}$ is bounded, we have

$$\|uC_\varphi f - uC_\varphi p_n\|_{\mathcal{B}^w} \leq \|uC_\varphi\|_{H(p,q,\phi)\to\mathcal{B}^w}\|f - p_n\|_{H(p,q,\phi)} \to 0$$

as $n \to \infty$. Since $\mathcal{B}_0^w$ is a closed subset of $\mathcal{B}^w$, we obtain $uC_\varphi(H(p,q,\phi)) \subset \mathcal{B}_0^w$, which implies the boundedness of the operator $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$. $\square$

Theorem 4. *Suppose φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, $0 < p, q < \infty$ and ϕ and w are normal on $[0,1)$. Then, $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0 \tag{28}$$

and

$$\lim_{|z|\to 1} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} = 0. \tag{29}$$

Proof. First, we assume that $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$ is compact. Taking the test function $f(z) \equiv 1$ we obtain that $u \in \mathcal{B}_0^w$. From this, taking $f(z) = z$, and using the boundedness of $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$ it follows that

$$\lim_{|z|\to 1} w(z)|u(z)\varphi'(z)| = 0. \tag{30}$$

Hence, if $\|\varphi\|_\infty < 1$, in view of $u \in \mathcal{B}_0^w$ and (30), we obtain

$$\lim_{|z|\to 1} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} \le C \lim_{|z|\to 1} \frac{w(z)|u'(z)|}{\phi(\|\varphi\|_\infty)(1-\|\varphi\|_\infty^2)^{1/p}} = 0$$

and

$$\lim_{|z|\to 1} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/p}} \le C \lim_{|z|\to 1} \frac{w(z)|u(z)\varphi'(z)|}{\phi(\|\varphi\|_\infty)(1-\|\varphi\|_\infty^2)^{1+1/p}} = 0.$$

from which the implication follows in this case.

Now assume $\|\varphi\|_\infty = 1$. Let $(\varphi(z_n))_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$. By using the functions $(f_{\varphi(z_n)})_{n\in\mathbb{N}}$ and $(g_{\varphi(z_n)})_{n\in\mathbb{N}}$ as in the proof of Theorem 2 we obtain

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|u(z)\varphi'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1+1/q}} = 0 \tag{31}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{w(z)|u'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0. \tag{32}$$

From $u \in \mathcal{B}_0^w$ and (30)-(32) and by employing Lemmas 5 and 6 the result follows.

Conversely, by taking the supremum in (15) over the unit ball in $H(p,q,\phi)$, then letting $|z| \to 1$, and using conditions (28) and (29) we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{H(p,q,\phi)}\le 1} w(z)|(uC_\varphi(f))'(z)| = 0,$$

which by Lemma 4 implies the compactness of $uC_\varphi : H(p,q,\phi) \to \mathcal{B}_0^w$. $\square$

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A NOTE ON THE FOURIER TRANSFORM OF p-ADIC q-INTEGRALS ON $\mathbb{Z}_p$

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Abstract. The p-adic q-integral ($= I_q$-integral) was defined by author in the previous paper [1, 3]. In this paper, we consider I_q-Fourier transform and investigate some properties which are related to this transform. By our results in this paper, we easily see that I_q-Fourier transform is exactly same I_0-Fourier transform when $q = 1$ due to Woodcock, see [15: p.105].

§1. Introduction

Let us denote $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}$ sets of positive integer, integer, rational and complex numbers respectively. Let p be prime and $x \in \mathbb{Q}$. Then $x = p^{v(x)}\frac{m}{n}$, where $m, n, v = v(x) \in \mathbb{Z}$, m and n are not divisible by p. Let $|x|_p = p^{-v(x)}$ and $|0|_p = 0$. Then $|x|_p$ is valuation on $\mathbb{Q}$ satisfying

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

Completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is denoted by $\mathbb{Q}_p$ and called the field of p-adic rational numbers. $\mathbb{C}_p$ is the completion of algebraic closure of $\mathbb{Q}_p$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is called the ring of p-adic rational integers(see [1, 2, 3, 4]). Let l be a fixed integer and let p be a fixed prime number. We set

$$X_l = \varprojlim_N (\mathbb{Z}/lp^N\mathbb{Z}), \text{ and } X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<lp \\ (a,p)=1}} (a + lp\mathbb{Z}_p),$$
$$a + lp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{lp^N}\},$$

where $N \in \mathbb{N}$ and $a \in \mathbb{Z}$ lies in $0 \leq a < lp^N$, cf. [1-20].

When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper, we assume that $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$,

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so that $q^x = \exp(x \log q)$ for each $x \in X$. We use the notation as $[x] = [x; q] = \frac{1-q^x}{1-q}$ for each $x \in X$. Hence $\lim_{q \to 1}[x] = x$, cf.[3]. For any positive integer N, we set

$$\mu_q(a + lp^N\mathbb{Z}_p) = \frac{q^a}{[lp^N]}, \text{ cf. [1-20]},$$

and this can be extended to a distribution on X. This distribution yields an integral as follows (see [3]):

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \int_X f(x)\, d\mu_q(x),$$

where $f \in UD(\mathbb{Z}_p) =$ the space of uniformly differentiable function on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$, cf. [3]. Let C_{p^n} be the cyclic group consisting of all p^n-th roots of unity in $\mathbb{C}_p$ for any $n \geq 0$ and T_p be the direct limit of C_{p^n} with respect to the natural morphism, hence T_p ia the union of all C_{p^n} with discrete topology. U_p denotes the group of all principal units in $\mathbb{C}_p$. For any $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have an integral $I_0(f)$ with respect to the so called invariant measure μ_0:

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x), \text{ cf. [2, 4] },$$

and the Fourier transform $\hat{f}_w = I_0(f\phi_w)$, where ϕ_w denotes a uniformly differentiable function on $\mathbb{Z}_p$ belonging to $w \in T_p$ defined by $\phi_w(x) = w^x$, cf. [4]. Now we introduce the convolution for any $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ due to Woodcock as follows:

$$f * g(x) = \sum_w \hat{f}_w \hat{g_w} \phi_{w^{-1}}(x), \text{ see [2, 4]}.$$

As known results, $f * g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, and $\widehat{(f * g)}_w = \hat{f}_w \hat{g_w}$, (see [4]). In this paper, we consider I_q-Fourier transform and investigate some properties which are related to this transform. By using our results in this paper, we easily see that I_q-Fourier transform is exactly same I_0-Fourier transform when $q = 1$ due to Woodcock, see [15: p.105].

§2. I_q-Integral Transforms

For any $f \in UD(X)$, the p-adic q-integral was defined by

$$I_q(f) = \int_X f(x)\ d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[lp^N]} \sum_{0 \leq x < lp^N} f(x) q^x, \text{ cf. [3] }.$$

Note that

$$I_0(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)\ d\mu_1(x) = \lim_{N \to \infty} \frac{1}{lp^N} \sum_{0 \leq x < lp^N} f(x),$$

and that

$$I_0(f_1) = I_0(f) + f'(0), \text{ where } f'(0) = \frac{d}{dx} f(x)|_{x=0} \text{ and } f_1(x) = f(x+1).$$

Let $T_p = \cup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} \mathbb{Z}/p^n\mathbb{Z}$, where $C_{p^n} = \{\xi \in X |\ \xi^{p^n} = 1\}$ is the cyclic group of order p^n, see [1]. For $\xi \in T_p$, we denote by $\phi_\xi : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto \xi^x$. If we take $f(x) = \phi_\xi(x)e^{tx}$, then we have that $\int_X e^{tx}\phi_\xi(x)d\mu_q(x) = \frac{t+\log q}{q\xi e^t - 1}$, cf. [2]. We now consider I_q-Fourier transform as follows:

$$(\hat{f_w})_q = I_q(\phi_w f) = \int_{\mathbb{Z}_p} \phi_w(x) f(x) d\mu_q(x), \text{ where } f \in UD(\mathbb{Z}_p),\ w \in T_p\ ,$$

and its inverse transform is derived by

$$\frac{\log q}{q-1} \lim_{n\to\infty} \sum_{x=0}^{p^n-1} w^{-x} I_q(\phi_w f) = \frac{\log q}{q-1} \lim_{n\to\infty} \sum_{x=0}^{p^n-1} w^{-x} \frac{1}{[p^n]} \sum_{z=0}^{p^n-1} w^z f(z) q^z = f(x)\phi_q(x).$$

Thus, we obtain the below proposition.

Proposition 1. *Let $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$. Then we have the inverse formula of I_q-Fourier transform as follows:*

$$f(x)\phi_q(x) = \frac{\log q}{q-1} \sum_w \phi_{w^{-1}} (\hat{f_w})_q, \text{ where } \sum_w = \lim_{n\to\infty} \sum_{w\in C_{p^n}}.$$

Remark. In [4], we note that if $\alpha \in U_p$, then ϕ_α is called locally analytic character and if $\alpha \in T_p$, then ϕ_α is called locally constant function. For $f, g \in UD(\mathbb{Z}_p)$, we consider the convolution of f, g by

$$f *_q g = \sum_w \widehat{(f_{wq^{-1}})}_q \widehat{(g_{wq^{-1}})}_q \phi_{w^{-1}}.$$

Thus, we note that

$$f *_q g \in UD(\mathbb{Z}_p), \text{ and } \widehat{((f *_q g)_{wq^{-1}})}_q = \widehat{(f_{wq^{-1}})}_q \widehat{(g_{wq^{-1}})}_q.$$

And we also see that

$$UD(\mathbb{Z}_p, \mathbb{C}_p)/\{f \in UD(\mathbb{Z}_p, \mathbb{C}_p) | f' = 0\} \cong C(\mathbb{Z}_p, \mathbb{C}_p),$$

where $C(\mathbb{Z}_p, \mathbb{C}_p)$ is the space of the continuous function from $\mathbb{Z}_p$ to $\mathbb{C}_p$. Another convolution $\otimes_q$ is induced from the above convolution $*_q$ by $(f *_q g)' = -f' \otimes_q g'$. Then, we also see that $f \otimes_q g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$. From these definitions, we can derive the below theorem.

Theorem 2. *For $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have*

$$f *_q g(z) = \left(\frac{q-1}{\log q}\right) I_q^{(x)}(f(x)g(z-x)\phi_{q^{-1}}(x)) - \left(\frac{q-1}{\log q}\right)^2 (f \otimes_q g'(z)),$$

where $I_q^{(x)}$ means the integration with respect to the variable x.

Since $\widehat{((f *_q g)_{wq^{-1}})}_q = \widehat{(f_{wq^{-1}})}_q \widehat{(g_{wq^{-1}})}_q$, for $w \in T_p$, we have

$$\int_{\mathbb{Z}_p} (f *_q g(x)) q^{-x} d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) q^{-x} d\mu_q(x) \int_{\mathbb{Z}_p} g(x) q^{-x} d\mu_q(x).$$

From this, we can derive the below worthwhile and interesting formula:

Theorem 3. *Let $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$. Then we have*

$$I_q^{(z)}(f \otimes_q g'(z)q^{-z}) = \left(\frac{\log q}{q-1}\right) I_q^{(z)}(I_q^{(x)}(f(x)g(z-x)q^{-x})q^{-z}) - \left(\frac{\log q}{q-1}\right)^2 I_q^{(z)}(f(z)q^{-z})I_q^{(z)}(g(z)q^{-z}).$$

Indeed Theorem 3 is a q-analogue result due to Woodcock ([15: p.105]), corresponding to the case $q = 1$.

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TWO FIXED POINT THEOREMS ON THREE COMPLETE UNIFORM SPACES

DURAN TURKOGLU

Abstract. In this paper we prove two new fixed point theorems for three single valued mappings with functions on three complete uniform spaces.

1. Introduction

Uniform spaces form a natural extension of metric spaces. An exact analogue of the well-known Banach contraction principle in uniform spaces was obtained independently by Acharya [1] and Tarafdar [6]. Since then a number of fixed point theorems for single-valued and multi-valued mappings using various contactive conditions in this setting have been obtained ([2], [3], [4], [5], [6]).

Let $(X,\mathcal{U}_1)$, $(Y,\mathcal{U}_2)$ and $(Z,\mathcal{U}_3)$ be uniform spaces. Families $\{d_i : i \in I\}$, $\{\rho_i : i \in I\}$ and $\{\sigma_i : i \in I\}$ of pseudometrics on X, Y and Z respectively, are called on associated families for uniformities $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ respectively, if families

$$\beta_1 = \{V_1(i,r) : i \in: I, r > 0\}$$

$$\beta_2 = \{V_2(i,r) : i \in: I, r > 0\}$$

$$\beta_3 = \{V_3(i,r) : i \in: I, r > 0\}$$

where

$$V_1(i,r) = \{(x,x') : x,x' \in X, d_i(x,x') < r\}$$

$$V_2(i,r) = \{(y,y') : y,y' \in Y, \rho_i(y,y') < r\}$$

$$V_3(i,r) = \{(z,z') : z,z' \in Z, \sigma_i(z,z') < r\}$$

are subbases for the uniformities $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$ respectively. We may assume that β_1, β_2 and β_3 themselves are a base by adjoining finite intersections of members of β_1, β_2 and β_3 if necessary. The corresponding families of pseudometrics are called an augmented associated families for $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$. For details the reader is referred to Tarafdar [6] and Thron [7].

Throughout this paper, $\Re_+$ stands for the non-negative reals.

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2. Main Results

We will also denote by $\Im_3$ the set of all real functions $f : \Re_+^3 \to \Re_+$ such that:
(i) f is upper semi-continuous in each coordinate variable;
(ii) If either $u \leq f(v, u, 0)$ or $u \leq f(v, 0, u)$ for all $u, v \geq 0$, then there exists a real constant $0 \leq c < 1$ such that $u \leq cv$.

Theorem 1. *Let $(X,\mathcal{U}_1)$, $(Y,\mathcal{U}_2),(Z,\mathcal{U}_3)$ be complete Hausdorff uniform spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$d_i(RSy, RSTx) \leq f(\sigma_i(Sy, STx), d_i(x, RSTx), d_i(x, RSy)) \tag{2.1}$$

$$\rho_i(TRz, TRSy) \leq g(d_i(Rz, RSy), \rho_i(y, TRSy), \rho_i(y, TRz)) \tag{2.2}$$

$$\sigma_i(STx, STRz) \leq h(\rho_i(Tx, TRz), \sigma_i(z, STRz), \sigma_i(z, STx)) \tag{2.3}$$

for all x in X, y in Y and z in Z, $\forall i \in I$, where $f, g, h \in \Im_3$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$, $Rw = u$.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X, Y and Z, respectively by $x_n = (RST)^n x_0, y_n = Tx_{n-1}, z_n = Sy_n$ for $n = 1, 2, ...$

Let $U_1 \in \mathcal{U}_1$ be arbitrary and let $V(h, p) \in \beta, h \in I$ and $p > 0$ be such that $V(h, p) \subseteq U_1$. Applying inequality (2.1) for $y = y_n$ and $x = x_n$ we have

$$d_i(x_n, x_{n+1}) \leq f(\sigma_i(z_n, z_{n+1}), d_i(x_n, x_{n+1}), 0)$$

which implies by (ii) that

$$d_i(x_n, x_{n+1}) \leq c_i^1 \sigma_i(z_n, z_{n+1}) \tag{2.4}$$

where $c_i^1 \in [0, 1)$. Applying inequality (2.3) for $x = x_{n-1}$ and $z = z_n$ we have

$$\sigma_i(z_n, z_{n+1}) \leq h(\rho_i(y_n, y_{n+1}), \sigma_i(z_n, z_{n+1}), 0)$$

which implies by (ii) that

$$\sigma_i(z_n, z_{n+1}) \leq c_i^3 \rho_i(y_n, y_{n+1}) \tag{2.5}$$

where $c_i^3 \in [0, 1)$. Applying inequality (2.2) for $z = z_{n-1}$ and $y = y_n$ we have

$$\rho_i(y_n, y_{n+1}) \leq g(d_i(x_{n-1}, x_n), \rho_i(y_n, y_{n+1}), 0)$$

which implies by (ii) that

$$\rho_i(y_n, y_{n+1}) \leq c_i^2 d_i(x_{n-1}, x_n) \tag{2.6}$$

where $c_i^2 \in [0, 1)$. It follows from inequalities (2.4), (2.5) and (2.6) that

$$d_i(x_{n+1}, x_n) \leq c_i^1 \sigma_i(z_n, z_{n+1}) \leq c_i^1 c_i^3 \rho_i(y_n, y_{n+1}) \leq ... \leq (c_i^1 c_i^2 c_i^3)^n d_i(x_0, x_1).$$

Since $0 \leq c_i^1 c_i^2 c_i^3 < 1$, it follows that there exists $p > 0$ such that $d_i(x_{n+1}, x_n) < p$ and hence $(x_{n+1}, x_n) \in V(h, p) \subseteq U_1$. The sequence $\{x_n\}$ is therefore a Cauchy sequence in the complete uniform space X, and so has a limit u in X. Similarly the sequences $\{y_n\}$ and $\{z_n\}$ is a Cauchy sequence in the complete uniform spaces Y and Z so has a limit v in Y and w in Z respectively.

Now suppose that S is continuous. Then $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$ and so

$$Sv = w. \tag{2.7}$$

Let $U_2 \in \mathcal{U}_1$ be arbitrary and let $V(j,t) \in \beta$, $j \in I$ and $t > 0$ be such that $V(j,t) \subseteq U_2$. Applying inequality (2.1) we now have

$$d_i(RSv, x_{n-1}) \leq f(\sigma_i(Sv, z_n), d_i(x_{n-1}, x_n), d_i(x_{n-1}, RSv)).$$

Letting n tend to infinity and using (i), it follows

$$d_i(RSv, u) \leq f(\sigma_i(Sv, w), 0, d_i(u, RSv))$$

using equation (2.7) we have

$$d_i(RSv, u) \leq f(0, 0, d_i(u, RSv)).$$

By (ii) follows that $d(u, RSv) \leq c_i^1.0 < t$. Hence $(u, RSv) \in V(j,t) \subseteq U_2$. Since X is Hausdorff, we have $u = RSv$ which implies by (2.7) that

$$u = RSv = Rw. \tag{2.8}$$

Let $U_3 \in \mathcal{U}_2$ be arbitrary and let $V(k,s) \in \beta$, $k \in I$ and $s > 0$ be such that $V(k,s) \subseteq U_3$. Applying inequality (2.2) we have

$$\rho_i(Tu, y_{n+1}) \leq g(d_i(u, x_n), \rho_i(y_n, y_{n+1}), \rho_i(y_n, TRw)).$$

Letting n tend to infinity and using (i), it follows that

$$\rho_i(Tu, v) \leq g(0, 0, \rho_i(v, TRw))$$

which implies by (2.8) that

$$\rho_i(Tu, v) \leq g(0, 0, \rho_i(v, Tu)).$$

By (ii) follows that $\rho_i(Tu, v) \leq 0 < s$. Hence $(v, Tu) \in V(k,s) \subseteq U_3$. Since X is Hausdorff, we have

$$Tu = v. \tag{2.9}$$

It now follows from equations (2.7), (2.8) and (2.9)

$$\begin{aligned} TRSv &= TRw = Tu = v, \\ STRw &= STu = Sv = w, \\ RSTu &= RSv = Rw = u. \end{aligned}$$

The same results of course will hold if R or T is continuous instead of S.

We now prove the uniqueness of the fixed point u. Supoose that RST has a second fixed point u' . Let $U_4 \in \mathcal{U}_1$ be arbitrary and let $V(l,r) \in \beta$, $l \in I$ and $r > 0$ be such that $V(l,r) \subseteq U_4$. Using inequality (2.1), we have

$$d_i(RSTu, RSTu') \leq f(\sigma_i(STu', STu), d_i(u, RSTu), d_i(u, RSTu'))$$

and so

$$d_i(u, u') \leq f(\sigma_i(STu, STu'), 0, d_i(u, u')).$$

By (ii) we have

$$d_i(u, u') \leq c_i^1 \sigma_i(STu, STu'). \tag{2.10}$$

Further, using inequality (2.3), we have succesively

$$\sigma_i(STRSTu, STu') \leq h(\rho_i(Tu', TRSTu), 0, \sigma_i(STu, STu'))$$

and so

$$\sigma_i(STu, STu') \leq h(\rho_i(Tu', Tu), 0, \sigma_i(STu, STu')).$$

By (ii) we have

$$\sigma_i(STu, STu') \leq c_i^3 \rho_i(Tu, Tu'). \tag{2.11}$$

Finally, using inequality (2.2), we have

$$\rho_i(Tu, Tu') \leq c_i^2 d_i(u, u'). \tag{2.12}$$

By (2.10), (2.11) and (2.12) we have

$$d_i(u, u') \leq (c_i^1 c_i^2 c_i^3) d_i(u, u')$$

which implies $d_i(u, u') = 0$ and $(u, u') \in V(l, r) \subseteq U_4$. Since X is Hausdorff, we have $u = u'$. The fixed point u of RST is therefore unique. Similarly, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of the theorem. □

Corollary 1. *[8] Let (X, d), $(Y, \rho), (Z, \sigma)$ be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$d(RSy, RSTx) \leq f(\sigma(Sy, STx), d(x, RSTx), d(x, RSy))$$

$$\rho(TRz, TRSy) \leq g(d(Rz, RSy), \rho(y, TRSy), \rho(y, TRz))$$

$$\sigma(STx, STRz) \leq h(\rho(Tx, TRz), \sigma(z, STRz), \sigma(z, STx))$$

for all x in X, y in Y and z in Z, where $f, g, h \in \Im_3$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$, $Rw = u$.

Proof. When we replace the uniform spaces $(X, \mathcal{U}_1)$, $(Y, \mathcal{U}_2), (Z, \mathcal{U}_3)$ in Theorem1 by (X, d), $(Y, \rho), (Z, \sigma)$ metric spaces, then proof can be obtained easily. □

We will also denote by $\Im_4$ the set of all real functions $f : \Re_+^4 \to \Re_+$ such that:

(iii) f is upper semi-continuous in each coordinate variable;

(iv) If either $u \leq f(v, u, 0, w)$ or $u \leq f(v, 0, u, w)$ for all $u, v \geq 0$, then there exists a real constant $0 \leq c < 1$ such that $u \leq c \max\{v, w\}$.

We now generalize Theorem 2 as follows:

Theorem 2. *Let $(X, \mathcal{U}_1)$, $(Y, \mathcal{U}_2), (Z, \mathcal{U}_3)$ be complete Hausdorff uniform spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$d_i(RSTx, RSy) \leq f(\rho_i(y, Tx), d_i(x, RSTx), d_i(x, RSy), \sigma_i(Sy, STx)) \tag{2.13}$$

$$\rho_i(TRz, TRSy) \leq g(\sigma_i(z, Sy), \rho_i(y, TRSy), \rho_i(y, TRz), d_i(Rz, RSy)) \tag{2.14}$$

$$\sigma_i(STRz, STx) \leq h(d_i(x, Rz), \sigma_i(z, STRz), \sigma_i(z, STx), \rho_i(Tx, TRz)) \tag{2.15}$$

for all x in X, y in Y and z in Z, $i \in I$, where $f, g, h \in \Im_4$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$, $Rw = u$.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X, Y and Z, respectively, by

$$x_n = (RST)^n x_0 \ , \ y_n = Tx_{n-1}, \ z_n = Sy_n \text{ for } n = 1, 2, 3, ...$$

Let $U_1 \in \mathcal{U}_1$ be arbitrary and let $V(h,p) \in \beta, h \in I$ and $p > 0$ be such that $V(h,p) \subseteq U_1$Applying inequality (2.14) and using property (iv) for $z = z_{n-1}$ and $y = y_n$ we have

$$\rho_i(y_n, y_{n+1}) = \rho_i(TRz_{n-1}, TRSy_n) \leq g(\sigma_i(z_{n-1}, z_n), \rho_i(y_n, y_{n+1}), 0, d_i(x_{n-1}, x_n))$$

and it follows that

$$\rho_i(y_n, y_{n+1}) \leq c_i \max\{d_i(x_{n-1}, x_n), \sigma_i(z_{n-1}, z_n)\} \tag{2.16}$$

which implies by (iv) and inequality (2.16) that

$$\begin{aligned} \sigma_i(z_n, z_{n+1}) &\leq c_i \max\{d_i(x_{n-1}, x_n), \rho_i(y_n, y_{n+1})\} \\ &\leq c_i \max\{d_i(x_{n-1}, x_n), \sigma_i(z_{n-1}, z_n)\}. \end{aligned} \tag{2.17}$$

Applying inequality (2.13) for $y = y_n$ and $x = x_n$ we have

$$d_i(x_n, x_{n+1}) = d_i(RSTx_n, RSy_n) \leq f(\rho_i(y_n, y_{n+1}), d_i(x_n, x_{n+1}), 0, \sigma_i(z_n, z_{n+1}))$$

which implies by (iv) and inequality (2.16) and (2.17) that

$$\begin{aligned} d_i(x_n, x_{n+1}) &\leq c_i \max\{\rho_i(y_n, y_{n+1}), \sigma_i(z_n, z_{n+1})\} \\ &\leq c_i \max\{d_i(x_{n-1}, x_n), \sigma_i(z_{n-1}, z_n)\}. \end{aligned} \tag{2.18}$$

It now follows easily by induction on using inequalities (2.16), (2.17) and (2.18) that

$$\begin{aligned} d_i(x_n, x_{n+1}) &\leq c_i^{n-1} \max\{d_i(x_1, x_2), \sigma_i(z_1, z_2)\}, \\ \rho_i(y_n, y_{n+1}) &\leq c_i^{n-1} \max\{d_i(x_1, x_2), \sigma_i(z_1, z_2)\} \\ \sigma_i(z_n, z_{n+1}) &\leq c_i^{n-1} \max\{d_i(x_1, x_2), \sigma_i(z_1, z_2)\}. \end{aligned}$$

Since $0 \leq c_i < 1$, it follows that there exists $p > 0$ such that $d_i(x_{n+1}, x_n) < p$ and hence $(x_{n+1}, x_n) \in V(h,p) \subseteq U_1$. The sequence $\{x_n\}$ is therefore a Cauchy sequence in the complete uniform space X, and so has a limit u in X. Similarly the sequences $\{y_n\}$ and $\{z_n\}$ is a Cauchy sequence in the complete uniform spaces Y and Z so has a limit v in Y and w in Z respectively.
Now suppose that S is continuous. Then $\lim Sy_n = \lim z_n$ and so

$$Sv = w. \tag{2.19}$$

Let $U_2 \in \mathcal{U}_1$ be arbitrary and let $V(j,t) \in \beta$, $j \in I$ and $t > 0$ be such that $V(j,t) \subseteq U_2$. Applying inequality (2.13) for $y = v$ and $x = x_n$ we now have

$$d_i(RSv, x_{n+1}) \leq f(\rho_i(v, Tx_n), d_i(x_n, x_{n+1}), d_i(x_n, RSv), \sigma_i(Sv, STx_n)).$$

Letting n tend to infinity and using (iii) it follows

$$d_i(RSv, u) \leq f(0, 0, d_i(RSv, u), 0)$$

which implies by (iv) that $d_i(RSv, u) = 0 < t$. Hence $(RSv, u) \in V(j,t) \subseteq U_2$. Since X is Hausdorff, we have

$$RSv = u. \tag{2.20}$$

Using equation (2.19) this gives us

$$Rw = u. \tag{2.21}$$

Let $U_3 \in \mathcal{U}_2$ be arbitrary and let $V(k,s) \in \beta, k \in I$ and $s > 0$ be such that $V(k,s) \subseteq U_3$. Using equation (2.20) and inequality (2.14) for $z = Sv$ and $y = y_n$, we have

$$\rho_i(Tu, y_{n+1}) \leq g(\sigma_i(Sv, Sy_n), \rho_i(y_n, TRSy_n), \rho_i(y_n, TRSv), d_i(RSv, RSy_n)).$$

Letting n tend to infinity and using (iii) it follows

$$\rho_i(Tu, v) \leq g(0, 0, \rho_i(v, Tu), 0)$$

which implies (ii) that $\rho_i(Tu, v) = 0 < s$. Hence $(Tu, v) \in V(k,s) \subseteq U_3$. Since X is Hausdorff, we have

$$Tu = v. \tag{2.22}$$

It follows from equations (2.19),(2.21) and (2.22) that

$$\begin{aligned} TRSv &= TRw = Tu = v, \\ STRw &= STu = Sv = w, \\ RSTu &= RSv = Rw = u. \end{aligned}$$

The same results of course hold if R or T is continuous instead of S.We now prove the uniqueness of the fixed point u. Supoose that RST has a second fixed point $u^{'}$. Let $U_4 \in \mathcal{U}_1$ be arbitrary and let $V(l,r) \in \beta, l \in I$ and $r > 0$ be such that $V(l,r) \subseteq U_4$. Then using inequality (2.13) for $y = Tu$ and $x = u^{'}$ we have

$$d_i(u, u^{'}) = d_i(RSTu, RSTu') \leq f(\rho_i(Tu, Tu'), 0, d_i(u, u^{'}), \sigma_i(STu, STu^{'}))$$

which implies by (ii) that

$$d_i(u, u^{'}) \leq c_i \max\{\rho_i(Tu, Tu'), \sigma_i(STu, STu^{'})\}. \tag{2.23}$$

Further, using inequality (2.14) for $z = STu$ and $y = Tu'$ we have

$$\rho_i(Tu, Tu') \leq g(\sigma_i(STu, STu^{'}), 0, \rho_i(Tu, Tu'), d_i(u, u'))$$

which implies by (ii) that

$$\rho_i(Tu, Tu') \leq c_i \max\{\sigma_i(STu, STu^{'}), d_i(u, u')\}. \tag{2.24}$$

inequalities (2.23) and (2.24) implies that

$$d_i(u, u') \leq c_i \sigma_i(STu, STu'). \tag{2.25}$$

Finally, using inequality (2.15) and property (ii), we have

$$\sigma_i(STu, STu') \leq h(d_i(u, u'), 0, \sigma_i(STu, STu'), \rho_i(Tu, Tu'))$$

which implies by (ii)

$$\sigma_i(STu, STu') \leq c_i \max\{d_i(u, u'), \rho_i(Tu, Tu')\}. \tag{2.26}$$

It now follows from inequalities (2.24),(2.25) and (2.26) that

$$d_i(u, u') \leq c_i \sigma_i(STu, STu') \leq c_i^2 \sigma_i(STu, STu'),$$

since $0 \leq c_i < 1$, we have $d(u, u') \leq 0 < r$. Hence $(u, u') \in V(l,r) \subseteq U_4$. so $u = u'$, since X is Hausdorff, we have $u = u'$. The fixed point u of RST is therefore unique. Similarly, it can proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of theorem. □

Corollary 2. *[8] Let (X,d), $(Y,\rho),(Z,\sigma)$be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$d(RSTx,RSy)\leq f(\rho(y,Tx),d(x,RSTx),d(x,RSy),\sigma(Sy,STx))$$
$$\rho(TRz,TRSy)\leq g(\sigma(z,Sy),\rho(y,TRSy),\rho(y,TRz),d(Rz,RSy))$$
$$\sigma(STRz,STx)\leq h(d(x,Rz),\sigma(z,STRz),\sigma(z,STx),\rho(Tx,TRz))$$

for all x in X, y in Y and z in Z, where $f,g,h\in\Im_4$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu=v$, $Sv=w$, $Rw=u$.

Proof. When we replace the uniform spaces $(X,\mathcal{U}_1)$, $(Y,\mathcal{U}_2),(Z,\mathcal{U}_3)$ in Theorem2 by (X,d), $(Y,\rho),(Z,\sigma)$ metric spaces, then proof can be obtained easily. □

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Existences of solutions to a certain class of functional optimization problems*

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Abstract. The paper studies the optimization problem of a certain class of nonlinear functionals. Based on different constrains and initial conditions, several models are considered. The existences of the optimal solutions to these models are obtained using the method of converting an optimization problem on the infinite dimensional space to one on a finite dimensional space.

Keywords. optimization, polynomial, compact set.

MR(2000) Subject Classification 90C30, 90C46

1 Problems and main results

The optimization problem of nonlinear functionals has been well studied because of its wide applications (See [1 - 6]). A class of functional optimization problems can be summarized from shape designs and property analysis of machine components, such as intake and exhaust cams of internal combustion engineers, microwave antennas, and ship bodies. Given a set of discrete data points (x_i, y_i), $i = 0, 1, 2, \ldots, n$. For convenience, assume $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Let $f(x)$ be an approximation function with some continuous derivative requirements imposed. According to the

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requirements of the real problems, seeking $f(x)$ results in minimizing the following functional

$$F(f(x)) = \int_0^1 [f^{(m)}(x)]^2 dx, \tag{1}$$

with

$$\sum_{i=0}^{n} \left[\frac{f(x_i) - y_i}{\delta_i} \right]^2 \leq S. \tag{2}$$

Some other constraints and initial conditions may apply too. Here the integer $m \geq 0$ determines the smoothness of the approximation function, the higher the value the better the smoothness, and the constants $\delta_i > 0$, $i = 0, 1, 2, \ldots, n$, and $S > 0$ control the approximation of $f(x_i)$ to y_i, $i = 0, 1, 2, \ldots, n$. In general, δ_i is given according to the individual accuracy of the approximation of $f(x_i)$ to y_i, the more accurate the approximation the smaller the value of δ_i, and S is given according to the overall accuracy of the approximation to all y_i's, $i = 0, 1, 2, \ldots, n$, the more accurate the approximation the smaller the value of S.

Different models can be obtained by adding some extra constraints and initial conditions. Some specific models have been well applied on profile modelling and property analysis of a certain class of machine components, and the design and property analysis of intake and exhaust cams of internal combustion engines. Others have been conveniently used in curve fitting with different smoothness. In paper [2], concrete forms and computation methods of the optimal solutions to these models are given, based on the assumption that the optimal solutions exist. But the existences of the optimal solutions have not been well studied. This problem is hard because the function spaces considered in the models are of infinitely dimensional.

A motivation of this paper is to study the existences of the optimal solutions of the models for the functional defined in (1). We study three models with different constraints and initial conditions, and give the proofs of the existences of the optimal solutions to these models by converting the infinite dimensional problems into ones with finite dimensions.

For every model studied in the paper, we assume that the set of functions restricted by the constraints and initial conditions is nonempty.

Let m be a positive integer and I be an interval, denote $C^{2m}I$ the set of functions having the $2m$th continuous derivative in I. Denote F the functional defined in (1).

Model I

Let $m \geq 3$. Consider the minimization problem of F over $C^{2m}[0,1]$ with the constraints (2) and

$$B \leq f''(x) \leq A, \quad x \in [0,1], \tag{3}$$

where $A > 0$, and $B < 0$. We have the following theorem.

Theorem 1. *Denote* $M_1 = \{f \in C^{2m}[0,1] \mid f$ *satisfies conditions (2) and (3)* $\}$.

1) If f *is a minimum point of* F *on* M_1*, then* f *is a polynomial with order no more than* $2m$.

2) F *has an optimal solution on* M_1 *if* M_1 *is not empty.*

The following model is more general than Model I, with a looser continuous derivative requirement on f.

Model II

This model is similar to Model I except that f belongs to $C^{2m-2}[0,1]$ with $m \geq 3$, and

$$f \in C^{2m}(x_{i-1}, x_i), \quad (x_{i-1}, x_i) \subset [0,1], \quad i = 1, 2, \ldots, n. \tag{4}$$

We state the result in the following theorem.

Theorem 2. *Denote* $M_2 = \{f \in C^{2m-2}[0,1] \mid f$ *satisfies the conditions (2), (3), and (4)*$\}$.

1) If f *is a minimum point of* F *on* M_2*, then* f *is a piece-wise function on* $[0,1]$*, with each piece over* (x_{i-1}, x_i)*,* $i = 1, 2, \ldots, n$*, being a polynomial with order no more than* $2m$.

2) F *has an optimal solution in* M_2 *if* M_2 *is not empty.*

In the above two models, initial conditions are not required. The model considered next takes some initial condition into consideration.

Model III

Let $m \geq 3$. Consider the minimization problem of F defined in (1). Suppose $f(x) \in C^{2m-2}[0,1]$, and $f(x)$ satisfies (4). For the constraints, suppose that $f(x)$ not only satisfies condition (2) but also that for the

data points (x_i, y_i) and the controlling constants δ_i used in condition (2),

$$f^{(2m-1)}(x_{i-1}+0) - f^{(2m-1)}(x_{i-1}-0) = (-1)^{m+1}\lambda\frac{f(x_{i-1}-y_{i-1})}{\delta_{i-1}}, \quad (5)$$

$i = 1, 2, \ldots, n$, where λ is some given real number. In addition, we consider the following initial condition

$$f^{(k)}(1) = f_{1,k}, \quad k = 0, 1, 2, \ldots, 2m-1, \quad (6)$$

where $f_{1,k} > 0$, $k = 0, 1, 2, \ldots, 2m-1$ are some given constants.

Theorem 3. *1) Suppose $f \in C^{2m-2}[0,1]$ satisfies the conditions (4) and (5), then there exists a piecewise function g such that g is a polynomial with order no more than $2m$ in every subinterval (x_{i-1}, x_i), $i = 1, 2, \ldots, n$, and*

$$F(g(x)) = F(f(x)),$$
$$g^{(k)}(1) = f^{(k)}(1), k = 0, 1, \ldots, 2m-1,$$
$$g^{(k)}(x_{i-1}) = f^{(k)}(x_{i-1}), i = 1, 2, \ldots, n, k = 0, 1, \ldots, 2m-2,$$
$$g^{(2m-1)}(x_i+0) - g^{(2m-1)}(x_i-0) = (-1)^{n+1}\lambda\frac{f(x_i)-y_i}{\delta_i},$$

$i = 1, 2, \ldots, n-1$, where λ is some given real number.

2) Denote $M_3 = \{f \in C^{(2m-2)}[0,1] \,|\, f$ satisfies (2), (4), (5), and (6) $\}$. The minimization problem of F has an optimal solution in M_3 if M_3 is not empty.

The proofs of Theorem 1 and Theorem 2 are similar, and we show them in Section 2. The proof of Theorem 3 is shown in Section 3.

2 Proof of Theorem 2

Let I be an open interval in $[0,1]$. Denote l_I the left end point of I, and r_I the right end point of I. Define $\Phi_0(I)$ as the set of functions ϕ in $C^\infty[0,1]$ satisfying the following conditions:

i) $\exists\alpha > 0$, such that $l_I + \alpha < r_I - \alpha$, and for every $x \in [0, l_I+\alpha] \cup [r_I-\alpha, 1]$, $\phi(x) = 0$, and $\phi(x)$ is not identical to 0 in $(l_I+\alpha, r_I-\alpha)$.

ii) $\int_0^1 \phi(x)dx = 0$.

We have the following lemmas.

Lemma 1. *Suppose $g \in C[0,1]$, and I is an open interval in $[0,1]$. Also suppose the functional*

$$G_g(\phi) = \int_0^1 g(x)\phi(x)dx$$

is non-negative (or non-positive) on $\Phi_0(I)$, then $g(x)$ is constant in I.

Proof. It is obvious that $-\phi \in \Phi_0(I)$ for every $\phi \in \Phi_0(I)$. Hence, the condition that $G_g(\phi)$ is non-negative (or non-positive) on $\Phi_0(I)$ implies that $G_g(\phi)$ is zero on $\Phi_0(I)$. This gives that for every $\phi \in \Phi_0(I)$, $\int_0^1 g(x)\phi(x)dx = 0$. We only need to prove Lemma 1 when $g(x)$ is not constantly zero. Suppose there exists $x_0 \in I$ such that $g(x_0) > 0$. (If no such x_0 exists, we can consider $-g(x)$.)

If $g(x)$ is not constant on I, then there exists $x_1, x_2 \in I$, such that $x_1 \neq x_2$, and $0 < g(x_1) < g(x_2)$. Let $\epsilon = \frac{1}{3}[g(x_2) - g(x_1)]$, then there exists $\delta > 0$ such that $[x_1 - \delta, x_1 + \delta] \cap [x_2 - \delta, x_2 + \delta] = \emptyset$, $[x_i - \delta, x_i + \delta] \subset I$, $i = 1, 2$, and for every $x \in [x_i - \delta, x_i + \delta]$, $g(x) > 0$ and $|g(x) - g(x_i)| \leq \epsilon$, $i = 1, 2$. For every $0 < \eta < \delta$, there exists a $\phi \in C^\infty[0,1]$, such that $0 \leq \phi(x) \leq 1$, $\{x \in [0,1] \,|\, \phi(x) \neq 0\} \subset (x_1 - \delta, x_1 + \delta)$, and for every $x \in [x_1 - \delta + \eta, x_1 + \delta - \eta]$, $\phi(x) = 1$.

Define ψ as

$$\psi(x) = \begin{cases} \phi(x), & x \in [x_1 - \delta, x_1 + \delta]; \\ -\phi(x - x_2 + x_1), & x \in [x_2 - \delta, x_2 + \delta]; \\ 0, & x \in [0,1] \setminus ([x_1 - \delta, x_1 + \delta] \cup [x_2 - \delta, x_2 + \delta]). \end{cases}$$

It is easy to verify that $\psi(x) \in \Phi_0(I)$ and

$$\begin{aligned} &\int_0^1 g(x)\psi(x)dx = \int_{x_1-\delta}^{x_1+\delta} g(x)\psi(x)dx + \int_{x_2-\delta}^{x_2+\delta} g(x)\psi(x)dx \\ \leq &2\delta(g(x_1) + \epsilon) - 2(\delta - \eta)(g(x_2) - \epsilon). \end{aligned}$$

Let $\eta \to 0$, get

$$\begin{aligned} &\int_0^1 g(x)\psi(x)dx \leq 2\delta(g(x_1) + \epsilon) - \delta(g(x_2) - \epsilon) \\ =&2\delta(g(x_1) - g(x_2) + 2\epsilon) \\ =&2\delta(-3\epsilon + 2\epsilon) = -2\delta\epsilon < 0, \end{aligned}$$

which contradicts the requirement for g in the lemma. Therefore, the assumption does not hold and $g(x)$ is constant in I. This completes the proof of the lemma. □

Lemma 2. *Suppose $g \in C[0,1]$, if the set $g([0,1])$ is at most countable, then g is constant on $[0,1]$.*

Proof. If not, let $\alpha = \min_{x\in[0,1]} g(x)$ and $\beta = \max_{x\in[0,1]} g(x)$, then $\alpha < \beta$. But g is continuous on $[0,1]$, we have $[\alpha,\beta] \subset g([0,1])$, contradicting the assumption that $g([0,1])$ is at most countable. □

Now we go to the proof of Theorem 1. We prove part 1) first.

Proof of part 1) of Theorem 1. Let f be a minimum point of F on M_1, and $Z(A) = \{x \in [0,1] \,|\, f''(x) = A\}$, $Z(B) = \{x \in [0,1] \,|\, f''(x) = B\}$. Let $Z = Z(A) \cup Z(B)$, and $Q = \{x_0, x_1, \ldots, x_n\}$.

If $f''(x) = 0$, $x \in [0,1]$, then f is a polynomial with order no more than 2. Suppose f'' is not identical to zero on $[0,1]$. Denote

$$G = [0,1] \setminus (Z \cup Q).$$

It is easy to see that G is a non-empty open set in the real line. Hence G is the union of at most countable mutually disjoint open intervals. Let $G = \bigcup_{\lambda\in\Lambda} I_\lambda$, where Λ is an at most countable set, and the set $\Gamma = \{I_\lambda | \lambda \in \Lambda\}$ is a set of mutually disjoint open intervals.

For every $I \in \Gamma$ and every $\phi \in \Phi_0(I)$, it is not hard to verify that when a positive number t is sufficiently small, $f + t\phi \in M_1$. Since f is a minimum point of F, we have

$$F(f + t\phi) \geq F(f),$$

which gives

$$t \int_0^1 [\phi^{(m)}(x)]^2 dx + 2 \int_0^1 f^{(m)}(x)\phi^{(m)}(x)dx \geq 0.$$

Let $t \to 0$, get

$$\int_0^1 f^{(m)}(x)\phi^{(m)}(x)dx \geq 0.$$

Integration by parts and the definition of $\Phi_0(I)$ give

$$(-1)^m \int_0^1 f^{(2m)}(x)\phi(x)dx \geq 0.$$

This means for every $\phi \in \Phi_0(I)$, $\int_0^1 f^{(2m)}(x)\phi(x)dx$ does not change its sign. By Lemma 1, $f^{(2m)}$ is constant in I. Thus the set $g^{(2m)}(G)$ is at most countable.

For $x \in [0,1] \setminus G$, consider the following two cases.

Case 1. x is not a cumulation point of Z.

There exist $I, J \in \Gamma$ such that $I \cup \{x\} \cup J$ is a open interval. By the continuity of $f^{(2m)}$, $f^{(2m)}(x) \in f^{(2m)}(G)$.

Case 2. x is a cumulation point of Z.

In every neighborhood U of x, there are infinitely many points in Z. Without loss of generality, suppose U contains infinitely many points in $Z(A)$, then there exists a non-repeated sequence $u_1, u_2, \ldots$, such that $\forall j = 1, 2, \ldots$, $f^{(2m)}(u_j) = f^{(2m)}(u_1)$ and $\lim_{j\to\infty} u_j = x$. Repeatedly apply Roll's Theorem, get a non-repeated sequence $v_1, v_2, \ldots$ in $[0,1]$ converging to x, such that $f^{(2m)}(v_j) = 0$, $j = 1, 2, \ldots$. Therefore, $f^{(2m)}(x) = 0$.

From the above two cases, $f^{(2m)}([0,1] \setminus G) \subset f^{(2m)}(G) \cup \{0\}$, so $f^{(2m)}([0,1])$ is at most countable. By Lemma 2, $f^{(2m)}$ is constant, and hence f is a polynomial with order no more than $2m$.

Proof of part 2) of Theorem 1. For a non-negative integer k, denote $P_k(x)$ the set of polynomials with order no more than k. Consider the mapping

$$\begin{aligned} T : & P_{2m}(x) \to \mathbb{R}^{2m+1} \\ & \sum_{j=0}^{2m} a_j x^j \to (a_0, a_1, \ldots, a_{2m}). \end{aligned}$$

By part 1) of Theorem 1, the minimization problem of F on M_1 is equivalent to the minimization problem of the function

$$g(t_0, t_1, \ldots, t_{2m}) = \int_0^1 (p_m^m t_m + p_{m+1}^m t_{m+1} x + \cdots + p_{2m}^{2m} t_{2m} x^m)^2 dx,$$

where $p_j^m = \frac{j!}{(j-m)!}$, $j = m, m+1, \ldots, 2m$, under the constraints

$$\sum_{i=0}^{n} \left(\frac{t_0 + t_1 x_i + t_2 x_i^2 + \cdots + t_{2m} x_i^{2m} - y_i}{\delta_i} \right)^2 \le S, \quad \text{and}$$

$$B \le (t_0 + t_1 x + t_2 x^2 + \cdots + t_{2m} x^{2m})'' \le A, x \in [0,1].$$

By computations, get

$$g(t_0, t_1, \ldots, t_{2m}) = \sum_{k,j=m}^{2m} \frac{p_k^m p_j^m}{k+j+1-2m} t_k t_j, \quad \text{and}$$

and the constraints become

$$\sum_{i=0}^{n} \left(\frac{t_0 + t_1 x_i + t_2 x_i^2 + \cdots + t_{2m} x_i^{2m} - y_i}{\delta_i} \right)^2 \leq S, \tag{7}$$

$$B \leq 2t_2 + 3t_3 x + \cdots + 2m(2m-1)t_{2m} x^{2m-2} \leq A, x \in [0,1]. \tag{8}$$

Let $M_1' = \{(t_0, t_1, \ldots, t_{2m}) \in \mathbb{R}^{2m+1} | (t_0, t_1, \ldots, t_{2m})$ satisfies (7) and (8)$\}$.

It is easy to see that M_1' is closed. To see that M_1' is bounded, fix a sequence of mutually unequal numbers, $\bar{x}_2, \bar{x}_3, \ldots, \bar{x}_{2m}$, in $(0,1)$, and make the following linear transformation in $\mathbb{R}^{2m+1}$,

$$\begin{cases} z_0 = t_0 \\ z_1 = t_0 + t_1 + t_2 + \cdots + t_{2m} \\ z_2 = 2t_2 + 3 \cdot 2t_3 \bar{x}_2 + \cdots + 2m(2m-1)t_{2m} \bar{x}_2^{2m-2} \\ z_3 = 2t_2 + 3 \cdot 2t_3 \bar{x}_3 + \cdots + 2m(2m-1)t_{2m} \bar{x}_3^{2m-2} \\ \cdots \\ z_{2m} = 2t_2 + 3 \cdot 2t_3 \bar{x}_{2m} + \cdots + 2m(2m-1)t_{2m} \bar{x}_{2m}^{2m-2}. \end{cases} \tag{9}$$

We get

$$\begin{cases} \left(\frac{z_0 - y_0}{\delta_0}\right)^2 + \left(\frac{z_1 - y_n}{\delta_n}\right)^2 \leq S \\ B \leq z_2 \leq A \\ \cdots \\ B \leq z_{2m} \leq A \end{cases} \tag{10}$$

where the first inequality is obtained from (7) and the first two equations of (9), others are from (8) and the corresponding equations of (9).

The set of points satisfying the requirements in (10) is bounded in $\mathbb{R}^{2m+1}$. Its inverse image contains the set M_1', so M_1' is also bounded. Therefore, M_1' is a non-empty compact set. Since the continuous function $g(t_0, t_1, \ldots, t_{2m})$ can reach the smallest value on M_1', F also has an optimal solution on M_1.

The proof of Theorem 1 is complete.

The proof the Theorem 2 is similar.

Proof of Theorem 2. Let f be a minimum point of F on M_2. If $f''(x) = 0$, $x \in [0,1]$, then f is a polynomial with order no more than 2. Suppose f'' is not identical to zero on $[0,1]$. As in the proof of Theorem 1, denote

$$G = [0,1] \setminus (Z \cup Q) = \bigcup_{\lambda \in \Lambda} I_\lambda.$$

For every $I \in \Gamma \cap [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, and every $\phi \in \Phi_0(I)$, it is easy to verify that when a positive number t is small enough, $f + t\phi \in M_2$. Hence

$$F(f + t\phi) \geq F(f),$$

which gives

$$t \int_0^1 [\phi^{(m)}(x)]^2 dx + 2 \int_0^1 f^{(m)}(x)\phi^{(m)}(x)dx \geq 0.$$

Let $t \to 0$, get

$$\int_0^1 f^{(m)}(x)\phi^{(m)}(x)dx \geq 0.$$

This is in fact

$$\int_{x_{i-1}}^{x_i} f^{(m)}(x)\phi^{(m)}(x)dx \geq 0, \quad i = 1, 2, \ldots, n.$$

Integration by parts and the definition of $\Phi_0(I)$ give

$$(-1)^m \int_{x_{i-1}}^{x_i} f^{(2m)}(x)\phi(x)dx \geq 0.$$

We can use a similar method to what is used in the proof of part 1) of Theorem 1 to get that f is a polynomial with order no more than $2m$ in the interval (x_{i-1}, x_i), and hence f is a piecewise polynomial in $[0,1]$. This completes the proof of part 1) of Theorem 2. The proof of part 2) of Theorem 2 is quite similar to that of part 2) of Theorem 1, so we omit the details here. The proof of Theorem 2 is complete.

3 Proof of Theorem 3

Proof. We prove part 1) first. Let $f(x) = f_i(x)$, $x \in (x_i, x_{i+1})$, $i = 0, 1, \ldots, n-1$, and

$$f_i(x) = t_0^i + t_1^i(x - x_i) + t_2^i(x - x_i)^2 + \cdots + t_{2m}^i(x - x_i)^{2m},$$

where t_k^i, $k = 0, 1, 2, \ldots, 2m$, $i = 0, 1, \ldots, n-1$, are coefficients.

For f to satisfy the conditions in the lemma, we need, for every i, $i = 0, 1, \ldots, n-1$,

$$\sum_{i=0}^{n-1} \int_{x_i}^{x_i+1} [f^{(m)}(x)]^2 = F(f), \tag{11}$$

$$f_{n-1}^{(k)}(1) = f^{(k)}(1), k = 0, 1, \ldots, 2m-1, \tag{12}$$

$$f_{i+1}^{(k)}(x_{i+1}) = f^{(k)}(x_{i+1}), i = 0, 1, \ldots, 2m-2, k = 0, 1, \ldots, 2m-1, \tag{13}$$

and

$$f^{(2m-1)}(x_i + 0) - f^{(2m-1)}(x_i - 0) = (-1)^{m+1}\lambda \frac{f(x_i) - y_i}{\delta_i^2}. \tag{14}$$

From (12), we obtain

$$\begin{cases} t_0^{n-1} + t_1^{n-1}(1 - x_{n-1}) + \cdots + t_{2m}^{n-1}(1 - x_{n-1})^{2m} = f(1), \\ t_1^{n-1} + 2t_2^{n-1}(1 - x_{n-1}) + \cdots + 2mt_{2m}^{n-1}(1 - x_{n-1})^{2m-1} = f'(1), \\ 2t_2^{n-1} + \cdots + 2m(2m-1)t_{2m}^{n-1}(1 - x_{n-1})^{2m-2} = f''(1), \\ \ldots \\ (2m-1)!t_{2m-1}^{n-1} + (2m)!(1 - x_{n-1})t_{2m}^{n-1} = f^{(2m-1)}(1). \end{cases}$$

Notice that $t_1^{n-1}, t_2^{n-2}, \ldots, t_{2m}^{n-1}$ can be expressed in terms of t_0^{n-1}.

From (13) and (14), we have

$$\begin{cases} t_0^i + t_1^i(x_{i+1} - x_i) + \cdots + t_{2m}^i(x_{i+1} - x_i)^{2m} = t_0^{i+1}, \\ t_1^i + 2t_2^i(x_{i+1} - x_i) + \cdots + 2mt_{2m}^i(x_{i+1} - x_i)^{2m-1} = t_1^{i+1}, \\ 2t_2^i + \cdots + 2m(2m-1)t_{2m}^i(x_{i+1} - x_i)^{2m-2} = 2t_2^{i+1}, \\ \ldots \\ (2m-2)!t_{2m-2}^i + (2m-1)!t_{2m-1}^i(x_{i+1} - x_i) \\ \qquad +(2m)!/2t_{2m}^i(x_{i+1} - x_i)^2 = (2m-2)t_{2m-2}^{i+1}, \\ (2m-1)!t_{2m-2}^i + (2m)!t_{2m}^i(x_{i+1} - x_i) - (2m-1)!t_{2m-1}^{i+1} \\ \qquad = (-1)^{m+1}\frac{t_0^{i+1} - y_{i+1}}{\delta_i^2}. \end{cases}$$

Also notice that $t_1^i, t_2^i, \ldots, t_{2m}^i$ can be expressed in terms of t_0^i and t_0^{i+1}. Thus, all the $t_1^i, t_2^i, \ldots, t_{2m}^i$, $i = 0, 1, \ldots, n-1$ can be expressed in terms of t_0^i, $i = 0, 1, \ldots, n-1$. This also means there are $(2m+1)\times n$ variables and $2mn$ equations, and n free variables $t_0^0, t_0^1, \ldots, t_0^{n-1}$ in (12), (13), and (14). Since there exist $t_0^0, t_0^1, \ldots, t_0^{n-1}$ satisfying (11), there exists a piecewise function g satisfying the requirements in part 1) of the theorem.

Then we prove part 2) of Theorem 3. Denote $Q_{2m}(x)$ the set of piecewise polynomials with order no more than $2m$. Each $f \in Q_{2m}(x)$ has the form

$$f(x) = t_0^i + t_1^i(x - x_i) + \cdots + t_{2m}^i(x - x_i)^{2m},$$

$x \in (x_i, x_{i+1})$, $i = 0, 1, \ldots, n-1$.

Consider the following mapping

$$\begin{aligned} T: \quad & Q_{2m}(x) \to \mathbb{R}^{(2m+1)\times n} \\ & f(x) \to (t_0^0, t_1^0, \ldots, t_{2m}^0, t_0^1, t_1^1, \ldots, t_{2m}^1, \ldots, t_0^{n-1}, t_1^{n-1}, \ldots, t_{2m}^{n-1}). \end{aligned}$$

By part 1) of Theorem 3, the minimization problem of F on M_3 is equivalent to that of

$$\begin{aligned} &\sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (p_m^m t_m^i + p_{m+1}^m t_{m+1}^i x + \cdots + p_{2m}^m t_{2m}^i x^m)^2 dx \\ =&G(t_0^0, t_1^0, \ldots, t_{2m}^0, t_0^1, t_1^1, \ldots, t_{2m}^1, \ldots, t_0^{n-1}, t_1^{n-1}, \ldots, t_{2m}^{n-1}) \end{aligned}$$

in $\mathbb{R}^{(2m+1)\times n}$, where $p_j^m = \frac{j!}{(j-m)!}$, $j = m, m+1, \ldots, 2m$. The constraints become

$$\sum_{i=0}^{n-1} \left(\frac{t_0^i - y_i}{\delta_i} \right)^2 + (f_{1,0} - y_i)^2 \le S, \tag{15}$$

$$\begin{cases} t_0^{n-1} + t_1^{n-1}(1 - x_{n-1}) + \ldots t_{2m}^{n-1}(1 - x_{n-1})^{2m} = f_{1,0}, \\ t_1^{n-1} + 2t_2^{n-1}(1 - x_{n-1} + \ldots 2m t_2^{n-1}(1 - x_{n-1})^{2m-1} = f_{1,1}, \\ \ldots \\ (2m-1)! t_{2m-1}^{n-1} + (2m)! t_{2m}^{n-1}(1 - x_{n-1}) = f_{1,2m-1}, \end{cases} \tag{16}$$

and

$$\begin{cases} t_0^I + t_1^i(x_{i+1}-x_i) + \cdots + t_{2m}^i(x_{i+1}-x_i)^{2m} = t_0^{i+1}, \\ t_1^i + t_2^i(x_{i+1}-x_i) + \cdots + 2mt_{2m}^i(x_{i+1}-x_i)^{2m-1} = t_1^{i+1}, \\ \cdots \\ (2m-2)!t_{2m-2}^i + (2m-1)!t_{2m-1}^i(x_{i+1}-x_i) \\ \qquad +(2m)!/2t_{2m}^i(x_{i+1}-x_i)^2 = (2m-2)!t_{2m-2}^{i+1}, \\ (2m-1)!t_{2m-2}^i + (2m)!(x_{i+1}-x_i) \\ \qquad = (2m-1)!t_{2m-1}^i + (-1)^m\lambda\frac{t_0^{i+1}-y_i}{\delta_i}, \end{cases} \tag{17}$$

$i = 0, 1, \ldots, n-1$. It is easy to check that there are $2mn$ equations and $(2m+1) \times n$ unknowns in (16) and (17), and hence there are n free variables $t_0^0, t_0^1, \ldots, t_0^{n-1}$. By this the minimization problem of F on M_3 is converted to a minimization problem in $\mathbb{R}^n$. The set of points $(t_0^0, g_0^1, \ldots, t_0^{n-1})$ satisfying (15) is bounded and closed, so it is compact. Therefore F has an optimal solution on M_3. The proof is done. □

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Generalized monosplines and inequalities for the remainder term of quadrature formulas

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In this paper we studied some quadrature formulas witch are obtained using connection between the monosplines and the quadrature formulas. For the remainder term we give some inequalities.

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1 Introduction

In recent years a number of authors have considered generalization of some known and some new quadrature rules ([1], [2], [3], [4], [5], [6], [7], [8], [9]). For example, P. Cerone and S.S. Dragomir in [4] give a generalization of the midpoint quadrature rule:

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1}[1+(-1)^k]\frac{(b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)+(-1)^n\int_a^b K_n(t)f^{(n)}(t)dt \tag{1}$$

where

$$K_n(t)=\begin{cases}\dfrac{(t-a)^n}{n!}\ , & t\in\left[a,\dfrac{a+b}{2}\right]\\ \dfrac{(t-b)^n}{n!}\ , & t\in\left(\dfrac{a+b}{2},b\right]\end{cases}$$

and $f\in C^{n-1}[a,b]$ and $f^{(n-1)}$ is absolutely continuous.

We observe that for $n=1$ we get the mid-point rule

$$\int_a^b f(t)dt=(b-a)f\left(\frac{a+b}{2}\right)-\int_a^b K_1(t)f'(t)dt\,.$$

In this paper, we will study the case of the quadrature formulae with the weight function $w(t)=(b-t)(t-a)$.

We denote

$$W_p^n[a,b]:=\left\{f\in C^{n-1}[a,b]\,,\ f^{(n-1)}\text{ absolutely continuous }\,,\ \left\|f^{(n)}\right\|_p<\infty\right\}$$

with

$$\|f\|_p:=\left\{\int_a^b|f(x)|^p\,dx\right\}^{\frac{1}{p}}\ \text{for}\ 1\le p<\infty\,,$$
$$\|f\|_\infty:=\operatorname{supvrai}_{x\in[a,b]}|f(x)|\ .$$

Definition 1 *The function $s(x)$ is called a spline function of degree n with knots $\{x_i\}_{i=1}^{m-1}$ if $a:=x_0<x_1<\cdots<x_{m-1}<x_m:=b$ and*

i) for each $i=0,\dots,m-1$, $s(x)$ coincides on (x_i,x_{i+1}) with a polynomial of degree not greater then n;

ii) $s(x),s'(x),\dots,s^{(n-1)}(x)$ are continuous functions on $[a,b]$.

Definition 2 *Functions of the form*

$$M_n(t)=v(t)+s_{n-1}(t)\,,$$

where $s_{n-1}(t)$ is a spline of degree $n-1$ and v is the n^{th} integral of weight function $w:[a,b]\to\mathbb{R}$, are called generalized monosplines.

2 Main results

Let $m\in\mathbb{N}\,,\ m\ge 2$ and $(\Delta_m)_{m\in\mathbb{N}}$ be a division of $[a,b]$,

$$\Delta_m:a=x_0<x_1<x_2<\cdots<x_{m-1}<x_m=b$$

and $(\xi_i)_{i=\overline{1,m}}$ a system of intermediate points,

$$\xi_1 < \xi_2 < \cdots < \xi_m\,,\ \xi_i \in [x_{i-1}, x_i] \ \text{ for } \ i = \overline{1,m}\,.$$

Let

$$\begin{aligned} M_n(t) &= \frac{(t-a)^{n+1}}{(n+1)!}(b-a) - \frac{2}{(n+2)!}(t-a)^{n+2} + \sum_{k=0}^{n-1} A_{k,0}\frac{(t-a)^{n-k-1}}{(n-k-1)!} \\ &+ \sum_{i=1}^{m-1}\sum_{k=0}^{n-1} A_{k,i}\frac{(t-x_i)_+^{n-k-1}}{(n-k-1)!}\,, \end{aligned} \tag{2}$$

where

$$\begin{aligned} A_{k,0} &= (b-\xi_1)(\xi_1-a)\frac{(a-\xi_1)^{k+1}}{(k+1)!} + (b-2\xi_1+a)\frac{(a-\xi_1)^{k+2}}{(k+2)!} - 2\frac{(a-\xi_1)^{k+3}}{(k+3)!}\,, \\ A_{k,i} &= \frac{(b-\xi_{i+1})(\xi_{i+1}-a)(x_i-\xi_{i+1})^{k+1} - (b-\xi_i)(\xi_i-a)(x_i-\xi_i)^{k+1}}{(k+1)!} \\ &+ \frac{(b-2\xi_{i+1}+a)(x_i-\xi_{i+1})^{k+2} - (b-2\xi_i+a)(x_i-\xi_i)^{k+2}}{(k+2)!} \\ &- 2\frac{(x_i-\xi_{i+1})^{k+3} - (x_i-\xi_i)^{k+3}}{(k+3)!}\,, \\ &k = \overline{0,n-1}\,,\ \ i = \overline{1,m-1}\,, \end{aligned}$$

be the generalized monospline of degree n.

Lemma 1 *The generalized monospline, defined in (2),has the representation*

$$M_n(t) = \begin{cases} (b-\xi_i)(\xi_i-a)\dfrac{(t-\xi_i)^n}{n!} + (b-2\xi_i+a)\dfrac{(t-\xi_i)^{n+1}}{(n+1)!} - 2\dfrac{(t-\xi_i)^{n+2}}{(n+2)!}, \\ \qquad\qquad t \in [x_{i-1},\ x_i)\ ,\ i = \overline{1,m-1} \\ (b-\xi_m)(\xi_m-a)\dfrac{(t-\xi_m)^n}{n!} + (b-2\xi_m+a)\dfrac{(t-\xi_m)^{n+1}}{(n+1)!} - 2\dfrac{(t-\xi_m)^{n+2}}{(n+2)!}, \\ \qquad\qquad t \in [x_{m-1}, x_m] \end{cases} \tag{3}$$

Proof. If $t \in [x_0,\ x_1)$, then

$$\begin{aligned}
M(t) &= \frac{(t-a)^{n+1}}{(n+1)!}(b-a) - \frac{2}{(n+2)!}(t-a)^{n+2} \\
&+ (b-\xi_1)(\xi_1-a)\sum_{k=0}^{n-1}\frac{(a-\xi_1)^{k+1}}{(k+1)!}\frac{(t-a)^{n-k-1}}{(n-k-1)!} \\
&+ (b-2\xi_1+a)\sum_{k=0}^{n-1}\frac{(a-\xi_1)^{k+2}}{(k+2)!}\frac{(t-a)^{n-k-1}}{(n-k-1)!} \\
&- 2\sum_{k=0}^{n-1}\frac{(a-\xi_1)^{k+3}}{(k+3)!}\frac{(t-a)^{n-k-1}}{(n-k-1)!} \\
&= \frac{(t-a)^{n+1}}{(n+1)!}(b-a) - \frac{2}{(n+2)!}(t-a)^{n+2} \\
&+ \frac{(b-\xi_1)(\xi_1-a)}{n!}\cdot\left[(t-\xi_1)^n-(t-a)^n\right] \\
&+ \frac{b-2\xi_1+a}{(n+1)!}\cdot\left[(t-\xi_1)^{n+1}-(t-a)^{n+1}-(n+1)(a-\xi_1)(t-a)^n\right] \\
&- \frac{2}{(n+2)!}\left[(t-\xi_1)^{n+2}-(t-a)^{n+2}-(n+2)(a-\xi_1)(t-a)^{n+1}\right. \\
&- \left.\frac{(n+1)(n+2)}{2}\cdot(a-\xi_1)^2(t-a)^n\right] \\
&= (b-\xi_1)(\xi_1-a)\frac{(t-\xi_1)^n}{n!} + (b-2\xi_1+a)\frac{(t-\xi_1)^{n+1}}{(n+1)!} - 2\frac{(t-\xi_1)^{n+2}}{(n+2)!}\,.
\end{aligned}$$

If $t \in [x_{i-1}\ , x_i)\ ,\ i=\overline{2,m-1}$, then

$$\begin{aligned}
M_n(t) &= \frac{(t-a)^{n+1}}{(n+1)!}(b-a) - \frac{2}{(n+2)!}(t-a)^{n+2} \\
&+ \sum_{k=0}^{n-1}A_{k,0}\frac{(t-a)^{n-k-1}}{(n-k-1)!} + \sum_{j=1}^{i-1}\sum_{k=0}^{n-1}A_{k,j}\frac{(t-x_j)_+^{n-k-1}}{(n-k-1)!} \\
&= (b-\xi_1)(\xi_1-a)\frac{(t-\xi_1)^n}{n!} + (b-2\xi_1+a)\frac{(t-\xi_1)^{n+1}}{(n+1)!} - 2\frac{(t-\xi_1)^{n+2}}{(n+2)!} \\
&+ \sum_{j=1}^{i-1}\sum_{k=0}^{n-1}A_{k,j}\frac{(t-x_j)^{n-k-1}}{(n-k-1)!}\,.
\end{aligned}$$

If denote $S = \sum_{j=1}^{i-1}\sum_{k=0}^{n-1}A_{k,j}\frac{(t-x_j)^{n-k-1}}{(n-k-1)!}$, then we have

$$
\begin{aligned}
S &= \sum_{j=1}^{i-1}\Bigg\{(b-\xi_{j+1})(\xi_{j+1}-a)\sum_{k=0}^{n-1}\frac{(x_j-\xi_{j+1})^{k+1}}{(k+1)!}\cdot\frac{(t-x_j)^{n-k-1}}{(n-k-1)!} \\
&- (b-\xi_j)(\xi_j-a)\cdot\sum_{k=0}^{n-1}\frac{(x_j-\xi_j)^{k+1}}{(k+1)!}\cdot\frac{(t-x_j)^{n-k-1}}{(n-k-1)!} \\
&+ (b-2\xi_{j+1}+a)\sum_{k=0}^{n-1}\frac{(x_j-\xi_{j+1})^{k+2}}{(k+2)!}\cdot\frac{(t-x_j)^{n-k-1}}{(n-k-1)!} \\
&- (b-2\xi_j+a)\sum_{k=0}^{n-1}\frac{(x_j-\xi_j)^{k+2}}{(k+2)!}\cdot\frac{(t-x_j)^{n-k-1}}{(n-k-1)!} \\
&- 2\sum_{k=0}^{n-1}\frac{(x_j-\xi_{j+1})^{k+3}}{(k+3)!}\cdot\frac{(t-x_j)^{n-k-1}}{(n-k-1)!} \\
&+ 2\sum_{k=0}^{n-1}\frac{(x_j-\xi_j)^{k+3}}{(k+3)!}\cdot\frac{(t-x_j)^{n-k-1}}{(n-k-1)!}\Bigg\} \\
&= \sum_{j=1}^{i-1}\Bigg\{\frac{(b-\xi_{j+1})(\xi_{j+1}-a)}{n!}\left[(t-\xi_{j+1})^n-(t-x_j)^n\right] \\
&- \frac{(b-\xi_j)(\xi_j-a)}{n!}\left[(t-\xi_j)^n-(t-x_j)^n\right] \\
&+ \frac{(b-2\xi_{j+1}+a)}{(n+1)!}\left[(t-\xi_{j+1})^{n+1}-(t-x_j)^{n+1}-(n+1)\cdot(x_j-\xi_{j+1})(t-x_j)^n\right] \\
&- \frac{(b-2\xi_j+a)}{(n+1)!}\left[(t-\xi_j)^{n+1}-(t-x_j)^{n+1}-(n+1)\cdot(x_j-\xi_j)(t-x_j)^n\right] \\
&- \frac{2}{(n+2)!}\Big[(t-\xi_{j+1})^{n+2}-(t-x_j)^{n+2}-(n+2)(x_j-\xi_{j+1})(t-x_j)^{n+1} \\
&- \frac{(n+1)(n+2)}{2}(x_j-\xi_{j+1})^2(t-x_j)^n\Big]+\frac{2}{(n+2)!}\left[(t-\xi_j)^{n+2}-(t-x_j)^{n+2}\right. \\
&- \left.(n+2)(x_j-\xi_j)(t-x_j)^{n+1}-\frac{(n+1)(n+2)}{2}(x_j-\xi_j)^2(t-x_j)^n\Big] \\
&= S_1+S_2+S_3\,,
\end{aligned}
$$

where

$$
\begin{aligned}
S_1 &= \frac{1}{n!}\sum_{j=1}^{i-1}\left[(b-\xi_{j+1})(\xi_{j+1}-a)(t-\xi_{j+1})^n-(b-\xi_j)(\xi_j-a)(t-\xi_j)^n\right] \\
&+ \frac{1}{(n+1)!}\sum_{j=1}^{i-1}\left[(b-2\xi_{j+1}+a)(t-\xi_{j+1})^{n+1}-(b-2\xi_j+a)(t-\xi_j)^{n+1}\right] \\
&- \frac{2}{(n+2)!}\sum_{j=1}^{i-1}\left[(t-\xi_{j+1})^{n+2}-(t-\xi_j)^{n+2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n!}(b-\xi_i)(\xi_i-a)(t-\xi_i)^n + \frac{1}{(n+1)!}(b-2\xi_i+a)(t-\xi_i)^{n+1} \\
&- \frac{2}{(n+2)!}(t-\xi_i)^{n+2} - \left[\frac{1}{n!}(b-\xi_1)(\xi_1-a)(t-\xi_1)^n \right. \\
&+ \left. \frac{1}{(n+1)!}(b-2\xi_1+a)(t-\xi_1)^{n+1} - \frac{2}{(n+2)!}(t-\xi_1)^{n+2}\right].
\end{aligned}
$$

$$
\begin{aligned}
S_2 &= \sum_{j=1}^{i-1}\left\{-\frac{1}{n!}(b-\xi_{j+1})(\xi_{j+1}-a)(t-x_j)^n + \frac{b-2\xi_{j+1}+a}{(n+1)!}\left[-(t-x_j)^{n+1}\right.\right. \\
&- (n+1)(x_j-\xi_{j+1})(t-x_j)^n] + \frac{2}{(n+2)!}\left[(t-x_j)^{n+2}\right. \\
&+ \left.\left.(n+2)(x_j-\xi_{j+1})(t-x_j)^{n+1} + \frac{(n+1)(n+2)}{2}(x_j-\xi_{j+1})^2(t-x_j)^n\right]\right\} \\
&= -\sum_{j=1}^{i-1}\left[\frac{1}{n!}(b-x_j)(x_j-a)(t-x_j)^n \right. \\
&+ \left. \frac{1}{(n+1)!}(a+b-2x_j)(t-x_j)^{n+1} - \frac{2}{(n+2)!}(t-x_j)^{n+2}\right].
\end{aligned}
$$

$$
\begin{aligned}
S_3 &= \sum_{j=1}^{i-1}\left\{\frac{1}{n!}(b-\xi_j)(\xi_j-a)(t-x_j)^n + \frac{b-2\xi_j+a}{(n+1)!}\left[(t-x_j)^{n+1}\right.\right. \\
&+ (n+1)(x_j-\xi_j)(t-x_j)^n] - \frac{2}{(n+2)!}\left[(t-x_j)^{n+2}\right. \\
&+ \left.\left.(n+2)(x_j-\xi_j)(t-x_j)^{n+1} + \frac{(n+1)(n+2)}{2}(x_j-\xi_j)^2(t-x_j)^n\right]\right\} \\
&= \sum_{j=1}^{i-1}\left[\frac{1}{n!}(b-x_j)(x_j-a)(t-x_j)^n + \frac{1}{(n+1)!}(a+b-2x_j)(t-x_j)^{n+1} \right. \\
&- \left. \frac{2}{(n+2)!}(t-x_j)^{n+2}\right].
\end{aligned}
$$

Therefore

$$
M_n(t) = (b-\xi_i)(\xi_i-a)\frac{(t-\xi_i)^n}{n!} + (b-2\xi_i+a)\frac{(t-\xi_i)^{n+1}}{(n+1)!} - 2\frac{(t-\xi_i)^{n+2}}{(n+2)!}.
$$

If $t \in [x_{m-1}, x_m]$, then

$$\begin{aligned} M_n(t) &= \frac{(t-a)^{n+1}}{(n+1)!}(b-a) - \frac{2}{(n+2)!}(t-a)^{n+2} \\ &+ \sum_{k=0}^{n-1} A_{k,0}\frac{(t-a)^{n-k-1}}{(n-k-1)!} + \sum_{j=1}^{m-1}\sum_{k=0}^{n-1} A_{k,j}\frac{(t-x_j)^{n-k-1}}{(n-k-1)!} \\ &= (b-\xi_m)(\xi_m-a)\frac{(t-\xi_m)^n}{n!} + (b-2\xi_m+a)\frac{(t-\xi_m)^{n+1}}{(n+1)!} - 2\frac{(t-\xi_m)^{n+2}}{(n+2)!}. \end{aligned}$$

Lemma 2 *If $f \in W_1^n[a,b]$, then*

$$\begin{aligned} \int_a^b w(t)f(t)dt &= \sum_{k=0}^{n-1}(-1)^{k+1}A_{k,0}f^{(k)}(a) + \sum_{i=1}^{m-1}\sum_{k=0}^{n-1}(-1)^{k+1}A_{k,i}f^{(k)}(x_i) \\ &+ \sum_{k=0}^{n-1}(-1)^k A_{k,m}f^{(k)}(b) + \mathcal{R}[f]\,, \end{aligned} \tag{4}$$

where

$$\begin{aligned} w(t) &= (b-t)(t-a)\,, \\ A_{k,0} &= (b-\xi_1)(\xi_1-a)\frac{(a-\xi_1)^{k+1}}{(k+1)!} + (b-2\xi_1+a)\frac{(a-\xi_1)^{k+2}}{(k+2)!} - 2\frac{(a-\xi_1)^{k+3}}{(k+3)!}\,, \\ A_{k,i} &= \frac{(b-\xi_{i+1})(\xi_{i+1}-a)(x_i-\xi_{i+1})^{k+1} - (b-\xi_i)(\xi_i-a)(x_i-\xi_i)^{k+1}}{(k+1)!} \\ &+ \frac{(b-2\xi_{i+1}+a)(x_i-\xi_{i+1})^{k+2} - (b-2\xi_i+a)(x_i-\xi_i)^{k+2}}{(k+2)!} \\ &- 2\frac{(x_i-\xi_{i+1})^{k+3} - (x_i-\xi_i)^{k+3}}{(k+3)!} \\ A_{k,m} &= \left[(b-\xi_m)(\xi_m-a)\frac{(b-\xi_m)^{k+1}}{(k+1)!} + (b-2\xi_m+a)\frac{(b-\xi_m)^{k+2}}{(k+2)!} - 2\frac{(b-\xi_m)^{k+3}}{(k+3)!}\right], \\ & k = \overline{0, n-1}\ ,\ \ i = \overline{1, m-1}\,, \end{aligned}$$

$$\mathcal{R}[f] = (-1)^n \int_a^b M_n(t)f^{(n)}(t)dt \tag{5}$$

and

$$M_n(t) = \begin{cases} (b-\xi_i)(\xi_i-a)\dfrac{(t-\xi_i)^n}{n!} + (b-2\xi_i+a)\dfrac{(t-\xi_i)^{n+1}}{(n+1)!} - 2\dfrac{(t-\xi_i)^{n+2}}{(n+2)!}, \\ \qquad\qquad t \in [x_{i-1},\ x_i)\ ,\ i = \overline{1, m-1} \\ (b-\xi_m)(\xi_m-a)\dfrac{(t-\xi_m)^n}{n!} + (b-2\xi_m+a)\dfrac{(t-\xi_m)^{n+1}}{(n+1)!} - 2\dfrac{(t-\xi_m)^{n+2}}{(n+2)!}, \\ \qquad\qquad t \in [x_{m-1}, x_m] \end{cases} \tag{6}$$

Proof. Let

$$\begin{aligned} M_n(t) &= \frac{(t-a)^{n+1}}{(n+1)!}(b-a) - \frac{2}{(n+2)!}(t-a)^{n+2} + \sum_{k=0}^{n-1} A_{k,0}\frac{(t-a)^{n-k-1}}{(n-k-1)!} \\ &+ \sum_{i=1}^{m-1}\sum_{k=0}^{n-1} A_{k,i}\frac{(t-x_i)_+^{n-k-1}}{(n-k-1)!}, \end{aligned} \tag{7}$$

be the generalized monospline of degree n and let

$$\begin{aligned} \int_a^b w(t)f(t)dt &= \sum_{k=0}^{n-1}(-1)^{k+1}A_{k,0}f^{(k)}(a) + \sum_{i=1}^{m-1}\sum_{k=0}^{n-1}(-1)^{k+1}A_{k,i}f^{(k)}(x_i) \\ &+ \sum_{k=0}^{n-1}(-1)^k A_{k,m}f^{(k)}(b) + \mathcal{R}[f]\,, \end{aligned} \tag{8}$$

be the quadrature formulae, where $w(t) = (b-t)(t-a)$.

Between the generalized monospline (7) and the quadrature formulae (8) there is a connection, namely, the coefficients $\{A_{k,i}\}_{k=0\ \ i=0}^{n-1\ \ m-1}$ of the quadrature formulae are the same with the coefficients of monospline (7), $A_{k,m} = M_n^{(n-k-1)}(b)$, $k = \overline{0, n-1}$ and the remainder term of quadrature formulae have the representation

$$\mathcal{R}[f] = (-1)^n \int_a^b M_n(t)f^{(n)}(t)dt\,, \quad f \in W_1^n[a,b]\,.$$

If we choose the generalized monospline defined in (2) we find

$$\begin{aligned} A_{k,m} &= M_n^{(n-k-1)}(b) = \frac{(b-a)^{k+3}}{(k+2)!} - \frac{2}{(k+3)!}(b-a)^{k+3} \\ &+ \sum_{j=0}^{k} A_{j,0}\frac{(b-a)^{k-j}}{(k-j)!} + \sum_{i=0}^{m-1}\sum_{j=0}^{k} A_{j,i}\frac{(b-x_i)^{k-j}}{(k-j)!} \\ &= (b-\xi_m)(\xi_m-a)\frac{(b-\xi_m)^{k+1}}{(k+1)!} + (b-2\xi_m+a)\frac{(b-\xi_m)^{k+2}}{(k+2)!} - 2\frac{(b-\xi_m)^{k+3}}{(k+3)!} \end{aligned}$$

and from Lemma 1 and the connection between quadrature formulae and monospline, we obtain the quadrature formulae (4).

Remark 1 *If we choose $\xi_1 = a$ and $\xi_2 = b$, then the quadrature formulae (4) is open type.*

In case of equidistance nodes, namely

$$\begin{aligned} x_i &= a + 2ih\,, \quad i = \overline{0, m} \\ \xi_i &= a + (2i-1)h\,, \quad i = \overline{1, m}\,, \end{aligned}$$

where $h = \dfrac{b-a}{2m}$ we give some inequalities for the remainder term. In this case we have the quadrature formulae

$$\begin{aligned}\int_a^b w(t)f(t)dt &= \sum_{k=0}^{n-1}(-1)^{k+1}A_{k,0}f^{(k)}(a) + \sum_{i=1}^{m-1}\sum_{k=0}^{n-1}(-1)^{k+1}A_{k,i}f^{(k)}(x_i) \\ &+ \sum_{k=0}^{n-1}(-1)^{k}A_{k,m}f^{(k)}(b) + \mathcal{R}[f], \end{aligned} \tag{9}$$

where

$$\begin{aligned} w(t) &= (b-t)(t-a)\,, \\ A_{k,0} &= (-1)^{k+1}\frac{h^{k+3}}{(k+1)!}\left\{2m-1-\frac{2(m-1)}{k+2}-\frac{2}{(k+2)(k+3)}\right\}, \\ A_{k,i} &= \frac{h^{k+3}}{(k+1)!}\left\{(2m-2i-1)(2i+1)(-1)^{k+1}-(2m-2i+1)(2i-1)\right\} \\ &+ \frac{h^{k+3}}{(k+2)!}\left\{(2m-4i-2)(-1)^k-(2m-4i+2)\right\}-\frac{2h^{k+3}}{(k+3)!}\left\{(-1)^{k+1}-1\right\}, \\ A_{k,m} &= \frac{h^{k+3}}{(k+1)!}\left\{2m-1-\frac{2(m-1)}{k+2}-\frac{2}{(k+2)(k+3)}\right\}, \\ & k=\overline{0,n-1}\,,\ \ i=\overline{1,m-1}\,. \end{aligned}$$

$$\mathcal{R}[f] = (-1)^n\int_a^b M_n(t)f^{(n)}(t)dt \tag{10}$$

and

$$M_n(t) = \begin{cases} P_{n,i}(t)\,,\ t\in[a+(2i-2)h,\ a+2ih)\ ,\ \ i=\overline{0,m-1}\,, \\ P_{n,m}(t)\,,\ t\in[a+(2m-2)h, a+2mh]\ , \end{cases} \tag{11}$$

where

$$\begin{aligned} P_{n,i}(t) &= (2m-2i+1)(2i-1)\frac{h^2}{n!}\left[t-a-(2i-1)h\right]^n \\ &+ (2m-4i+2)\frac{h}{(n+1)!}\cdot\left[t-a-(2i-1)h\right]^{n+1} \\ &- \frac{2}{(n+2)!}\left[t-a-(2i-1)h\right]^{n+2}\ ,\ \ i=\overline{1,m}\,. \end{aligned}$$

Theorem 1 *The generalized monospline of degree n, $M_n(t)$, $n>1$, defined in (11) verifies:*

$$\int_a^b M_n(t)dt = 0\,,\ \textit{if } n \textit{ is odd} \tag{12}$$

$$\int_a^b |M_n(t)|\,dt = \frac{2mh^{n+3}}{(n+1)!}\cdot\left\{\frac{2m^2+1}{3}-\frac{2}{(n+2)(n+3)}\right\} \tag{13}$$

$$\max_{t\in[a,b]}|M_n(t)| = \begin{cases} \dfrac{h^{n+2}}{n!}\left[m^2-\dfrac{n}{n+2}\right]\,,\ \textit{if } m \textit{ is even} \\ \dfrac{h^{n+2}}{n!}\left[m^2-\dfrac{2}{(n+1)(n+2)}\right]\,,\ \textit{if } m \textit{ is odd} \end{cases} \tag{14}$$

Proof. If n is odd number, we have

$$\begin{aligned}\int_a^b M_n(t)dt &= \sum_{i=1}^{m}\int_{a+(2i-2)h}^{a+2ih} P_{n,i}(t)dt \\ &= \sum_{i=1}^{m}\Bigg\{(2m-2i+1)(2i-1)\frac{h^2}{(n+1)!}\left[t-a-(2i-1)h\right]^{n+1} \\ &+ (2m-4i+2)\frac{h}{(n+2)!}\cdot\left[t-a-(2i-1)h\right]^{n+2} \\ &- \frac{2}{(n+3)!}\left[t-a-(2i-1)h\right]^{n+3}\Bigg\}\Bigg|_{a+(2i-2)h}^{a+2ih} \\ &= \frac{h^{n+3}}{(n+1)!}\sum_{i=1}^{m}\Bigg[(2m-2i+1)(2i-1)+\frac{2m-4i+2}{n+2} \\ &- \frac{2}{(n+2)(n+3)}\Bigg]+(-1)^n\frac{h^{n+3}}{(n+1)!}\sum_{i=1}^{m}\Big[(2m-2i+1)(2i-1) \\ &- \frac{2m-4i+2}{n+2}-\frac{2}{(n+2)(n+3)}\Big]=\frac{4h^{n+3}}{(n+2)!}\sum_{i=1}^{m}(m-2i+1)=0\,.\end{aligned}$$

$$\begin{aligned}\int_a^b |M_n(t)|\,dt &= \sum_{i=1}^{m}\int_{a+(2i-2)h}^{a+2ih}|P_{n,i}(t)|\,dt \\ &= \frac{2h^{n+3}}{(n+1)!}\sum_{i=1}^{m}\left[(2m-2i+1)(2i-1)-\frac{2}{(n+2)(n+3)}\right] \\ &= \frac{2mh^{n+3}}{(n+1)!}\cdot\left\{\frac{2m^2+1}{3}-\frac{2}{(n+2)(n+3)}\right\}\,.\end{aligned}$$

$$\max_{t\in[a,b]}|M_n(t)| = \max_{i=\overline{1,m}}\left\{\max_{t\in[a+(2i-2)h,a+2ih]}|P_{n,i}(t)|\right\}$$
$$= \max_{i=1,\left[\frac{m+1}{2}\right]}\left\{\frac{h^{n+2}}{n!}\left[(2i-1)(2m-2i+1)+\frac{2m-4i+2}{n+1}-\frac{2}{(n+1)(n+2)}\right]\right\}, \text{ where}$$
$\left[\frac{m+1}{2}\right]$ is the integer part of $\frac{m+1}{2}$.

We denote $g(i)=\frac{h^{n+2}}{n!}\left[(2i-1)(2m-2i+1)+\frac{2m-4i+2}{n+1}-\frac{2}{(n+1)(n+2)}\right]$.

Because $g(i)$ is increasing function, we have

$$\max_{i=1,\left[\frac{m+1}{2}\right]} g(i) = g\left(\left[\frac{m+1}{2}\right]\right) = \begin{cases} \frac{h^{n+2}}{n!}\left[m^2-\frac{n}{n+2}\right], & \text{if m is even} \\ \frac{h^{n+2}}{n!}\left[m^2-\frac{2}{(n+1)(n+2)}\right], & \text{if m is odd} \end{cases}$$

Theorem 2 *If $f\in W_1^n[a,b]$, $n>1$ and there exist numbers γ_n,Γ_n such that $\gamma_n\le f^{(n)}(t)\le\Gamma_n$, $t\in[a,b]$, then*

$$|\mathcal{R}[f]|\le\frac{mh^{n+3}(\Gamma_n-\gamma_n)}{(n+1)!}\left\{\frac{2m^2+1}{3}-\frac{2}{(n+2)(n+3)}\right\},\ \textit{if } n \textit{ is odd} \quad (15)$$

and

$$|\mathcal{R}[f]|\le\frac{2mh^{n+3}}{(n+1)!}\left\{\frac{2m^2+1}{3}-\frac{2}{(n+2)(n+3)}\right\}\left\|f^{(n)}\right\|_\infty,\ \textit{if } n \textit{ is even.} \quad (16)$$

Proof. Let n be odd. Using relations (10) and (12) we can written

$$\mathcal{R}[f]=(-1)^n\int_a^b M_n(t)f^{(n)}(t)dt=(-1)^n\int_a^b M_n(t)\left[f^{(n)}(t)-\frac{\gamma_n+\Gamma_n}{2}\right]dt,$$

such that we have

$$|\mathcal{R}[f]|\le\max_{t\in[a,b]}\left|f^{(n)}(t)-\frac{\gamma_n+\Gamma_n}{2}\right|\int_a^b|M_n(t)|\,dt. \quad (17)$$

We also have

$$\max_{t\in[a,b]}\left|f^{(n)}(t)-\frac{\gamma_n+\Gamma_n}{2}\right|\le\frac{\Gamma_n-\gamma_n}{2}. \quad (18)$$

From (13), (17) and (18) we have

$$|\mathcal{R}[f]|\le\frac{mh^{n+3}(\Gamma_n-\gamma_n)}{(n+1)!}\left\{\frac{2m^2+1}{3}-\frac{2}{(n+2)(n+3)}\right\}.$$

Let n be even. Then we have

$$|\mathcal{R}[f]|\le\left\|f^{(n)}\right\|_\infty\cdot\int_a^b|M_n(t)|\,dt=\frac{2mh^{n+3}}{(n+1)!}\left\{\frac{2m^2+1}{3}-\frac{2}{(n+2)(n+3)}\right\}\left\|f^{(n)}\right\|_\infty.$$

Theorem 3 *Let $f \in W_1^n[a,b], n > 1$ and let n, m be odd. If there exist a real number γ_n such that $\gamma_n \leq f^{(n)}(t)$, then*

$$|\mathcal{R}[f]| \leq \frac{2mh^{n+3}}{n!}\left[m^2 - \frac{2}{(n+1)(n+2)}\right]\cdot(T_n - \gamma_n), \tag{19}$$

where

$$T_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}.$$

If there exist a real number Γ_n such that $f^{(n)}(t) \leq \Gamma_n$, then

$$|\mathcal{R}[f]| \leq \frac{2mh^{n+3}}{n!}\left[m^2 - \frac{2}{(n+1)(n+2)}\right]\cdot(\Gamma_n - T_n). \tag{20}$$

Proof. Using relation (12) we can written

$$|\mathcal{R}[f]| = \left|\int_a^b \left(f^{(n)}(t) - \gamma_n\right) M_n(t)dt\right|.$$

From (14) we have

$$\begin{aligned}
|\mathcal{R}[f]| &\leq \max_{t\in[a,b]} |M_n(t)| \cdot \int_a^b \left(f^{(n)}(t) - \gamma_n\right) dt \\
&= \frac{h^{n+2}}{n!}\left[m^2 - \frac{2}{(n+1)(n+2)}\right]\left[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma_n(b-a)\right] \\
&= \frac{2mh^{n+3}}{n!}\left[m^2 - \frac{2}{(n+1)(n+2)}\right](T_n - \gamma_n).
\end{aligned}$$

In a similar way we can prove that (20) holds.

Theorem 4 *Let $f \in W_1^n[a,b], n > 1$, n a odd number and let m be even. If there exist a real number γ_n such that $\gamma_n \leq f^{(n)}(t)$, then*

$$|\mathcal{R}[f]| \leq \frac{2mh^{n+3}}{n!}\left[m^2 - \frac{n}{n+2}\right]\cdot(T_n - \gamma_n), \tag{21}$$

where

$$T_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}.$$

If there exist a real number Γ_n such that $f^{(n)}(t) \leq \Gamma_n$, then

$$|\mathcal{R}[f]| \leq \frac{2mh^{n+3}}{n!}\left[m^2 - \frac{n}{n+2}\right]\cdot(\Gamma_n - T_n). \tag{22}$$

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ON A NEW SYSTEM OF NONLINEAR VARIATIONAL INEQUALITIES AND ALGORITHMS

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Abstract. The purpose of this paper is to introduce and study the existence of solutions and approximation solvability for a class of new generalized nonlinear mixed variational inequalities in Hilbert spaces. An equivalence of this system of generalized nonlinear mixed variational inequalities on a closed convex cone to a system of nonlinear complementarity problems is also considered.

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1. Introduction and Preliminaries

Recently Verma [5-8], Kim and Kim [3-4] and Nie et al [2] introduced some system of nonlinear strongly monotone variational inequalities and studied the approximate solvability of this system based on a system of projection methods.

Projection methods have been applied widely to problems arising especially from complementarity, convex quadratic programming and variational problems.

The purpose of this paper is to consider, based on the resolvent method, the existence of solutions and approximation solvability of a class of new system of generalized nonlinear mixed variational inequalities and complementarity problem in the setting of Hilbert spaces. The results presented in this paper generalize, improve and unify the corresponding results of Verma [5-8], Kim and Kim [3-4] and Nie et al [2].

Throughout this paper we always assume that H is a Hilbert space with inner product $\langle , \cdot , \rangle$ and norm $|| \cdot ||$, K is a nonempty closed convex subset of H, $T, S : H \times H \to H$ are two given mappings and $\phi : H \to R \cup \{\infty\}$ is a proper convex lower semi-continuous function.

We consider the following problem: find $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq \rho(\phi(x^*) - \phi(x)), \\ \langle \eta S(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq \eta(\phi(y^*) - \phi(x)), \end{cases} \tag{1.1}$$

for all $x \in H$, which is called *the system of generalized nonlinear mixed variational inequality*, where $\rho > 0$ and $\eta > 0$ are two any constants.

Special cases of the Problem (1.1).

(I). If $T(x, y) = Ay + By$, $S(x, y) = Cx + Dx$, where $A, B, C, D : H \to H$ be four single-valued mappings, then the problem (1.1) reduces to finding $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho(Ay^* + By^*) + x^* - y^*, x - x^* \rangle \geq \rho(\phi(x^*) - \phi(x)), \\ \langle \eta(Cx^* + Dx^*) + y^* - x^*, x - y^* \rangle \geq \eta(\phi(y^*) - \phi(x)), \end{cases} \tag{1.2}$$

for all $x \in H$, which was considered in Kim and Kim [3].

(II). If $T(x, y) = S(x, y)$ and $\phi = \delta_K$ (the indicator function of a nonempty closed convex subset K *in* H), then the problem (1.1) is reduced to finding $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \eta T(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq 0, \end{cases} \tag{1.3}$$

for all $x \in K$, which was considered in Verma [8].

(III) If $T(x, y) = S(x, y)$, then the problem (1.1) is reduced to finding $x^*, y^* \in H$ such that

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - y^*, x - x^* \rangle \geq \rho(\phi(x^*) - \phi(x)), \\ \langle \eta T(x^*, y^*) + y^* - x^*, x - y^* \rangle \geq \eta(\phi(y^*) - \phi(x)), \end{cases} \tag{1.4}$$

for all $x \in H$. This kind of system of generalized nonlinear mixed variational inequalities is a generalization of problem (1.3), where $\rho > 0$ and $\eta > 0$ are two any constants.

(IV). If $\eta = 0$, then the problem (1.1) is reduced to finding x^* *in* H such that

$$\langle T(x^*, x^*), x - x^* \rangle \geq \phi(x^*) - \phi(x), \quad \forall \ x \in K. \tag{1.5}$$

(V). If K is a closed convex cone in H and $\phi = \delta_K$ (the indicator function of K), then the problem (1.1) is reduced to finding $x^*, y^* \in K$ such that $T(y^*, x^*) \in K^*$, $S(y^*, x^*) \in K^*$ and

$$\begin{cases} \langle \rho T(y^*, x^*) + x^* - y^*, x^* \rangle = 0, \\ \langle \eta S(x^*, y^*) + y^* - x^*, y^* \rangle = 0, \end{cases} \tag{1.6}$$

where K^* is the polar cone to K defined by

$$K^* = \{f \in H : \langle f, x \rangle \geq 0, \quad \forall x \in K\}$$

(VI) In (1.2) if $\phi = \delta$ (the indicator function of a nonempty closed convex subset K of H), then the problem (1.2) reduces to finding $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho(Ay^* + By^*) + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \eta(Cx^* + Dx^*) + y^* - x^*, x - y^* \rangle \geq 0, \end{cases} \tag{1.7}$$

for all $x \in H$, which is called the system of generalized nonlinear variational inequality, and it has been considered in Kim and Kim [3].

(VII). In (1.2) if $\phi = \delta_K$ (the indicator function of a nonempty closed convex cone K of H), then the problem (1.2) reduces to finding $x^*, y^* \in K$ such that $Ay^* + By^* \in K^*$, $Cx^* + Bx^* \in K^*$ and

$$\begin{cases} \langle \rho(Ay^* + By^*) + x^* - y^*, x^* \rangle = 0, \\ \langle \eta(Cx^* + Dx^*) + y^* - x^*, y^* \rangle = 0. \end{cases} \tag{1.8}$$

This is a class of new system of generalized nonlinear complementarity problem.

For the sake of convenience, we first recall some definitions and Lemmas.

Definition 1.1. A mapping $T : H \to H$ is said to be $k-$strongly monotone if, there exists an constant $k > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq k||x - y||^2, \quad \forall x, y \in H$$

Definition 1.2. A mapping $T : H \to H$ is said to be $\gamma-$Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$||Tx - Ty|| \leq \gamma||x - y||, \ \ \forall x, y \in H.$$

Lemma 1.1 [1]. *For a given $u \in H$, the point $z \in H$ satisfies the following inequality:*

$$\langle u - z, v - u\rangle \geq \rho(\phi(u) - \phi(v)), \ \ \forall v \in H$$

if and only if

$$u = J_{\phi}^{\rho}(z),$$

where $J_{\phi}^{\rho} = (I + \rho\partial\phi)^{-1}$ and $\partial\phi$ denotes the subdifferential of a proper convex lower semi-continuous function ϕ.

Lemma 1.2. *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers satisfying the following condition: there exists a nonnegative integer n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \ \ \forall n \geq n_0$$

where

$$t_n \in [0, 1], \ \sum_{n=1}^{\infty} t_n = \infty, \ b_n = 0(t_n), \ \sum_{n=1}^{\infty} c_n < \infty.$$

Then $a_n \to 0$ $(n \to \infty)$.

From Lemma 1.1, we can obtain the following result:

Lemma 1.3. *For given x^x, $y^* \in H$, (x^*, y^*) is a solution of the problem (1.1) if and only if*

$$\begin{cases} x^* = J_{\phi}^{\rho}(y^* - \rho T(y^*, x^*)), \\ y^* = J_{\phi}^{\eta}(x^* - \eta S(x^*, y^*)). \end{cases} \tag{1.9}$$

Definition 1.3. Let $T : H \to H$ be a mapping, $\{\alpha_n\}$ be a sequence in [0, 1] with $\sum_{n=1}^{\infty} \alpha_n = \infty$. For arbitrarily chosen initial points x_0, $y_0 \in H$ compute the sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J_{\phi}^{\rho}[y_n - \rho T(y_n, x_n)], \\ y_n = J_{\phi}^{\eta}[x_n - \eta S(x_n, y_n)], \end{cases} \tag{1.10}$$

which is called the Mann type iterative sequence of T

2. Main Results

In this section, we present the convergence analysis for resolvent methods in the context of the approximate solvability of problem (1.1). We have the following result.

Theorem 2.1. *Let $\phi : H \to R\cup\{+\infty\}$ be a proper convex lower semi-continuous function and $T, S : H \times H \to H$ be two mappings satisfying the following conditions:*

(i) the mappings $x \mapsto T(x, y)$ and $x \mapsto S(x, y)$ are k_1 and k_2-strongly monotone respectively;

(ii) the mappings $x \mapsto T(x, y)$ and $y \mapsto T(x, y)$ are r_1 and r_2 Lipschitz continuous respectively;

(iii) the mappings $x \mapsto S(x, y)$ and $y \mapsto S(x, y)$ are c_1 and c_2 Lipschitz continuous respectively.

(iv)

$$\begin{cases} 0 < \rho < \dfrac{2(k_1 - r_2)}{(r_1 + r_2)^2}, \ \text{and} \ \ k_1 > r_2; \\ 0 < \eta < \min\{\dfrac{2(k_2 - c_2)}{(c_1 + c_2)^2}, \ \dfrac{-c_2 + \sqrt{c_2^2 + 4(c_2^2 + c_1c_2)}}{2(c_2^2 + c_1c_2)}\} \ \text{and} \ \ k_2 > c_2, \end{cases} \tag{2.1}$$

If $(x^*, y^*) \in H$ is a solution, then the iterative sequence $\{x_n\}$, $\{y_n\}$ defined by (1.8) converge strongly to x^* and y^* respectively.

Proof. Since $(x^*, y^*) \in H$ is a solution of problem (1.1), it follows from Lemma 1.3 that

$$\begin{cases} x^* = J_\phi^\rho(y^* - \rho T(y^*, x^*)), \\ y^* = J_\phi^\eta(x^* - \eta S(x^*, y^*)). \end{cases} \tag{2.2}$$

In view of the nonexpansiveness of the mapping J_ϕ^ρ, it follows from (1.8) that

$$\begin{aligned} &||x_{n+1} - x^*|| \\ &= ||(1 - \alpha_n)x_n + \alpha_n J_\phi^\rho[y_n - \rho T(y_n, x_n)] \\ &\quad - [(1 - \alpha_n)x^* + \alpha_n J_\phi^\rho[y^* - \rho T(y^*, x^*)]|| \\ &\le (1 - \alpha_n)||x_n - x^*|| + \alpha_n||y_n - y^* - \rho(T(y_n, x_n) - T(y^*, x^*))||. \end{aligned} \tag{2.3}$$

Now we consider the second term on the right side of (2.3). we have

$$\begin{aligned} &||y_n - y^* - \rho(T(y_n, x_n) - T(y^*, x^*))||^2 \\ &\qquad = ||y_n - y^*||^2 + \rho^2||T(y_n, x_n) - T(y^*, x^*)||^2 \\ &\qquad\quad - 2\rho\langle T(y_n, x_n) - T(y^*, x^*), y_n - y^*\rangle. \end{aligned} \tag{2.4}$$

By condition (ii), the mappings $x \mapsto T(x,y)$ and $y \mapsto T(x,y)$ are r_1- and r_2-Lipschitz continuous respectively, hence we have

$$\begin{aligned}
&\rho^2||T(y_n,x_n)-T(y^*,x^*))||^2\\
&\le \rho^2\{||T(y_n,x_n)-T(y^*,x_n)||+||T(y^*,x_n)-T(y^*,x^*))||\}^2\\
&\le \rho^2\{r_1||y_n-y^*||+r_2||x_n-x^*||\}^2\\
&= \rho^2\{r_1^2||y_n-y^*||^2+r_2^2||x_n-x^*||^2+2r_1r_2||y_n-y^*||||x_n-x^*||\}\\
&\le \rho^2\{r_1^2||y_n-y^*||^2+r_2^2||x_n-x^*||^2+r_1r_2[||y_n-y^*||^2+||x_n-x^*||^2]\}\\
&= \rho^2(r_1^2+r_1r_2)||y_n-y^*||^2+\rho^2(r_2^2+r_1r_2)||x_n-x^*||^2.
\end{aligned} \tag{2.5}$$

Again by condition (i), the mappings $x \mapsto T(x,y)$ is k_1-strongly monotone , therefore we have

$$\begin{aligned}
&-2\rho\langle T(y_n,x_n)-T(y^*,x^*),y_n-y^*\rangle\\
&= -2\rho\langle T(y_n,x_n)-T(y^*,x_n)+T(y^*,x_n)-T(y^*,x^*),y_n-y^*\rangle\\
&\le -2\rho k_1||y_n-y^*||^2+2\rho r_2||x_n-x^*||||y_n-y^*||\\
&\le -2\rho k_1||y_n-y^*||^2+\rho r_2\{||x_n-x^*||^2+||y_n-y^*||^2\}\\
&= \rho(r_2-2k_1)||y_n-y^*||^2+\rho r_2||x_n-x^*||^2.
\end{aligned} \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4) and simplifying the resulting result we have

$$\begin{aligned}
||y_n-y^*-\rho(T(y_n,x_n)-T(y^*,x^*))||^2&\\
\le [1+\rho^2(r_1^2+r_1r_2)+\rho(r_2-2k_1)]||y_n-y^*||^2&\\
+[\rho^2(r_2^2+r_1r_2)+\rho r_2]||x_n-x^*||^2&
\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.3) and simplifying it we have

$$||x_{n+1}-x^*|| \le (1-\alpha_n)||x_n-x^*||+\alpha_n\sqrt{\delta||y_n-y^*||^2+\gamma||x_n-x^*||^2}, \tag{2.8}$$

where

$$\begin{cases}\delta = 1+\rho^2(r_1^2+r_1r_2)+\rho(r_2-2k_1)\\ \gamma = \rho^2(r_2^2+r_1r_2)+\rho r_2.\end{cases} \tag{2.9}$$

By condition (iv), it is easy to see that $0<\delta+\gamma<1$.

Now we give an estimation for $||y_n-y^*||$. By condition (iii), the mappings $x \mapsto S(x,y)$ and $y \mapsto S(x,y)$ are c_1- and c_2-Lipschitz continuous, respectively, and by condition (i) the mapping $x \mapsto S(x,y)$ is k_2-strongly monotone, and so

we have

$$\begin{aligned}
&||y_n - y^*||^2 \\
&= ||J_\phi^\eta[x_n - \eta S(x_n, y_n)] - J_\phi^\eta[x^* - \eta S(x^*, y^*)]||^2 \\
&\le ||x_n - x^* - \eta[S(x_n, y_n) - S(x^*, y^*)]||^2 \\
&= ||x_n - x^*||^2 + \eta^2||S(x_n, y_n) - S(x^*, y^*)||^2 \\
&\quad - 2\eta\langle S(x_n, y_n) - S(x^*, y^*), x_n - x^*\rangle \\
&= ||x_n - x^*||^2 + \eta^2\{||S(x_n, y_n) - S(x^*, y_n) + S(x^*, y_n) - S(x^*, y^*)||\}^2 \\
&\quad - 2\eta\langle S(x_n, y_n) - S(x^*, y_n) + S(x^*, y_n) - S(x^*, y^*), x_n - x^*\rangle \\
&\le ||x_n - x^*||^2 + \eta^2\{c_1||x_n - x^*|| + c_2||y_n - y^*||\}^2 \\
&\quad - 2\eta k_2||x_n - x^*||^2 + 2\eta||x_n - x^*||c_2||y_n - y^*|| \\
&\le ||x_n - x^*||^2 + \eta^2\{(c_1^2 + c_1c_2)||x_n - x^*||^2 + (c_2^2 + c_1c_2)||y_n - y^*||^2\} \\
&\quad + (c_2 - 2k_2)\eta||x_n - x^*||^2 + c_2\eta||y_n - y^*||^2 \\
&= \theta_1||x_n - x^*||^2 + \theta_2||y_n - y||^2,
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
\theta_1 &= 1 + \eta^2(c_1^2 + c_1c_2) + (c_2 - 2k_2)\eta \\
\theta_2 &= \eta^2(c_2^2 + c_1c_2) + c_2\eta.
\end{aligned}$$

This implies that

$$||y_n - y^*||^2 \le \frac{\theta_1}{1 - \theta_2}||x_n - x^*||^2.$$

By condition (iv) it is easy to prove that $0 < \theta_2 < 1$ and $0 < \frac{\theta_1}{1-\theta_2} < 1$. This implies that

$$||y_n - y^*|| \le ||x_n - x^*|| \tag{2.11}$$

Substituting (2.11) into (2.8) we have

$$||x_{n+1} - x^*|| \le (1 - \alpha_n(1 - \omega))||x_n - x^*||, \tag{2.12}$$

where $\omega^2 = (\delta + \gamma) < 1$.

Let $a_n = ||x_n - x^*||$, $t_n = \alpha_n(1 - \omega) \in [0, 1]$, $b_n = 0$, $c_n = 0$. Then $\sum_{n=0}^{\infty} t_n = \infty$ and the conditions in Lemma 1.2 are satisfied. Hence we have $x_n \to x^*$. This completes the proof of Theorem 2.1. □

If $T(x, y) = S(x, y)$, then the following result can be obtained from Theorem 2.1 immediately.

Theorem 2.2. *Let $\phi : H \to R \cup \{+\infty\}$ be a proper convex lower semi-continuous function and $T : H \times H \to H$ be a mapping satisfying the following conditions:*

(i) the mappings $x \mapsto T(x, y)$ is k_1 strongly monotone;

(ii) the mappings $x \mapsto T(x,y)$ and $y \mapsto T(x,y)$ are r_1 and r_2 Lipschitz respectively;

(iii)

$$\begin{cases} 0 < \rho < \dfrac{2(k_1 - r_2)}{(r_1 + r_2)^2}, \ \text{and} \ \ k_1 > r_2 \\ 0 < \eta < \min\{\dfrac{2(k_1 - r_2)}{(r_1 + r_2)^2}, \ \dfrac{-r_2 + \sqrt{r_2^2 + 4(r_2^2 + r_1 r_2)}}{2(r_2^2 + r_1 r_2)}\} \ \ \text{and} \ \ k_1 > r_2. \end{cases} \tag{2.13}$$

If $(x^*, y^*) \in H$ is a solution of the problem (1.4), then for any given $x_0,\ y_0 \in H$ the iterative sequence $\{x_n\}$, $\{y_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J_\phi^\rho [y_n - \rho T(y_n, x_n)], \\ y_n = J_\phi^\eta [x_n - \eta T(x_n, y_n)], \end{cases} \tag{2.14}$$

where $\{\alpha_n\}$ is a real sequence in [0, 1] with $\sum_{n=0}^{\infty} \alpha_n = \infty$, converges strongly to x^* and y^* respectively.

Remark 1. Theorem 2.2 generalizes and improves the corresponding result in Nie et al [2] and Verma [5-8].

If $T(x,y) = Ay + By$ and $S(x,y) = Cx + Dx$, then from Theorem 2.1 we can obtain the following result:

Theorem 2.3. *Let $\phi : H \to R \cup \{+\infty\}$ be a proper convex lower semi-continuous function and $A, B, C, D : H \to H$ be four mappings satisfying the following conditions:*

(i) mapping $A + B : H \to H$ is k_1- strongly monotone and $C + D : H \to H$ is k_2-strongly monotone;

(ii) mappings $A, B, C, D : H \to H$ are l_1-, l_2-, l_3-, l_4-Lipschitz continuous respectively;

(iii)

$$\begin{cases} 0 < \rho < \dfrac{2k_1}{(l_1 + l_2)^2}, \\ 0 < \eta < \dfrac{2k_2}{(l_3 + l_4)^2}, \end{cases} \tag{2.15}$$

Then the problem (1.2) has a unique solutio $(x^*, y^*) \in H$, and for any given $x_0 \in H$, the iterative sequences $\{x_n\}$, $\{y_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J_\phi^\rho [y_n - \rho(A + B)(y_n)], \\ y_n = J_\phi^\eta [x_n - \eta(C + D)(x_n)], \end{cases} \tag{2.16}$$

where $\{\alpha_n\}$ a real sequence with $\sum_{n=0}^{\infty} \alpha_n = \infty$ converge strongly to x^* and y^* respectively.

Proof. (I) First we prove the existence of solution for the problem (1.2).

Define a mapping $F : H \to H$ by

$$F(x) = J_\phi^\rho[J_\phi^\eta(x - \eta(C+D)(x)) - \rho(A+B)(J_\phi^\eta(x - \eta(C+D)(x)))]. \quad (2.17)$$

For any $x, y \in H$, let

$$X = J_\phi^\eta(x - \eta(C+D)(x)),$$
$$Y = J_\phi^\eta(y - \eta(C+D)(y)),$$

therefore we have

$$\begin{aligned} ||F(x) - F(y)||^2 &= ||J_\phi^\rho[X - \rho(A+B)(X)] - J_\phi^\rho[Y - \rho(A+B)(Y)]||^2 \\ &\le ||X - Y - \rho[(A+B)(X) - (A+B)(Y)]||^2 \\ &= ||X - Y||^2 - 2\rho\langle(A+B)(X) - (A+B)(Y), X - Y\rangle \\ &\quad + \rho^2||(A+B)(X) - (A+B)(Y)||^2 \\ &\le ||X - Y||^2 - 2\rho k_1||X - Y||^2 \\ &\quad + \rho^2\{||AX - AY|| + ||BX - BY||\}^2 \\ &\le ||X - Y||^2 - 2\rho k_1||X - Y||^2 + \rho^2(l_1 + l_2)^2||X - Y||^2 \\ &= [1 - 2\rho k_1 + \rho^2(l_1 + l_2)^2]||X - Y||^2 \\ &= \sigma_1^2||X - Y||^2, \end{aligned} \quad (2.18)$$

where

$$\sigma_1 = \sqrt{1 - 2\rho k_1 + \rho^2(l_1 + l_2)^2} < 1 \ \ (\textit{by condition } (2.15).)$$

Next we give an estimate for $||X - Y||$. We have

$$\begin{aligned} ||X - Y||^2 &= ||J_\phi^\eta(x - \eta(C+D)(x)) - J_\phi^\eta(y - \eta(C+D)(y))|| \\ &\le ||x - y - \eta[(C+D)(x) - (C+D)(y)]||^2 \\ &= ||x - y||^2 - 2\eta\langle(C+D)(x) - (C+D)(y), x - y\rangle \\ &\quad + \eta^2||(C+D)(x) - (C+D)(y)||^2 \\ &\le ||x - y||^2 - 2\eta k_2||x - y||^2 + \eta^2(l_3 + l_4)^2||x - y||^2 \\ &= \sigma_2^2||x - y||^2, \end{aligned} \quad (2.19)$$

where $\sigma_2 = \sqrt{1 - 2\eta k_2 + \eta^2(l_3 + l_4)^2} < 1$ (by condition (2.15)).

Substituting (2.19) into (2.18) we have

$$||F(x) - F(y)|| \le \sigma_1\sigma_2||x - y||, \ \ \forall x, y \in H.$$

This implies that $F : H \to H$ is a Banach contraction mapping. Therefore there exists a unique fixed point $x^* \in H$ of F such that

$$x^* = F(x^*). \quad (2.20)$$

Letting

$$y^* = J_\phi^\eta(x^* - \eta(C+D)(x^*))$$

hence we have

$$x^* = J_\phi^\rho[y^* - \rho(A+B)(y^*)].$$

By Lemma 1.3 (x^*, y^*) is the unique solution of the problem (1.2).

(II) Next we prove the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (2.16) converge strongly to x^* and y^* respectively. In fact,we have

$$\begin{aligned}
&||x_{n+1} - x^*|| \\
&= ||(1-\alpha_n)(x_n - x^*) + \alpha_n J_\phi^\rho[y_n - \rho(A+B)(y_n)] \\
&\quad - \alpha_n J_\phi^\rho[y^* - \rho(A+B)(y^*)]|| \\
&\leq (1-\alpha_n)||x_n - x^*|| + \alpha_n||y_n - y^* - \rho[(A+B)(y_n) - (A+B)(y^*)]||
\end{aligned} \tag{2.21}$$

Now we consider the second term on the right side of (2.21). We have

$$\begin{aligned}
&||y_n - y^* - \rho[(A+B)(y_n) - (A+B)(y^*)]||^2 \\
&= ||y_n - y^*||^2 - 2\rho\langle(A+B)(y_n) - (A+B)(y^*), y_n - y^*\rangle \\
&\quad + \rho^2||(A+B)(y_n) - (A+B)(y^*)||^2 \\
&\leq ||y_n - y^*||^2 - 2\rho k_1||y_n - y^*||^2 \\
&\quad + \rho^2\{||A(y_n) - A(y^*)|| + ||B(y_n) - B(y^*)||\}^2 \\
&\leq ||y_n - y^*||^2 - 2\rho k_1||y_n - y^*||^2 + \rho^2(l_1+l_2)^2||y_n - y^*||^2 \\
&= [1 - 2\rho k_1 + \rho^2(l_1+l_2)^2]||y_n - y^*||^2.
\end{aligned}$$

This implies that

$$||y_n - y^* - \rho[(A+B)(y_n) - (A+B)(y^*)]|| \leq \sigma_1||y_n - y^*|| \tag{2.22}$$

where $\sigma_1 = \sqrt{1 - 2\rho k_1 + \rho^2(l_1+l_2)^2} < 1$ (by condition (2.15)). Substituting (2.22) into (2.21) we have

$$||x_{n+1} - x^*|| \leq (1-\alpha_n)||x_n - x^*|| + \alpha_n\sigma_1||y_n - y^*|| \tag{2.23}$$

Now we give an estimate for $||y_n - y^*||$. We have

$$\begin{aligned}
&||y_n - y^*|| \\
&= ||J_\phi^\eta[x_n - \eta(C+D)(x_n)] - J_\phi^\eta[x_* - \eta(C+D)(x_*)]|| \\
&\leq ||x_n - x^* - \eta[(C+D)(x_n) - (C+D)(x_*)]||
\end{aligned} \tag{2.24}$$

Since

$$\begin{aligned}
&||x_n - x^* - \eta[(C+D)(x_n) - (C+D)(x_*)]||^2 \\
&= ||x_n - x^*||^2 - 2\eta\langle(C+D)(x_n) - (C+D)(x_*)\rangle \\
&\quad + \eta^2||(C+D)(x_n) - (C+D)(x_*)||^2 \\
&\leq ||x_n - x^*||^2 - 2\eta k_2||x_n - x_*||^2 + \quad \eta^2(l_3+l_4)^2||x_n - x_*||^2 \\
&= [1 - 2\eta k_2 + \eta^2(l_3+l_4)^2]||x_n - x_*||^2
\end{aligned}$$

This implies that

$$||x_n - x^* - \eta[(C+D)(x_n) - (C+D)(x_*)]|| \leq \sigma_2||x_n - x_*||. \tag{2.25}$$

where $\sigma_2 = \sqrt{1 - 2\eta k_2 + \eta^2(l_3+l_4)^2} < 1$ (by condition (2.15)).

Substituting (2.15) into (2.24) we have

$$||y_n - y^*|| \leq \sigma_2||x_- x^*|| \tag{2.26}$$

Substituting (2.26) into (2.23) we have

$$||x_{n+1} - x^*|| \leq (1 - \alpha_n(1 - \sigma_1\sigma_2))||x_n - x^*||.$$

By Lemma 1.2, we know that $||x_n - x^*|| \to 0$, i.e., $x_n \to x^*$ $(as\ n \to \infty)$ This completes the proof of Theorem 2.3. □

Remark. Theorem 2.3 generalizes and improves the corresponding results in Kim and Kim [3].

As a direct conclusion of Theorem 2.3, we have the following result.

Theorem 2.4. *Let $\phi = \delta_K$ be the indicator function of a nonempty closed convex cone $K \subset H$. Let $A, B, C, D : H \to H$ be four mappings satisfying the following conditions:*

(i) the mapping $A + B : H \to H$ is k_1- strongly monotone and the mapping $C + D : H \to H$ is k_2-strongly monotone;

(ii) mappings $A, B, C, D : H \to H$ are l_1-, l_2-, l_3-, l_4-Lipschitz continuous respectively;

(iii)

$$\begin{cases} 0 < \rho < \dfrac{2k_1}{(l_1+l_2)^2}, \\ 0 < \eta < \dfrac{2k_2}{(l_3+l_4)^2}. \end{cases} \tag{2.27}$$

Then the system of generalized nonlinear complementarity problem (1.8) has a unique solution $(x^*, y^*) \in H$*, and for any given* $x_0 \in H$*, the iterative sequences* $\{x_n\}$*,* $\{y_n\}$ *defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_\phi^\rho[y_n - \rho(A+B)(y_n)], \\ y_n = J_\phi^\eta[x_n - \eta(C+D)(x_n)], \end{cases} \tag{2.28}$$

where $\{\alpha_n\}$ *is a real sequence with* $\sum_{n=0}^{\infty} \alpha_n = \infty$*, converge strongly to* x^* *and* y^* *respectively.*

Proof. It follows from Theorem 2.3 that there exist $x^*, y^* \in K$ such that

$$\begin{cases} \langle \rho(Ay^* + By^*) + x^* - y^*, x - x^* \rangle \geq 0, \\ \langle \eta(Cx^* + Dx^*) + y^* - x^*, x - y^* \rangle \geq 0. \end{cases} \tag{2.29}$$

for all $x \in K$ and the iterative sequences $\{x_n\}$ and $\{y_n\}$ defined by (2.28) converge strongly to x^* and y^*, respectively. Since K is a cone in H. Therefore (2.29) is equivalent to $Ay^* + By^*,\ Cx^* + Dx^* \in K^*$ and

$$\begin{cases} \langle \rho(Ay^* + By^*) + x^* - y^*, x^* \rangle = 0, \\ \langle \eta(Cx^* + Dx^*) + y^* - x^*, y^* \rangle = 0. \end{cases} \tag{2.30}$$

This completes the proof of Theorem 2.4. □

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ON $\mathcal{L}$-FUZZY NORMED SPACES

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Abstract. The purpose of this paper to introduce $\mathcal{L}$-fuzzy normed spaces due to Saadati and Vaezpour [14]. Also $\mathcal{L}$-fuzzy Banach spaces are defined and open mapping and closed graph theorems are proved. Furthermore the concept of quotient of $\mathcal{L}$-fuzzy Banach space is given.

1. Introduction

Since the introduction of the concept of fuzzy set by Zadeh [18] in 1965, many authors have introduced the concept of fuzzy metric space in different ways [2, 9,11,12]. George and Veeramani [6,8] modified the concept of fuzzy metric space introduced by Kromosil and Michalek [12] and defined a Hausdorff topology on this fuzzy metric space. Using to idea of $\mathcal{L}$-fuzzy sets [7], Saadati et al. [16] introduced the notion of $\mathcal{L}$-fuzzy metric spaces with the help of continuous t-norms as a generalization of fuzzy metric space due to George and Veeramani [6] and intuitionistic fuzzy metric space due to Park and Saadati [13,15]. Recently, Saadati [17] proved some known results of metric spaces including Uniform continuity theorem and Ascoli–Arzela theorem for $\mathcal{L}$-fuzzy metric spaces. He also proved that every $\mathcal{L}$-fuzzy metric space has a countably locally finite basis and used this result to conclude that every $\mathcal{L}$-fuzzy metric space is metrizable.

In this paper we define $\mathcal{L}$-fuzzy normed space due to Saadati and Vaezpour [14]. We also define $\mathcal{L}$-fuzzy Banach spaces, and prove some theorems especially open mapping and closed graph theorems. In last section we give the concept of quotient of $\mathcal{L}$-fuzzy Banach space.

2. Preliminaries

Definition 1 ([7]). *Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a non-empty set called universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.*

Lemma 1 ([4]). *Consider the set L^* and operation $\leq_{L^*}$ defined by $L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2$ and $x_1 + x_2 \leq 1\}$, $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, for every (x_1, x_2), $(y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.*

Definition 2 ([1]). *An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ on a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.*

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Classically, a triangular norm T on $([0,1],\leq)$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(x,1) = x$, for all $x \in [0,1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_\mathcal{L} = \inf L$ and $1_\mathcal{L} = \sup L$.

Definition 3. *A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:*

(i) $(\forall x \in L)(\mathcal{T}(x, 1_\mathcal{L}) = x)$; (boundary condition)
(ii) $(\forall (x,y) \in L^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x)$; (commutativity)
(iii) $(\forall (x,y,z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z)$; (associativity)
(iv) $(\forall (x,x',y,y') \in L^4)(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x,y) \leq_L \mathcal{T}(x',y'))$ (monotonicity).

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be continuous if for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have $\lim_n \mathcal{T}(x_n, y_n) = \mathcal{T}(x,y)$.

For example, $\mathcal{T}(x,y) = \min(x,y)$ and $\mathcal{T}(x,y) = xy$ are two continuous t-norms on $[0,1]$.

A t-norm can also be defined recursively as an $(n+1)$-ary operation ($n \in \mathbb{N}$) by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x_1, ..., x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, ..., x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in L$.

Definition 4 ([3])**.** *A t-norm $\mathcal{T}$ on L^* is called t-representable if and only if there exist a t-norm T and a t-conorm S on $[0,1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$,*

$$\mathcal{T}(x,y) = (T(x_1, y_1), S(x_2, y_2)).$$

Definition 5 ([16])**.** *A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_\mathcal{L}) = 1_\mathcal{L}$ and $\mathcal{N}(1_\mathcal{L}) = 0_\mathcal{L}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for $x \in L$, then $\mathcal{N}$ is called an involutive negation.*

The negation N_s on $([0,1],\leq)$ defined as, for all $x \in [0,1]$, $N_s(x) = 1 - x$, is called the standard negation on $([0,1],\leq)$. We show $(N_s(x), x) = \mathcal{N}_s(x)$.

Definition 6 ([16])**.** *The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an $\mathcal{L}$-fuzzy metric space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions for every x, y, z in X and t, s in $(0, +\infty)$:*

(a) $\mathcal{M}(x,y,t) >_L 0_\mathcal{L}$;
(b) $\mathcal{M}(x,y,t) = 1_\mathcal{L}$ for all $t > 0$ if and only if $x = y$;
(c) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;
(d) $\mathcal{T}(\mathcal{M}(x,y,t), \mathcal{M}(y,z,s)) \leq_L \mathcal{M}(x,z,t+s)$;
(e) $\mathcal{M}(x,y,\cdot) : (0,+\infty) \longrightarrow L$ is continuous.

In this case $\mathcal{M}$ is called an $\mathcal{L}$-fuzzy metric.

Henceforth, we assume that $\mathcal{T}$ is a continuous t-norm on lattice $\mathcal{L}$ such that for every $\mu \in L \backslash \{0_\mathcal{L}, 1_\mathcal{L}\}$, there is a $\lambda \in L \backslash \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that

$$\mathcal{T}^{n-1}(\mathcal{N}(\lambda), ..., \mathcal{N}(\lambda)) >_L \mathcal{N}(\mu).$$

3. $\mathcal{L}$-FUZZY NORMED SPACES

Definition 7. *The 3-tuple $(X, \mu, \mathcal{T})$ is said to be an $\mathcal{L}$-fuzzy normed space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and μ is an $\mathcal{L}$-fuzzy set on $X \times (0, +\infty)$ satisfying the following conditions for every x, y in X and $t, s > 0$:*

(a) $\mu(x,t) >_L 0_{\mathcal{L}}$;
(b) $\mu(x,t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = 0$;
(c) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;
(d) $\mathcal{T}(\mu(x,t), \mu(y,s)) \leq_L \mu(x, z, t+s)$;
(e) $\mu(x, \cdot) : (0, +\infty) \longrightarrow L$ is continuous;
(f) $\lim_{t\to\infty} \mu(x,t) = 1_{\mathcal{L}}$.

In this case μ is called an $\mathcal{L}$-fuzzy norm. If $\mu = \mu_{\mu,\nu}$ is an intuitionistic fuzzy set then the 3-tuple $(X, \mu_{\mu,\nu}, \mathcal{T})$ is said to be an intuitionistic fuzzy normed space.

Let $(X, \mu, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space. For $t > 0$, we define the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as

$$B(x, r, t) = \{y \in X : \mu(x - y, t) >_L \mathcal{N}(r)\}.$$

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let τ_μ denote the family of all open subsets of X. Then τ_μ is called the $\mathcal{L}$-fuzzy topology induced by the $\mathcal{L}$-fuzzy norm μ.

Example 1. *Let $(X, ||\cdot||)$ be a normed space. Define $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X \times (0, +\infty)$ defined as follows:*

$$\mu_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{kt^n}{kt^n + m||x||}, \frac{m||x||}{kt^n + m||x||}\right),$$

for all $t, k, m, n \in \mathbb{R}^+$. Then $(X, \mu_{\mu,\nu}, \mathcal{T})$ is an intuitionistic fuzzy normed space. If $h = m = n = 1$ then $(X, \mu_{\mu,\nu}, \mathcal{T})$ is a standard intuitionistic fuzzy normed space. Also, if we define

$$\mu_{\mu,\nu}(x,t) = (\mu(x,t), \nu(x,t)) = \left(\frac{t}{t + m||x||}, \frac{||x||}{t + ||x||}\right),$$

where $m > 1$. Then $(X, \mu_{\mu,\nu}, \mathcal{T})$ is an intuitionistic fuzzy normed space in which $\mu_{\mu,\nu}(0,t) = 1_{L^*}$ and $\mu_{\mu,\nu}(x,t) <_{L^*} 1_{L^*}$ for $x \neq 0$.

Definition 8. *A sequence $\{x_n\}$ in an $\mathcal{L}$-fuzzy normed space $(X, \mu, \mathcal{T})$ is called a Cauchy sequence, if for each $\varepsilon \in L\backslash\{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$*

$$\mu(x_m - x_n, t) >_L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}$ is said to be convergent to $x \in X$ in the $\mathcal{L}$-fuzzy normed space $(X, \mu, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mu} x$) if $\mu(x_n - x, t) \longrightarrow 1_{\mathcal{L}}$ as $n \longrightarrow \infty$ for every $t > 0$. An $\mathcal{L}$-fuzzy normed space is said to be complete iff every Cauchy sequence is convergent.

Lemma 2. *Let $(X, \mu, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and let $x, y \in X$ and $t > 0$. Then*

(i) $\mu(x,t)$ is nondecreasing with respect to t for each $x \in X$.

(ii) $\mu(x-y,t)=\mu(y-x,t)$.

Proof. (a) Let $t<s$. Then $k=s-t>0$ and we have

$$\mu(x,t)=\mathcal{T}(\mu(x,t),1_{\mathcal{L}})=\mathcal{T}(\mu(x,t),\mu(0,k))\leq_L \mu(x,s).$$

Hence $\mu(x,t)$ is nondecreasing.

(b) $\mu(x-y,t)=\mu((-1)(y-x),t)=\mu\left(y-x,\frac{t}{|-1|}\right)=\mu(y-x,t)$. □

Lemma 3. *Let $(X,\mu,\mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space. If we define*

$$\mathcal{M}(x,y,t)=\mu(x-y,t),$$

then $\mathcal{M}$ is an $\mathcal{L}$-fuzzy metric on X, which is called the $\mathcal{L}$-fuzzy metric induced by the $\mathcal{L}$-fuzzy norm μ.

Definition 9. *The $\mathcal{L}$-fuzzy normed space $(X,\mu,\mathcal{T})$ is said to be an $\mathcal{L}$-fuzzy Banach space whenever X is complete with respect to the $\mathcal{L}$-fuzzy metric induced by the $\mathcal{L}$-fuzzy norm.*

Lemma 4. *An $\mathcal{L}$-fuzzy metric $\mathcal{M}$ which is induced by an $\mathcal{L}$-fuzzy normed space $(X,\mu,\mathcal{T})$ has the following properties for all $x,y,z\in X$ and every scalar $\alpha\neq 0$:*

(i) $\mathcal{M}(x+z,y+z,t)=\mathcal{M}(x,y,t)$,
(ii) $\mathcal{M}(\alpha x,\alpha y,t)=\mathcal{M}\left(x,y,\frac{t}{|\alpha|}\right)$.

Proof. (i) $\mathcal{M}(x+z,y+z,t)=\mu((x+z)-(y+z),t)=\mu(x-y,t)=\mathcal{M}(x,y,t)$.

(ii) $\mathcal{M}(\alpha x,\alpha y,t)=\mu(\alpha x-\alpha y,t)=\mu\left(x-y,\frac{t}{|\alpha|}\right)=\mathcal{M}\left(x,y,\frac{t}{|\alpha|}\right)$. □

Lemma 5. *Let $(X,\mu,\mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and let $x,y\in X$ and $t>0$. Then*

(i) The function $(x,y)\longrightarrow x+y$ is continuous,
(ii) The function $(\alpha,x)\longrightarrow \alpha x$ is continuous.

Proof. (i) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X with $x_n\longrightarrow x$ and $y_n\longrightarrow y$. Then $n\longrightarrow\infty$

$$\mu((x_n+y_n)-(x+y),t)\geq_L \mathcal{T}\left(\mu\left(x_n-x,\frac{t}{2}\right),\mu\left(y_n-y,\frac{t}{2}\right)\right)\longrightarrow 1_{\mathcal{L}}$$

for all $t>0$. This completes the proof.

(ii) Now if $x_n\longrightarrow x$ and $\alpha_n\longrightarrow\alpha$ where $\alpha_n\neq 0$ then

$$\begin{aligned}\mu(\alpha_n x_n-\alpha x,t) &= \mu(\alpha_n(x_n-x)+x(\alpha_n-\alpha),t)\\ &\geq {}_L\mathcal{T}\left(\mu\left(\alpha_n(x_n-x),\frac{t}{2}\right),\mu\left(x(\alpha_n-\alpha),\frac{t}{2}\right)\right)\\ &= \mathcal{T}\left(\mu\left(x_n-x,\frac{t}{2\alpha_n}\right),\mu\left(x,\frac{t}{2(\alpha_n-\alpha)}\right)\right)\longrightarrow 1_{\mathcal{L}},\end{aligned}$$

as $n\longrightarrow\infty$ for all $t>0$, which completes the proof. □

Theorem 1 (Open Mapping Theorem)**.** *If T is a continuous linear operator from the $\mathcal{L}$-fuzzy Banach space $(X,\mu_1,\mathcal{T})$ onto the $\mathcal{L}$-fuzzy Banach space $(X,\mu_2,\mathcal{T})$, then T is an open mapping.*

Proof. We will prove this theorem in several steps.

Step 1: Let E be a neigborhood of 0 in X. We show that $0 \in (\overline{T(E)})^o$. Let D be a balanced neigborhood of 0 such that $D + D \subset E$. Since $T(X) = Y$ and D absorbing, it follows that $Y = \cap_n T(nD)$. So by Theorem 3.17 in [6], there exists a positive integer $n_0 \in \mathbb{N}$ such that $\overline{T(n_0 D)}$ has nonempty interior. Therefore, $0 \in (\overline{T(D)})^o - (\overline{T(D)})^o$. On the other hand,

$$\begin{aligned}(\overline{T(D)})^o - (\overline{T(D)})^o &\subset (\overline{T(D)}) - (\overline{T(D)}) \\ &= (\overline{T(D) - T(D)}) \\ &\subset \overline{T(E)}.\end{aligned}$$

So the set $\overline{T(E)}$ includes the neigborhood $(\overline{T(D)})^o - (\overline{T(D)})^o$ of 0.

Step 2: It is shown $0 \in (T(E))^o$. Since $0 \in E$ and E is an open set, then there exist $\alpha \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t_0 \in (0, \infty)$ such that $B(0, \alpha, t_0) \subset E$. But for $\alpha \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, a sequence can be found such that $\varepsilon_n \longrightarrow 0_{\mathcal{L}}$ and

$$\mathcal{N}(\alpha) <_L \lim_n \mathcal{T}^{n-1}(\mathcal{N}(\varepsilon_1), ..., \mathcal{N}(\varepsilon_n)).$$

On the other hand $0 \in \overline{T(B(0, \varepsilon_n, t'_n))}$, where $t'_n = \frac{1}{2^n} t_0$, so by Step 1, there exist $\sigma_n \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t_n > 0$ such that $B(0, \sigma_n, t_n) \subset \overline{T(B(0, \varepsilon_n, t'_n))}$. Since the set $\{B(0, r_n, \frac{1}{n}) : n = 1, 2, ...\}$ is a countable base at zero where $r_n \longrightarrow 0_{\mathcal{L}}$ and $t'_n \to 0$ as $n \to \infty$, so t_n and σ_n can be chosen such that $t_n \, , \sigma_n \longrightarrow 0_{\mathcal{L}}$ as $n \to \infty$.

Now it is shown that $B(0, \sigma_1, t_1) \subset (T(E))^o$. Suppose $y_0 \in B(0, \sigma_1, t_1)$. Then $y_0 \in \overline{T(B(0, \varepsilon_1, t'))}$ and so for $\sigma_2 >_L 0_{\mathcal{L}}$ and $t_2 > 0$ the ball $B(y_0, \sigma_2, t_2)$ intersects $T(B(0, \varepsilon_1, t'))$. Therefore there exists $x_1 \in B(0, \varepsilon_1, t')$ such that $Tx_1 \in B(y_0, \sigma_2, t_2)$, i.e.,

$$\mu_2(y_0 - Tx_1, t_2) >_L \mathcal{N}(\sigma_2)$$

or equivalently $y_0 - Tx_1 \in B(0, \sigma_2, t_2) \subset \overline{T(B(0, \varepsilon_1, t'))}$ and by the similar argument there exist x_2 in $B(0, \varepsilon_2, t')$ such that

$$\mu_2(y_0 - (Tx_1 + Tx_2), t_3) = \mu_2((y_0 - Tx_1) - Tx_2, t_3) >_L \mathcal{N}(\sigma_3).$$

If this process is continued, it leads to a sequence $\{x_n\}$ such that $x_n \in B(0, \varepsilon_n, t'_n)$ and

$$\mu_2 \left(y_0 - \sum_{j=1}^{n-1} Tx_j, t_n \right) >_L \mathcal{N}(\sigma_n) \, .$$

Now if $n \in \mathbb{N}$ and $\{p_n\}$ is a positive and increasing sequence, then

$$\mu_1 \left(\sum_{j=1}^{n} x_j - \sum_{j=1}^{n+p_n} x_j, t \right) = \mu_1 \left(\sum_{j=n+1}^{n+p_n} x_j, t \right) \geq_L \mathcal{T}^{n-1}(\mu_1(x_{n+1}, t_1), ..., \mu_1(x_{n+p_n}, t_{p_n}))$$

where $t_1 + t_2 + ... + t_{p_n} = t$. By putting $t_0 = \min\{t_1, t_2, ..., t_{p_n}\}$, since $t'_n \to 0$ so there exists $n_0 \in \mathbb{N}$ such that $0 < t'_n \leq t_0$ for $n > n_0$. Therefore,

$$\begin{aligned}\mathcal{T}^{n-1}(\mu_1(x_{n+1}, t_0), ..., \mu_1(x_{n+p_n}, t_{p_n})) &\geq {}_L\mathcal{T}^{n-1}(\mu_1(x_{n+1}, t'_{n+1}), ..., \mu_1(x_{n+p_n}, t'_{n+p_n})) \\ &\geq {}_L\mathcal{T}^{n-1}(\mathcal{N}(\varepsilon_{n+1}), ..., \mathcal{N}(\varepsilon_{n+p_n})).\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\mu_1\left(\sum_{j=n+1}^{n+p_n}x_j,t\right)\geq_L\lim_{n\to\infty}\mathcal{T}^{n-1}(\mathcal{N}(\varepsilon_{n+1}),...,\mathcal{N}(\varepsilon_{n+p_n}))=1_{\mathcal{L}},$$

that is,

$$\lim_{n\to\infty}\mu_1\left(\sum_{j=n+1}^{n+p_n}x_j,t\right)\to 1_{\mathcal{L}}$$

for all $t>0$.So the sequence $\left\{\sum_{j=1}^{n}x_j\right\}$ is a Cauchy sequence and consequently the series $\left\{\sum_{j=1}^{\infty}x_j\right\}$ converges to some point $x_0\in X$, because X is complete space.

By fixing $t>0$, there exists $n_0\in\mathbb{N}$ such that $t>t_n$ for $n>n_0$, because $t_n\to 0$. So it follows:

$$\mu_2\left(y_0-T\left(\sum_{j=1}^{n-1}x_j\right),t\right)\geq_L\mu_2\left(y_0-T\left(\sum_{j=1}^{n-1}x_j\right),t_n\right)\geq_L\mathcal{N}(\sigma_n),$$

and thus

$$\mu_2\left(y_0-T\left(\sum_{j=1}^{n-1}x_j\right),t\right)\to 1_{\mathcal{L}}\ .$$

Therefore,

$$y_0=\lim_n T\left(\sum_{j=1}^{n-1}x_j\right)=Tx_0.$$

But

$$\begin{aligned}\mu_1(x_0,t_0) &\geq {}_L\limsup_n\mu_1\left(\sum_{j=1}^{n}x_j,t_0\right)\geq_L\limsup_n\mathcal{T}^{n-1}(\mu_1(x_1,t_1'),...,\mu_1(x_n,t_n'))\\ &\geq {}_L\limsup_n\mathcal{T}^{n-1}(\mathcal{N}(\varepsilon_1),...,\mathcal{N}(\varepsilon_n))=\mathcal{N}(\alpha).\end{aligned}$$

Hence $x_0\in B(0,\alpha,t_0)$.

Step 3: Let G be an open subset of X and $x\in G$. Then,

$$T(G)=Tx+T(-x+G)\supset Tx+(T(-x+G))^o.$$

Hence $T(G)$ would be open, because it includes a neigborhood of each of its point. □

Theorem 2 (Closed Graph Theorem)**.** *Let T be a linear operator from the $\mathcal{L}$-fuzzy Banach space $(X,\mu_1,\mathcal{T})$ into the $\mathcal{L}$-fuzzy Banach space $(Y,\mu_2,\mathcal{T})$. Suppose for every sequence $\{x_n\}$ in X such that $x_n\to x$ and $Tx_n\to y$ for some elements $x\in X$ and $y\in Y$ it follows $Tx=y$. Then T is continuous.*

Proof. At first it is proved that the $\mathcal{L}$-fuzzy norm μ which is defined on $X\times Y$ by,

$$\mu((x,y),t)=\mathcal{T}(\mu_1(x,t),\mu_2(y,t))$$

is a complete $\mathcal{L}$-fuzzy norm. For each $x, z \in X$, $y, u \in Y$ and $t, s > 0$ it follows:

$$\begin{aligned}\mathcal{T}[\mu((x,y),t), \mu((z,u),s)] &= \mathcal{T}[\mathcal{T}(\mu_1(x,t), \mu_2(y,t)), \mathcal{T}(\mu_1(z,s), \mu_2(u,s))] \\ &= \mathcal{T}[\mathcal{T}(\mu_1(x,t), \mu_1(z,s)), \mathcal{T}(\mu_2(y,t), \mu_2(u,s))] \\ &\leq {}_L\mathcal{T}(\mu_1(x+z, t+s), \mu_2(y+u, t+s)) \\ &= \mu((x+z, y+u), t+s).\end{aligned}$$

Now if $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$, then for every $\varepsilon \in L \backslash \{0_\mathcal{L}, 1_\mathcal{L}\}$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for $m, n > n_0$,

$$\mu((x_n, y_n) - (x_m, y_m), t) >_L \mathcal{N}(\varepsilon).$$

So for $m, n > n_0$,

$$\begin{aligned}\mathcal{T}(\mu_1(x_n - x_m, t), \mu_2(y_n - y_m, t)) &= \mu((x_n - x_m, y_n - y_m), t) \\ &= \mu((x_n, y_n) - (x_m, y_m), t) \\ &> {}_L\mathcal{N}(\varepsilon).\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively and there exist $x \in V$ and $y \in W$ such that $x_n \to x$ and $y_n \to y$ and consequently $(x_n, y_n) \to (x, y)$. Hence $(X \times Y, \mu, \mathcal{T})$ is a complete $\mathcal{L}$-fuzzy normed space. □

4. Quotient spaces

Definition 10. *Let $(X, \mu, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space, K be a linear manifold in X and let $Q : X \to X/K$ be the natural map, $Qx = x + K$. We define*

$$\mu(x + K, t) = \sup\{\mu(x + y, t) : y \in K\}.$$

Theorem 3. *If K is a closed subspace of $\mathcal{L}$-fuzzy normed space X, $\mu(x + K, t)$ is defined as above then,*

(i) μ is an $\mathcal{L}$-fuzzy norm on X/K.
(ii) $\mu(Qx, t) \geq_L \mu(x, t)$.
(iii) If $(X, \mu, \mathcal{T})$ is $\mathcal{L}$-fuzzy Banach space, then so is $(X/K, \mu, \mathcal{T})$.

Proof. (i) Let $(X, \mu, \mathcal{T})$ be an $\mathcal{L}$-fuzzy normed space and K is a closed subspace of X. Let $m, n \in K$ and $x, y \in X$.

(a) Since $\mu(x + y, t) >_L 0_\mathcal{L}$, then $\mu(x + K, t) >_L 0_\mathcal{L}$ for all $x \in X$, $y \in K$ and $t > 0$.

(b) Let $\mu(x + K, t) = 1_\mathcal{L}$. By definition there exist a sequence $\{x_n\}$ in K such that $\mu(x + x_n, t)$ converges to $1_\mathcal{L}$. So $x + x_n$ converges to $0_\mathcal{L}$ or equivalently x_n converges to $(-x)$ and since K is closed so $x \in K$ and $x + K = K$, the zero element of X/K.

(c) Let $\alpha \neq 0$. Then,

$$\begin{aligned}\mu(\alpha(x + K), t) &= \mu(\alpha x + K, t) = \sup\{\mu(\alpha x + \alpha y, t) : y \in K\} \\ &= \sup\left\{\mu\left(x + y, \frac{t}{|\alpha|}\right) : y \in K\right\} = \mu\left(x + K, \frac{t}{|\alpha|}\right).\end{aligned}$$

(d) Let $m, n \in K$ and $x, y \in X$. Then,

$$\begin{aligned}\mu((x + K) + (y + K), t) &= \mu((x + y) + K, t) \\ &\geq {}_L\mu((x + m) + (y + n), t) \\ &\geq {}_L\mathcal{T}(\mu(x + m, t_1), \mu(y + n, t_2))\end{aligned}$$

for $t_1 + t_2 = t$. Taking sup on both sides, we have,

$$\mu\left((x+K)+(y+K),t\right) \geq_L \mathcal{T}(\mu\left(x+K,t_1\right),\mu\left(y+K,t_2\right)).$$

(e) Clearly, $\mu\left(x+K,_\right):(0,\infty)\to L$ is continuous.

(f) Also, $\lim_{t\to\infty}\mu(x,t)=1_{\mathcal{L}}$ is holds.

Therefore $(X/K,\mu,\mathcal{T})$ is an $\mathcal{L}$-fuzzy normed space.

(ii) $\mu(Qx,t)=\mu(x+K,t)=\sup\{\mu(x+y,t):y\in K\}\geq_L\mu(x,t)$.

(iii) Let $\{x_n+K\}$ be a cauchy sequence in X/K. then there exists $\varepsilon_n\in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ such that ε_n converges to $0_{\mathcal{L}}$ and,

$$\mu\left((x_n+K)-(x_{n+1}+K),t\right)\geq_L\mathcal{N}(\varepsilon_n).$$

Let $y_1=0$. We choose $y_2\in K$ such that,

$$\mu(x_1-(x_2-y_2),t)\geq_L\mathcal{T}(\mu\left((x_1-x_2)+K,t\right),\mathcal{N}(\varepsilon_1)).$$

But

$$\mu\left((x_1-x_2)+K,t\right)\geq_L\mathcal{N}(\varepsilon_1).$$

Therefore,

$$\mu(x_1-(x_2-y_2),t)\geq_L\mathcal{T}(\mathcal{N}(\varepsilon_1),\mathcal{N}(\varepsilon_1)).$$

Now suppose y_{n-1} has been chosen, $y_n\in K$ can be chosen such that

$$\mu((x_{n-1}+y_{n-1})-(x_n+y_n),t)\geq_L\mathcal{T}[\mu\left((x_{n-1}-x_n)+K,t\right),\mathcal{N}(\varepsilon_{n-1})],$$

and therefore,

$$\mu((x_{n-1}+y_{n-1})-(x_n+y_n),t)\geq_L\mu\mathcal{T}(\mathcal{N}(\varepsilon_{n-1}),\mathcal{N}(\varepsilon_{n-1})).$$

Thus, $\{x_n+y_n\}$ is a Cauchy sequence in X. Since X is complete, there is an x_0 in X such that $x_n+y_n\to x_0$ in X. On the other hand,

$$x_n+K=Q(x_n+y_n)\longrightarrow Q(x_0)=x_0+K.$$

Therefore every Cauchy sequence $\{x_n+K\}$ is convergent in X/K and so X/K is complete and $(X/K,\mu,\mathcal{T})$ is an $\mathcal{L}$-fuzzy Banach space. □

Theorem 4. *Let K be a closed subspace of $\mathcal{L}$-fuzzy normed space $(X,\mu,\mathcal{T})$. If a couple of the spaces X, K, X/K are complete, so is third one.*

Proof. If X is an $\mathcal{L}$-fuzzy Banach space, so are X/K and K. Therefore all that needs to be checked is that X is complete whenever both K and X/K are complete. Suppose K and X/K are $\mathcal{L}$-fuzzy Banach spaces and let $\{x_n\}$ be a Cauchy sequence in X. Since

$$\mu((x_n-x_m)+K,t)\geq_L\mu(x_n-x_m,t)$$

whenever $m,n\in\mathbb{N}$, the sequence $\{x_n+K\}$ is Cauchy in X/K and so converges to $y+K$ for some $y\in X$. So there exist a sequence $\{\varepsilon_n\}$ in $L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ such that $\varepsilon_n\to 0_{\mathcal{L}}$ and

$$\mu((x_n-y)+K,t)>_L\mathcal{N}(\varepsilon_n)$$

for each $t>0$. Now by Theorem 3 there exists a sequence $\{y_n\}$ in X such that $y_n+K=(x_n-y)+K$ and

$$\mu(y_n,t)>_L\mathcal{T}[\mu((x_n-y)+K,t),\mathcal{N}(\varepsilon_n)]\ .$$

So $\lim_n\mu(y_n,t)\geq_L 1_{\mathcal{L}}$ and $\lim_n y_n=0$. Therefore $\{x_n-y_n-y\}$ is a Cauchy sequence in K and thus is convergent to a point $z\in K$ and this implies that $\{x_n\}$ converges to $z+y$ and X is complete. □

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LIMITS OF ZEROS OF POLYNOMIAL SEQUENCES

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ABSTRACT. In the present paper we consider $F_k(x) = x^k - \sum_{t=0}^{k-1} x^t$, the characteristic polynomial of the k-th order Fibonacci sequence, the latter denoted $G(k,l)$. We determine the limits of the real roots of certain odd and even degree polynomials related to the derivatives and integrals of $F_k(x)$, that form infinite sequences of polynomials, of increasing degree. In particular, as $k \to \infty$, the limiting values of the zeros are determined, for both odd and even cases. It is also shown, in both cases, that the convergence is monotone for sufficiently large degree. We give an upper bound for the modulus of the complex zeros of the polynomials for each sequence. This gives a general solution related to problems considered by Dubeau 1989, 1993, Miles 1960, Flores 1967, Miller 1971 and later by the second author in the present paper, and Narayan 1997.

Primary: 11B39, Fibonacci number

1. INTRODUCTION

The current work arose from consideration of sequences of polynomials [11] related to the asymptotic behavior of their zeros. It is based on the following infinite sequence of polynomials denoted as $\{F_k(x)\}_{k=1}^{\infty}$ for convenience in the present paper which for $k \geq 2$, comprise the characteristic polynomials of the k-th order Fibonacci sequence, denoted by $G(k,l)$ where for $l > k \geq 2$,

$$G(k,l) = \sum_{t=1}^{k} G(k, l-t),$$

and $G(k,1) = 1,\ G(k,t) = 2^{t-2},\ t = 2,3,\ldots,k$. For $k = 2$ we obtain the well-known Fibonacci sequence, $\{1,1,2,3,5,8,\ldots,F_{n-1} + F_{n-2} = F_n,\ldots\}$.

It is also well-known that

$$\lim_{k\to\infty} \frac{G(k,l+1)}{G(k,l)} = \phi_k,\ k \geq 2,$$

where ϕ_k is the positive zero of F_k. Number theoretic results concerning $G(k,l)$ are in [10]. A fractal described by A. Dias, in A. Posamentier and I. Lehman's new book [14] was first published in [10]. The significance of this fractal with respect to the present paper is that the fractal dimension is $\ln(\phi_2)/\ln 2$.

Miles 1960, [12] showed that the zeros of the sequence of polynomials $\{F_k(x)\}, k \geq 2$ are distinct, all but one lies in the unit disk and the latter is real and lies in the interval $(1,2)$. Miller [13], 1971 gave a different, shorter proof of this result. Flores 1967, [3], showed that $\phi_k \to 2$ monotonically as $k \to +\infty$ as did Dubeau, [1], [2]. In [11] the

sequences $\{F_k^{'}(x)\}$ and $\{F_k^{''}(x)\}$ were studied and we reproduce the following table for understanding and motivation:

TABLE 1. Does Interval Contain a Root, yes or no?

int/fn	F_{2k}	F_{2k+1}	F'_{2k}	F'_{2k+1}	F''_{2k}	F''_{2k+1}
$(-1,0)$	yes	no	no	yes	yes $(k>1)$	no
$(0,1]$	no	no	yes $(k=1)$	yes $(k=1)$	yes $(k=2)$	yes $(k=1,2)$
$(1,2)$	yes	yes	yes $(k>1)$	yes $(k>1)$	yes $(k>2)$	yes $(k>2)$

For the particular particular cases we find that $F_3{}'(1) = 0, F_2'(1/2) = 0, \quad F_2'' = 2,\ F_3{}''(1/3) = 0, \quad F_5{}''(1) = 0, \quad F_4{}''((1+\sqrt{11/3})/4) = 0.$

Note that in table 1, the number of negative roots is either 0 or 1 for odd and even degree respectively, while there is always a positive root in $(1,2)$ (for sufficiently large degree.) It was indicated in [11] as an open question as to whether this happens for higher derivatives and conjectured in [4].

In [11] it was also shown that $\lim_{k\to\infty}\theta_k = -1$ where θ_k is the negative zero of each term in $\{F_{2k}\},\ k \geq 1$. Similarly, by examining approximations to zeros, the same asymptotic result was shown to hold for the sequences $\{F_k^{'}(x)\}$ and $\{F_k^{''}(x)\}$.

In [4] a conjecture was also made concerning the real zeros of the of l-th derivatives of each member of the sequence $\{F_k\}_{k=2}^{\infty}$. Namely, the zeros of $\{F_k^{(l)}\}_{k=2}^{\infty}$ exhibit the same (monotonic) behavior. A conjecture that the complex zeros are all within the unit circle was also made.

In this paper the question in [11] is answered, as are the first two questions of [4], affirmatively. The cases of the complex zeros is still open, although we obtain an upper bound. The present work also answers the same questions and yields similar results for the l-th integral of $\{F_k\}$.

In the present paper then, we consider the following sets of infinite sequences of polynomials given by,

$$U = \{\{F_1, F_2, \ldots,\},\ \{F_1', F_2', \ldots,\}, \{F_1'', F_2'', \ldots,\}, \ldots,\},$$

and,

$$V = \left\{\{F_1, F_2, \ldots,\}, \left\{\int F_1 dx, \int F_2 dx, \ldots,\right\}, \left\{\int\int F_1 dxdx, \int\int F_2 dxdx, \ldots,\right\}, \ldots,\right\}$$

where $F_1(x) = x - 1$ and

$$F_k(x) = x^k - \sum_{t=0}^{k-1} x^t, k \geq 2.$$

The sets $U,\ V$ are related to certain recurrence relations [5], [6] having solutions that lead to combinatorial identities. These recurrence relations result from a factorization of $F_k(x)$, with unknown coefficients. Several combinatorial identities are in [7], [8], [9],

for example it is shown in [9] that for any $c \neq -1, 0$

$$\begin{aligned} \frac{\frac{1}{c^{2(n+1)}} - 1}{1+c} &= \frac{1}{c^{n+2}} \sum_{i=1}^{n+1} \binom{n+i}{2i-1} \frac{(1-c)^{2i-1}}{c^{i-1}} \\ &= -\frac{1}{c} + \frac{1}{c^2} + \cdots + \frac{1}{c^{2(n+1)}}, \ n \geq 0. \end{aligned} \tag{1.1}$$

If $c \to -1$ in (1.1) one obtains,

$$2(n+1) = \sum_{i=1}^{n+1} \binom{n+i}{2i-1} 2^{2i-1} (-1)^{n+i+1},$$

which is equivalent to a result in G. Pólya and G. Szegö, [15].

The outline of the paper is as follows: in the next sections, §2.1, §2.2, we give the three main results with proofs supported in several lemmas. The first result deals with the set of derivatives U. The first and second derivative cases were treated in [11]; the second and third results deal with the set of integrals V. The second result deals with the first integral for which the proof leads to the general case and so is included for interest and clarity of exposition.

2. Results

2.1. U **or derivative case.** Now we consider the infinite sequence of polynomials $\{F_k^{(l)}(x)\}$ of the l-th derivative of the sequence $\{F_k(x)\}$.

Definition 2.1. We specify the following degree j polynomial $D_j(x)$ to correspond with the l-th derivative of $F_{j+l}(x)$.

$$D_j(x) = F_{j+l}^{(l)}(x) = l! \left(\binom{j+l}{l} x^j - \sum_{t=0}^{j-1} \binom{t+l}{l} x^t \right), \quad j \geq 1, \tag{2.1}$$

with $D_0(x) = l!$.

Lemma 2.1. *The l-th derivative of $F_k(x)$ is given by,*

$$D_{k-l}(x) = \frac{\sum_{t=0}^{l+1} (-1)^t a_t x^{k+1-t} + (-1)^l l!}{(x-1)^{l+1}}, \quad x \neq 1, \tag{2.2}$$

where each a_i is a degree l polynomial in k with positive leading coefficient.

Proof. We can write

$$F_k(x) = \frac{x^{k+1} - 2x^k + 1}{x - 1}.$$

We obtain the first derivative of $F_k(x)$ given by

$$F_k'(x) = \frac{kx^{k+1} - (3k-1)x^k + 2kx^{k-1} - 1}{(x-1)^2}, \quad x \neq 1.$$

Hence the statement is true for $l = 1$. Suppose the statement is true for $1 \leq l \leq j$. We have

$$D_{k-j}(x) = \frac{\sum_{t=0}^{j+1}(-1)^t a_t x^{k+1-t} + (-1)^j j!}{(x-1)^{j+1}}, \tag{2.3}$$

where each a_i is a degree j polynomial in k with positive leading coefficient.

We obtain the next derivative of (2.3):

$$\begin{aligned} D_{k-j-1}(x) &= \frac{\left(\sum_{t=0}^{j+1}(-1)^t a_t(k+1-t)x^{k-t}\right)(x-1)^{j+1}}{(x-1)^{2j+2}} \\ &- \frac{(j+1)(x-1)^j\left(\sum_{t=0}^{j+1}(-1)^t a_t x^{k+1-t} + (-1)^j j!\right)}{(x-1)^{2j+2}} \\ &= \frac{\sum_{t=0}^{j+2}(-1)^t b_t x^{k+1-t} + (-1)^{j+1}(j+1)!}{(x-1)^{j+2}}, \end{aligned} \tag{2.4}$$

where

$$b_0 = a_0(k+1) - a_0(j+1) = a_0(k-j).$$

For $1 \leq t \leq j+1$, we obtain by comparing the coefficients of like powers of x in (2.4)

$$\begin{aligned} b_t &= a_t(k+1-t) + a_{t-1}(k+2-t) - (j+1)a_t \\ &= a_t(k-j-t) - a_{t-1}(k+2-t), \end{aligned}$$

and

$$b_{j+2} = a_{j+1}(k-j).$$

Hence the lemma follows. □

Lemma 2.2. *If $k-l$ is odd, then D_{k-l} has one positive root and no negative root. If $k-l$ is even, then $D_{k-l}(x)$ has one positive root and one negative root.*

Proof. Suppose $k-l$ is odd; if k is even then l is odd. From (2.2), with $-x \leftarrow x$, the numerator of $D_{k-l}(x)$ can be written

$$\sum_{t=0}^{l+1}(-1)^t a_t(-x)^{k+1-t} + (-1)^l l! = -\sum_{t=0}^{l+1} a_t x^{k+1-t} - l!. \tag{2.5}$$

If k is odd, then l is even, and

$$\sum_{t=0}^{l+1}(-1)^t a_t(-x)^{k+1-t} + (-1)^l l! = \sum_{t=0}^{l+1} a_t x^{k+1-t} + l!. \tag{2.6}$$

By inspection of (2.5), (2.6) and employing Descartes' rule, $D_{k-l}(x)$ has no negative roots. Suppose $k-l$ is even; if k is even then l is even, and,

$$\sum_{t=0}^{l+1}(-1)^t a_t(-x)^{k+1-t} + (-1)^l l! = -\sum_{t=0}^{l+1} a_t x^{k+1-t} + l!.$$

If k is odd then l is odd, and

$$\sum_{t=0}^{l+1}(-1)^t a_t(-x)^{k+1-t} + (-1)^l l! = \sum_{t=0}^{l+1} a_t x^{k+1-t} - l!.$$

By similar argument $D_{k-l}(x)$ has one negative root. Taking the l-th derivative of (2.1), it is easy to see by Descartes' rule that $D_{k-l}(x)$ has exactly one positive root. □

Denote by u_k the positive root of $D_k(x)$; for k even, denote by v_k the negative root of $D_k(x)$.

Theorem 2.1. *We have the following results for the set U and fixed l:*

(1) *Let $j = k - l$. Then*

$$\lim_{j\to\infty} u_j = 2.$$

All of the other complex roots of $D_j(x)$ are inside of $|z| < u_j$. For j even, we have

$$\lim_{j\to\infty} v_j = -1$$

(2) *If j is odd, then $D_j(x)$ has one positive root and no negative root. If j is even, then $D_j(x)$ has one positive root and one negative root.*
(3) *For $j \geq 2$, we have $u_{j+1} > u_j$.*
(4) *There exists a even number N_0, such that for even $n > N_0$, we have $v_{n+2} < v_n$.*

Proof. This theorem is proved by the following lemmas 2.2–2.5. □

Remark 2.1. The corresponding theorem has been proved in [11] for the first derivative and second derivative cases, .

Lemma 2.3. *Let $j = k - l$, fixed l. Then the positive roots u_j satisfy*

$$\lim_{j\to\infty} u_j = 2.$$

All of the the other complex roots of $D_j(x)$ are inside of open disk $|z| < u_j$. For j even, the negative roots v_j satisfy,

$$\lim_{j\to\infty} v_j = -1.$$

Proof. We have from (2.1)

$$D_{k-l}(x) - D_{k-1-l}(x) = (k-1)\cdots(k-l+1)x^{k-l-1}\left((x-2)k + 2l\right). \tag{2.7}$$

It follows that for any a, $1 < a < 2$,

$$\lim_{k\to\infty} D_{k-l}(a) - D_{k-1-l}(a) = -\infty.$$

Hence for any a, $1 < a < 2$, we have

$$\lim_{k\to\infty} D_{k-l}(a) = -\infty.$$

It is easy to see from (2.7) that

$$\lim_{k\to\infty} D_{k-l}(2) = \infty.$$

Hence by the intermediate value theorem, $1 < u_j < 2$ for all $j \geq j_0$ for sufficiently large j_0.

$$\lim_{j\to\infty} u_j = 2.$$

For j even, we have from (2.1)

$$D_{k-l}(x) - D_{k-l-2}(x) = (k-2)\cdots(k-l+1)x^{k-l-2}h_k(x), \tag{2.8}$$

where

$$h_k(x) = (x^2-x-2)k^2 + (-x^2+(l+1)x+2(2l+1))k - lx - 2l(l+1). \tag{2.9}$$

Hence if $a \leq -1$, we have from (2.8), (2.9),

$$\lim_{k\to\infty} (D_{k-l}(a) - D_{k-l-2}(a)) = \infty.$$

For sufficiently large k, if $-1 < a < 0$, we have

$$D_{k-l}(a) - D_{k-l-2}(a) < 0.$$

Hence for j even, we have

$$\lim_{j\to\infty} v_j = -1.$$

Notice for $j = k - l > 0$, we have

$$D_j(x) = k(k-1)\cdots(k-l+1)x^{k-l} - \sum_{s=l}^{k-1} s(s-1)\cdots(s-l-1)x^{s-l}. \tag{2.10}$$

Let $x_0 = \rho e^{i\theta}$ be a complex zero of $D_j(x)$. By applying triangle inequality to (2.10), we get $D_j(\rho) \leq 0$. We know that $D_j(x) < 0$ if $0 \leq x \leq u_j$ and $D_j(x) > 0$ if $x > u_j$. Since $\rho > 0$, we get $0 < \rho < u_j$. □

Lemma 2.4. *For $k \geq 2$, we have $u_{k+1} > u_k$.*

Proof. Solving

$$D_k(x) - D_{k-1}(x) = l!\binom{k+l}{l}x^k - 2l!\binom{k+l-1}{l}x^{k-1} = 0,$$

we get

$$x_k = \frac{2k}{k+l} = 2 - \frac{2l}{k+l}$$

Hence x_k converges monotonically to 2. We calculate

$$\begin{aligned} D_2(x_2) &= l!\left(\frac{(l+2)(l+1)}{2}\frac{4^2}{(l+2)^2} - (l+1)\frac{4}{l+2)} - 1\right) \\ &= l!\left(\frac{4(l+1}{l+2} - 1\right) \\ &= l!\frac{3l+2}{l+2} \\ &> 0. \end{aligned} \tag{2.11}$$

Since $x_3 > x_2$, we obtain

$$D_3(x_3) = D_2(x_3) > D_2(x_2) > 0. \tag{2.12}$$

Hence $u_3 > u_2$. Inductively, we get $u_{k+1} > u_k$. □

Lemma 2.5. *There exists an even number N_0, such that for even $n > N_0$, we have $v_{n+2} < v_n$.*

Proof. Solving

$$D_k(x) - D_{k-2}(x) = (k+l-2)\cdots(k+1)g_k(x) = 0, \tag{2.13}$$

where

$$g_k(x) = (k+l)(k+l-1)x^2 - k(k+l-1)x - 2k(k-1),$$

we get the negative root of (2.13)

$$x_k = \frac{k}{2(k+l)} - \frac{1}{2}\sqrt{\left(\frac{k}{k+l}\right)^2 + \frac{8k(k-1)}{(k+l)(k+l-1)}} \tag{2.14}$$

Consider the following function derived from (2.14)

$$f(x) = \frac{1}{2}(1-lx) - \frac{1}{2}\sqrt{(1-lx)^2 + \frac{8(1-lx)\,(1-(l+1)x)}{(1-x)}} \tag{2.15}$$

we find that

$$f'(0) = \frac{5l}{2} > 0.$$

Hence $f(x)$ is increasing on a neighborhood V of 0.

Since

$$\frac{1}{k+l} > \frac{1}{k+l+2},$$

we get

$$x_{k+2} = f\left(1/(k+l+2)\right) < f\left(1/(k+l)\right) = x_k.$$

First we claim that there exists a sufficiently large even number k_0, such that $v_{k_0} < v_{k_0-2}$. Otherwise, suppose there exists a j_0, such that for all even number $j > j_0$, $v_{j+2} \geq v_j$. Since $D_k(-1) \to \infty$ as $k \to \infty$, this contradicts the fact $\lim_{j\to\infty} v_j = -1$. Hence there exists an even number k_0, such that $v_{k_0} < v_{k_0-2}$. It follows that

$$D_{k_0}(x_{k_0}) > 0.$$

Otherwise, we have $v_{k_0} > v_{k_0-2}$, a contradiction. Since $x_{k_0+2} < x_{k_0}$, we get $D_{k_0+2}(x_{k_0+2}) = D_{k_0}(x_{k_0}) > 0$. It follows that $v_{k_0+2} < v_{k_0}$. Notice $\{x_k\}$ decreases to -1 also. Inductively, we have that $v_{k+2} < v_k$ for k sufficiently large and even. □

2.2. V or integral case.

2.2.1. *First Integral Case.* Now we consider the infinite sequence of polynomials $\{\int F_k(x)\}$ of the first integral of the sequence $\{F_k(x)\}$.

Definition 2.2. We specify the following degree $j+1$ polynomial $I_j(x)$ to correspond with the first integral of $F_j(x)$.

$$I_j(x) = \int F_j(x) = \frac{x^{j+1}}{j+1} - \frac{x^j}{j} - \cdots - x - 1, \tag{2.16}$$

for all $j \geq 1$.

Theorem 2.2. *The roots of $I_k(x)$ satisfy the following properties,*

(1) *$I_k(x)$ has a positive simple root ϕ_k satisfying $2 < \phi_k < 3$.*
(2) *For $k \geq 2$, we have $\phi_{k+1} < \phi_k$.*
(3)
$$\lim_{j\to\infty} \phi_k = 2.$$
(4) *If k is odd, then*
 (a) *$I_k(x)$ has a negative simple root θ_k satisfying $-2 < \theta_k < -1$.*
 (b)
$$\lim_{j\to\infty} \theta_k = -1.$$
 (c) *$\theta_k > \theta_{k-2}$ for $k \geq 17$.*
(5) *For k even, $I_k(x)$ has no negative root.*

Proof. We prove this theorem in the following lemmas 2.6–2.10. □

Lemma 2.6. *$I_k(x)$ has a positive simple root ϕ_k satisfying $2 < \phi_k < 3$. If k is odd, then $I_k(x)$ has a negative simple root θ_k satisfies $-2 < \theta_k < -1$.*

Proof. From Descartes' Rule, we get that the number of possible positive roots for each $I_k(x)$ is 1. If $a = 2$,
$$I_1(2) = \frac{2^2}{2} - 2 - 1 = -1 < 0$$
We find that for $k > 1$,
$$I_k(2) - I_{k-1}(2) = \frac{-2^{k+1}}{k(k+1)} < 0$$
Then for all $k \geq 1$, we have $I_k(2) < 0$. Hence the positive root $\phi_k > 2$. If $a = 3$, then
$$I_1(3) = \frac{3^2}{2} - 3 - 1 = \frac{1}{2} > 0$$
We have that for $k \geq 2$,
$$I_k(3) - I_{k-1}(3) = \frac{3^k(k-2)}{k(k+1)} \geq 0 \tag{2.17}$$
Then for all $k \geq 1$, we have $I_k(3) > 0$. Hence the positive root ϕ_k satisfies $2 < \phi_k < 3$. If $I_k'(\phi_k) = 0$, then by [13], $1 < \phi_k < 2$. Hence $I_k'(\phi_k) \neq 0$. Therefore, ϕ_k is a simple root of $I_k(x)$. For k odd, we can get that the number of variation for signs $I_k(-x)$ is k. Then by Descartes' Rule, we know the possible number of negative roots for $I_k(x)$ is $k, k-2, \ldots, k-2t, \ldots, 1$. By [13], we know $I_k'(x)$ only has one real root b_k for k odd. It follows that $I_k(x)$ is increasing if $x > b_k$ and decreasing if $x < b_k$. Hence we get the number of negative real roots for $I_k(x)$ is 1.
If $k = 1$ then
$$I_1(-1) = -1 + 1 + \frac{1}{2} = \frac{1}{2} > 0.$$
If $k = 3$ then
$$I_3(-1) = -1 + 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 0.$$

If $k = 5$ then

$$I_5(-1) = -1 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = -\frac{1}{20} < 0. \tag{2.18}$$

If $k > 5$ and k is odd then

$$I_k(-1) = \frac{1}{k+1} + \sum_{l=1}^{k} \frac{(-1)^{l-1}}{l} - 1 \leq -1 + \ln(2) + \frac{2}{k+1} < 0 \tag{2.19}$$

For k odd, from [13], $I_k'(x) < 0$ for $x < 0$, so $I_k(x)$ is decreasing for $x < 0$. Hence from (2.18)and (2.19) for all $k \geq 5$ and k odd, the negative real root θ_k of $I_k(x)$. satisfying $\theta_k < -1$.
Next we show $-2 < \theta_k$. From (2.16), we obtain

$$I_k(x) - I_{k-2}(x) = \frac{x^{k+1}}{k+1} - \frac{x^k}{k} - 2\frac{x^{k-1}}{k-1}.$$

Solving

$$\frac{x^2}{k+1} - \frac{x}{k} - \frac{2}{k-1} = 0, \tag{2.20}$$

yields the negative root,

$$x_{k1} = \frac{1}{k} - \sqrt{\left(\frac{1}{k}\right)^2 + \frac{8}{(k+1)(k-1)}}\frac{k+1}{2}, \tag{2.21}$$

It can be shown by direct calculation that for k odd and $k \geq 7$, $-1 > x_{k1} > -2$.
That implies that for k odd and $k \geq 7$,

$$I_k(-2) - I_{k-2}(-2) > 0 \tag{2.22}$$

We know

$$I_5(-2) = \frac{221}{15} > 0.$$

Hence for all $k \geq 5$ and k odd, we have $I_k(-2) > 0$. Therefore we get $-2 < \theta_k < -1$.
If $I_k'(\theta_k) = 0$, then by [13], $-1 < \theta_k < 0$. Hence $I_k'(\theta_k) \neq 0$. It follows that θ_k is a simple root of $I_k(x)$. □

Lemma 2.7. *Let ϕ_k be the positive root of $I_k(x)$. Then for $k \geq 2$, we have $\phi_{k+1} < \phi_k$.*

Proof. Denoted by b_k the positive real root of $I_k'(x)$. By [13], we know $1 < b_k < 2$. Hence $I_i(x)$, $i \geq 2$, is increasing if $x > 2$. It's easy to see that $I_i > I_{i-1}$ if $x > 2 + \frac{2}{i}$ and $I_i < I_{i-1}$ if $x < 2 + \frac{2}{i}$. Notice $2 + \frac{2}{i}$ converges to 2 decreasingly. From $I_3(2 + 2/3) < 0$, we get $\phi_3 < \phi_2$. Suppose for all $2 < i \leq k$, we have $I_i(2 + 2/i) < 0$. Then since $I_{k-1}(2 + 2/k) = I_k(2 + 2/k) < 0$ and I_{k+1} is increasing if $x > 2 + 2/(k + 1)$, we get $I_{k+1}(2 + 2/(k + 1)) < 0$. We know $I_{k+1} > I_k$ if $2 + 2/(k + 1) < x < 3$. We get $I_k(\phi_{k+1}) < 0$. Hence for $k \geq 2$, $\phi_{k+1} < \phi_k$. □

Lemma 2.8.

$$\lim_{k\to\infty} \phi_k = 2.$$

Proof. For any k,

$$I_k(x) - I_{k-1}(x) = x^k\left(\frac{x}{k+1} - \frac{2}{k}\right) = \frac{x^k[(x-2)k-2]}{k(k+1)}. \tag{2.23}$$

If $a > 2$, then for sufficiently large k,

$$(a-2)k - 2 > 1. \tag{2.24}$$

We know

$$\lim_{k\to\infty} \frac{x^k}{k(k+1)} = \infty. \tag{2.25}$$

Hence employing (2.24), (2.25) in (2.23), for any $a > 2$, yields

$$\lim_{k\to\infty} I_k(a) - I_{k-1}(a) = \infty. \tag{2.26}$$

Notice for any $k > 2$,

$$I_k(x) = \sum_{l=3}^{k} (I_l(x) - I_{l-1}(x)) + I_2(x). \tag{2.27}$$

It follows from (2.26), (2.27) that for any $a > 2$,

$$\lim_{k\to\infty} I_k(a) = \infty.$$

If $a = 2$, we have

$$I_k(2) - I_{k-1}(2) = \frac{-2.2^k}{k(k+1)}.$$

Then by a similar argument as above,

$$\lim_{k\to\infty} I_k(2) = -\infty.$$

By the Mean Value Theorem, we obtain

$$\lim_{k\to\infty} \phi_k = 2.$$

□

Lemma 2.9. *Let k be a odd number and θ_k be the negative root of $I_k(x)$. Then*

$$\lim_{k\to\infty} \theta_k = -1. \tag{2.28}$$

Moreover, for $k > 17$ and k is odd, $\theta_k > \theta_{k-2}$.

Proof. For $a < -1$, we have from (2.16)

$$I_k(a) - I_{k-2}(a) = a^{k-1}\left(\frac{a^2}{k+1} - \frac{a}{k} - \frac{2}{k-1}\right) = a^{k-1}\frac{k^2(a^2-a-2) - (a^2+2)k + a}{(k+1)k(k-1)} \tag{2.29}$$

For $a < -1$ and k sufficiently large, we have from (2.29)

$$k^2(a^2 - a - 2) - (a^2 + 2)k + a > 1$$

Since for $a < -1$ and k odd, by similar argument as lemma 2.8,

$$\lim_{k\to\infty} \frac{a^{k-1}}{(k+1)k(k-1)} = \infty,$$

we get

$$\lim_{k\to\infty} I_k(a) - I_{k-2}(a) = \infty.$$

Then by writing $I_k(x)$ as telescoping sum, $a < -1$, k odd, it follows that

$$\lim_{k\to\infty} I_k(a) = \infty. \tag{2.30}$$

Substituting $a = -1$ in (2.16) gives

$$I_k(-1) = \frac{(-1)^{k+1}}{k+1} + H_k(-1) - 1,$$

where $H_k(x)$ is the standard alternating sum.
Hence

$$\lim_{k\to\infty} I_k(-1) = \ln(2) - 1 < 0. \tag{2.31}$$

It follows that from Mean Value Theorem, (2.30), (2.31),

$$\lim_{k\to\infty} \theta_k = -1.$$

A calculator check with $k = 17$ in (2.21) yields

$$I_k(x_{k1}) = -0.0337812682 < 0.$$

From (2.21) we write

$$f(x) = -\sqrt{(1+x)^2 + 8\frac{1+x}{1-x}} + (1+x)$$

Taking the derivative of $f(x)$ gives

$$f'(x) = -\frac{1}{2}\frac{2(1+x) + 16/(1-x)^2}{\sqrt{(1+x)^2 + 8(1+x)/(1-x)}} + 1$$

It's easy to check that for $0 < x < 1$, $f'(x) < 0$. $f(x)$ is decreasing for $0 < x < 1$. Since $1/(k+2) < 1/k$, we get for $k \geq 7$,

$$x_k = \frac{\frac{1}{k} - \sqrt{\left(\frac{1}{k}\right)^2 + \frac{8}{(k+1)(k-1)}}}{\frac{2}{k+1}} < \frac{\frac{1}{k+2} - \sqrt{\left(\frac{1}{k+2}\right)^2 + \frac{8}{(k+3)(k+1)}}}{\frac{2}{k+3}} = x_{k+2}. \tag{2.32}$$

Hence x_k increases to -1. Denote by θ_k the negative real root of $I_k(x)$. Since $I_{17}(x_{17}) < 0$, we get $\theta_{17} < x_{17}$. It follows that $I_{19}(\theta_{17}) > 0$ since $I_{19}(x) > I_{17}(x)$ when $x < x_{17}$. Hence $\theta_{19} > \theta_{17}$; it follows that $\theta_k > \theta_{k-2}$ for $k \geq 17$. □

It is noted that for $1 < a < 2$, using similar methods, we can get

$$\lim_{k\to\infty} I_k(a) = -\infty.$$

Lemma 2.10. *For k even, the integral $I_k(x)$, (2.16), has no negative root.*

Proof. Let $k = 2l$, $x = -a$ for $0 < a < 1$. By rewriting (2.16) we get

$$I_k(x) = -\frac{a^{2l+1}}{2l+1} - \frac{a^{2l}}{2l} + \sum_{t=2}^{l} a^{2(t-1)}\left(\frac{a}{2t-1} - \frac{1}{2t-2}\right) + a - 1 < 0. \tag{2.33}$$

Hence, for k even, $I_k(x)$ has no negative root on $-1 < x < 0$. It is easy to check that $I_k(-1) < 0$.
By [13], for k even, $I_k'(x)$ has a negative root r_k satisfying $-1 < r_k < 0$. Hence $I_k(x)$ is increasing on $-\infty < x < -1$ so that for k even $I_k(x) < 0$. Therefore, for k even, $I_k(x)$ has no negative root. □

Lemma 2.11. *For any $k \geq 2$, the complex zeros of $I_k(z)$ satisfy the inequality $|z| < \phi_k < 3$.*

Proof. Let $z_0 = re^{i\theta}$ be a complex root of $I_k(z)$. Using the triangle inequality we obtain

$$I_k(r) \leq 0. \tag{2.34}$$

Note that equality holds only at $\theta = 0$, i.e $z_0 = \phi_k$. Since $I_k(x) < 0$ for $0 < x < \phi_k < 3$ and x real, we get $r < \phi_k < 3$. □

Lemma 2.12. *If $-1 < a < 1$, then*

$$|I_k(x)| \leq \frac{1}{1-|x|}.$$

Moreover,

$$\lim_{k\to\infty} I_k(x) = -1 + \ln(1-x).$$

Proof. If $-1 < a < 1$, then

$$\begin{aligned} |I_k(x)| &\leq \sum_{l=0}^{k}\left|\frac{x^{l+1}}{l+1}\right| + 1 \\ &\leq \sum_{l=0}^{k}\left|x^{l+1}\right| + 1 \\ &\leq \frac{1}{1-|x|}. \end{aligned}$$

The Taylor series expansion for $I_k(x)$ with $-1 < x < 1$, yields

$$\lim_{k\to\infty} I_k(x) = -1 + \ln(1-x).$$

□

2.2.2. *General Case.* Now we consider the infinite sequence of polynomials $\left\{\overbrace{\int\int\cdots\int}^{l+2} F_k(x)\right\}$ of the $(l+2)$-th integral of the sequence $\{F_k(x)\}$.

Definition 2.3. For $0 < l < k$, We specify the following degree $k+1$ polynomial $H_k(x)$ to correspond with the $(l+2)$-th integral of $F_{k-l-1}(x)$.

$$\begin{aligned} H_k(x) &= \overbrace{\int\int\cdots\int}^{l+2} F_{k-l-1}(x) \\ &= \frac{x^{k+1}}{(l+2)!\binom{k+1}{l+2}} - \sum_{t=l+2}^{k} \frac{x^t}{(l+2)!\binom{t}{l+2}} - \sum_{s=0}^{l+1} \frac{x^s}{s!} \end{aligned} \tag{2.35}$$

Let α_k be the positive root of $H_k(x)$. For k odd, denote by β_k the negative real root of $H_k(x)$.

We have the following

Theorem 2.3. *The roots of $H_k(x)$ satisfy the following properties,*

(1)
$$\lim_{k\to\infty} \alpha_k = 2.$$
Except α_k, all the other complex roots are inside $\{z : |z| < \alpha_k\}$. For k odd, we have
$$\lim_{k\to\infty} \beta_k = -1.$$
(2) *For sufficiently large even k, for any $x < 0$, $H_k(x) < 0$, i.e $H_k(x)$ has no negative real roots.*
(3) *For sufficiently large odd k, for any $x < 0$, $H_k(x)$ has one negative root.*
(4) $\alpha_{j+1} < \alpha_j$, $\forall j \geq l+3$,
(5) *there exists odd N_0, such that for all odd $n \geq N_0$, we have $\beta_{n+2} > \beta_n$.*

Proof. The theorem is proved using lemmas 2.13–2.17. □

Lemma 2.13.
$$\lim_{k\to\infty} \alpha_k = 2.$$
Except α_k, the other complex roots are inside $\{z : |z| < \alpha_k\}$. For k odd, we have
$$\lim_{k\to\infty} \beta_k = -1.$$

Proof. The proof uses similar idea as the previous section with some differences, we include for completeness.

$$H_k(x) - H_{k-1}(x) = \frac{x^k}{(l+3)!\binom{k+1}{l+3}} \left((x-2)k - lx - x - 2)\right). \tag{2.36}$$

It follows that for $a > 2$,

$$\lim_{k\to\infty} (H_k(a) - H_{k-1}(a)) = \infty.$$

Hence for $a > 2$,

$$\lim_{k\to\infty} H_k(a) = \infty.$$

It's easy to prove that

$$\lim_{k\to\infty} H_k(2) = -\infty.$$

Hence,

$$\lim_{k\to\infty} \alpha_k = 2.$$

Let $z = re^{i\theta}$. Then by triangle inequality,

$$H_k(r) \le 0 \tag{2.37}$$

Equality in (2.37) holds only at $\theta = 0$; it follows that $r < \alpha_k$. Since $z = 0$ is not the root of $H_k(z)$, we have $0 < r < \alpha_k$.
If k is odd, then

$$H_k(x) - H_{k-2}(x) = \frac{x^{k-1}}{(l+4)!\binom{k+1}{l+4}} h_k(x), \tag{2.38}$$

where

$$h_k(x) = (x^2 - x - 2)k^2 - \left((2l+3)x^2 + (l+1)x + 2\right)k + \left((l+1)(l+2)x^2 + (l+2)x\right). \tag{2.39}$$

Hence, if $a < -1$ and k odd, employing (2.38), (2.39) we have

$$\lim_{k\to\infty} (H_k(a) - H_{k-2}(a)) = \infty. \tag{2.40}$$

It follows from (2.40) that

$$\lim_{k\to\infty} H_k(a) = \infty. \tag{2.41}$$

For k odd, it is easy to see from (2.35) that for sufficiently large k,

$$H_k(-1) < 0 \tag{2.42}$$

Denote by β_k the negative real root of $H_k(x)$. We have from (2.41), (2.42)

$$\lim_{k\to\infty} \beta_k = -1.$$

□

Lemma 2.14. *For sufficiently large even k, for any $x < 0$, $H_k(x) < 0$.*

Proof. This result was shown for the first integral ($l = -1$ in (2.35)) in lemma 2.10. Now we consider the case $l \ge 0$ in (2.35).
For k and l both even, we obtain

$$H_k'(-1) = A + B + C - 1,$$

where

$$A = \frac{1}{(l+1)!\binom{k}{l+1}} + \frac{1}{(l+1)!\binom{k-1}{l+1}},$$

$$B = \sum_{d=l+3}^{k-2} \frac{l+1}{(l+2)!\binom{d}{l+2}},$$

$$C = \sum_{s=1}^{l/2} \frac{2s-1}{(2s)!}.$$

We note that $H_k'(-1) < 0$. Since$A > 0$, $B > 0$, and $C \geq \frac{1}{2}$, this implies for k sufficiently large even k and l even,

$$|H_k'(-1)| < \frac{1}{2}. \tag{2.43}$$

The same result (2.43) holds with a similar proof in the case of odd l and for sufficiently large even k. Let θ_k be the negative root of $H_k'(x)$ and let γ_k be the negative root of $H_k^{(3)}(x)$. We know from lemma 2.13,

$$\lim_{k\to\infty} \theta_k = -1, \ \lim_{k\to\infty} \gamma_k = -1. \tag{2.44}$$

Notice

$$H_k(\theta_k) = \int_0^{\theta_k} H_k'(x)dx - 1$$

and $H_k'(x)$ is decreasing on $x < 0$ so $|H_k'(x)| < 1$ since $H_k'(0) = -1$. $H_k'(x)$ is concave down on $\gamma_k < x < 0$ since $H_k^{(3)}(x) < 0$ on $\gamma_k < x < 0$.
Hence for sufficiently large even k, if $\theta_k < \gamma_k$, we obtain

$$\begin{aligned} H_k(\theta_k) &= \int_{\gamma_k}^{\theta_k} H_k'(x)dx + \int_0^{\gamma_k} H_k'(x)dx - 1 \\ &< |\theta_k - \gamma_k| + \frac{1}{2}\left(|H_k'(\gamma_k)| + 1\right) - 1. \end{aligned} \tag{2.45}$$

For $l = 0$, we know that $-2 < \theta_k < -1$ and $-1 < \gamma_k < 0$. Write $\gamma_k = -a_k$ and write $k = 2t$. Then by taking the derivative of (2.35),

$$H_k'(\gamma_k) = -1 + \sum_{s=1}^{t-1} {a_k}^{2s-1}\left(\frac{1}{2s-1} - \frac{a_k}{2s}\right) + \frac{a_k^{2t-1}}{2t-1} + \frac{a_k^{2t}}{2t}. \tag{2.46}$$

Hence since $-1 < H_k'(\gamma_k) < 0$ and by inspection of (2.46)

$$|H_k'(\gamma_k)| < \left|-1 + a_k - \frac{{a_k}^2}{2}\right| < 1. \tag{2.47}$$

It follows for sufficiently large k from (2.45),(2.47)

$$H_k(\theta_k) < 0.$$

Therefore $H_k(x) < 0$ for all $x < 0$.
For $l > 0$, we have $\gamma_k < -1$. If $\theta_k < \gamma_k$, then $|H_k'(\gamma_k)| < |H_k'(-1)| < \frac{1}{2}$ in (2.43). Hence for sufficiently large even k, we get

$$H_k(x) < 0. \tag{2.48}$$

If $\gamma_k < \theta_k$, then

$$H_k(\theta_k) < \frac{1}{2}|\theta_k| - 1 < 0. \tag{2.49}$$

It follows for sufficiently large even k, we get

$$H_k(x) < 0.$$

□

Lemma 2.15. *For sufficiently large odd k, for any $x < 0$, $H_k(x)$ has exactly one negative root.*

Proof. By Lemma 2.14, we know $H_k'(x) < 0$ for $x < 0$. Hence $H_k(x)$ is decreasing on $(-\infty, 0)$. Since $H_k(0) = -1$ and $\lim_{x\to-\infty} H_k(x) = \infty$, we get that $H_k(x)$ has only one root on $(-\infty, 0)$. □

Now we study the monotonicity of the positive root α_k of $H_k(x)$ in the following

Lemma 2.16. *For all $j \geq l + 3$, where $l \geq -1$ is a fixed integer, we have $\alpha_{j+1} < \alpha_j$.*

Proof. Solving for the zero of (2.36) for $k = l+3$ yields the intersection point $x = l+4 = k+1$. Next we show $H_k(x) < 0$ at the intersection point $x = k + 1$,

$$\begin{aligned}
H_k(k+1) &= \frac{(k+1)^{k+1}}{(k-1)!\binom{k+1}{k-1}} - \sum_{t=1}^{k} \frac{(k+1)^t}{t!} - 1 \\
&= \frac{(k+1)^k}{k!} - \sum_{t=1}^{k-1} \frac{(k+1)^t}{t!} - 1 \\
&= \frac{(k+1)^{k-1}}{(k-1)!}\left(\frac{k+1}{k} - 1\right) - \sum_{t=1}^{k-2} \frac{(k+1)^t}{t!} - 1 \\
&= \frac{(k+1)^{k-1}}{k!} - \sum_{t=1}^{k-2} \frac{(k+1)^t}{t!} - 1 \\
&= \frac{(k+1)^{k-2}}{(k-2)!}\left(\frac{-k^2+2k+1}{k(k-1)}\right) - \sum_{t=1}^{k-3} \frac{(k+1)^t}{t!} - 1 \\
&< 0.
\end{aligned} \tag{2.50}$$

The lemma follows the similar argument as lemma (2.4). □

We now consider the monotonicity of the negative root β_k of $H_k(x)$ in the following

Lemma 2.17. *There exists odd N_0, such that for all odd $n \geq N_0$, we have $\beta_{n+2} > \beta_n$.*

Proof. Solving the zero of (2.38) we get the negative real root

$$x_k = \frac{1}{2}\left(\frac{k+1}{k-l-1} - \sqrt{\frac{(k+1)^2}{(k-l-1)^2} + \frac{8(k+1)k}{(k-l-1)(k-l-2)}}\right). \tag{2.51}$$

We consider the function derived from (2.51)

$$f(x) = 1 + (l+2)x - \sqrt{(1+(l+2)x)^2 + 8\frac{(1+(l+2)x)\,(1+(l+1)x)}{1-x}}. \tag{2.52}$$

Taking the derivative of $f(x)$ gives

(2.53)

$$f'(x) = (l+2) - \frac{A+B+C}{2}\left(\sqrt{(1+(l+2)x)^2 + 8\frac{(1+(l+2)x)\,(1+(l+1)x)}{1-x}}\right)^{-1},$$

where

$$A = 2\,(1+(l+2)x)\,(l+2),$$

$$B = 8(l+2)\frac{1+(l+1)x}{1-x},$$

$$C = 8\,(1+(l+2)x)\,\frac{(l+1)(1-x)+(1+(l+1)x)}{(1-x)^2}.$$

Substituting $x = 0$ in (2.53) gives

$$f'(0) = -2(l+2) < 0. \tag{2.54}$$

It follows from (2.54) that there exists a neighborhood V of 0, such that $f(x)$ is decreasing on V.
Since

$$\frac{1}{k-l-1} > \frac{1}{k+2-l-1}, \tag{2.55}$$

we have

$$x_k = \frac{1}{2}f\left(1/(k-l-1)\right) < \frac{1}{2}f\left(1/(k+2-l-1)\right) = x_{k+2}. \tag{2.56}$$

It's easy to see that

$$\lim_{k\to\infty} x_k = -1. \tag{2.57}$$

We claim that there exists a sufficiently large odd number j_0, such that $\beta_{j_0+2} > \beta_{j_0}$. Otherwise, suppose there exists a k_0, such that for all odd number $n > k_0$, we always have $\beta_{n+2} \le \beta_n$. This contradicts the fact $\lim_{k\to\infty}\beta_k = -1$. It follows that

$$H_{j_0}(x_{j_0}) < 0. \tag{2.58}$$

Otherwise suppose $H_{j_0}(x_{j_0}) > 0$. Since $H_j(x)$ is decreasing on $x < 0$, we get $\beta_{j_0} > x_{j_0}$. Since $H_{j_0+2}(\beta_{j_0}) < 0$, we get $\beta_{j_0+2} < \beta_{j_0}$, a contradiction. Then the lemma follows the similar arguments as lemma 2.9. □

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The Functional Inequalities related to a Derivation and a Generalized Derivation

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Abstract

In this article, we investigate the functional inequalities concerned with a linear derivation and a generalized derivation.

1 Introduction

In 1940, the following problem was raised by Ulam [12] during a his talk in the Mathematics Colloquium at the University of Wisconsin:

Let G_1 be a group and G_2 a group with metric d. Then, for any $\epsilon > 0$ and a mapping $f : G_1 \to G_2$, is it possible to find a positive constant δ and a homomorphism $h : G_1 \to G_2$ such that $d(f(x), h(x)) < \epsilon$ for $x \in G_1$, if $d(f(xy), f(x)f(y)) < \delta$ for any $x, y \in G_1$?.

In the following year, for Banach spaces, the problem was first solved by Hyers [5] which states that if $\varepsilon > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon, \quad x, y \in X$$

then there exists a unique additive mapping $a : X \to Y$ such that

$$\|f(x) - a(x)\| \leq \varepsilon.$$

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for all $x \in X$.

A generalized version of the theorem of Hyers for approximately linear mappings was given by Rassias [10]. Since then, the stability problems of various functional equations have been extensively investigated by a number of mathematicians. Moreover, the problems concerned with the generalizations and the applications of the stability to a number of functional equations and mappings have been developed as well.

By regarding a large influence of Ulam, Hyers and Rassias on the investigation of stability problems of functional equations the stability phenomenon that was introduced and proved by Rassias [10] in the year 1978 is called the Hyers-Ulam-Rassias stability.

Let A be an algebra. An additive (linear) mapping $\mu : A \to A$ is called a *derivation* (*linear derivation*) if $\mu(xy) = x\mu(y) + \mu(x)y$ for $x, y \in A$. An additive (linear) mapping $\mu : A \to A$ is said to be a *generalized derivation* (*generalized linear derivation*) if there exists a derivation $\delta : A \to A$ satisfying $\mu(xy) = x\mu(y) + \delta(x)y$ for any $x, y \in A$.

The stability of linear derivation was investigated by Park in [7, 8]. The present paper is devoted to the study of the stability of linear derivations and generalized linear derivations.

The main purpose of this article is to establish the functional inequalities associated with the linear derivation and the generalized derivation.

2 The Linear Derivation

Theorem 2.1 *Let A be a Banach algebra. Suppose that $f : A \to A$ is a function with $f(0) = 0$, and that holds the inequality*

$$\|f(\lambda x + \lambda y - zw) + f(\lambda x - \lambda y) + 2\lambda f(-x) + zf(w) + f(z)w\| \le \theta, \quad (1)$$

for x, y, z and $w \in A$ and for any $\lambda \in \mathbf{T} = \{\lambda : |\lambda| = 1\}$. Then there exists a unique linear derivation $d : A \to A$ such that

$$\|f(x) - d(x)\| \le 2\theta, \qquad x \in A$$

and

$$x(f(y) - d(y)) = 0, \qquad x, y \in A.$$

Proof. Let's take $\lambda = 1$ and $w = 0$ in (1), then it becomes

$$\|f(x + y) + f(x - y) + 2f(-x)\| \le \theta. \quad (2)$$

If $x = 0$ and $y = -x$ in (2), it follows that

$$\|f(-x) + f(x)\| \le \theta. \quad (3)$$

Substituting x and y with $-x$ in (2), then we have

$$\|f(-2x) + 2f(x)\| \le \theta. \quad (4)$$

Combining (3) and (4), one can easily get that

$$\|2f(x) - f(2x)\| \le \|f(-2x) + 2f(x)\| + \| - [f(-2x) + f(2x)]\| \le 2\theta,$$

and, therefore, we obtain the following inequality.

$$\Big\|\frac{f(2x)}{2} - f(x)\Big\| \le \theta.$$

An induction implies that

$$\Big\|\frac{f(2^n x)}{2^n} - f(x)\Big\| \le 2\Big(1 - \frac{1}{2^n}\Big)\theta. \tag{5}$$

For $n > m$, (5) can be rewritten as follows.

$$\begin{aligned}\Big\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\Big\| &= \frac{1}{2^m}\Big\|\frac{f(2^{n-m}\cdot 2^m x)}{2^{n-m}} - f(2^m x)\Big\| \\ &\le \frac{1}{2^{m-1}}\Big(1 - \frac{1}{2^{n-m}}\Big)\theta.\end{aligned}$$

As $m \to \infty$, it can be easily verified that $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence. Since A is complete, the Cauchy sequence $\{\frac{f(2^n x)}{2^n}\}$ converges. Thus if

$$d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \qquad x \in A,$$

we have $||f(x) - d(x)|| \le 2\theta$ as $n \to \infty$ in (5).

Replacing x and y with $2^n x$ and $2^n y$ in (2), respectively, and then dividing both sides by 2^n. We get

$$\Big\|\frac{f(2^n(x+y))}{2^n} + \frac{f(2^n(x-y))}{2^n} + \frac{2f(-2^n x)}{2^n}\Big\| \le \frac{\theta}{2^n}.$$

As $n \to \infty$ in the above inequality, we also get

$$d(x+y) + d(x-y) + 2d(-x) = 0. \tag{6}$$

The equation (6) can be more simplified as $d(x) + d(-x) = 0$ if we take $y = 0$. This implies that d is an odd function and so $d(0) = 0$.

Similarly, substituting $y = x$ into (6), we obtain that

$$d(2x) = 2d(x). \tag{7}$$

Let $u = x + y$ and $v = x - y$ in the equation (6), then we can rewrite (6) as

$$d(u) + d(v) + 2d\Big(-\frac{u+v}{2}\Big) = 0$$

and, therefore, we have

$$2d\Big(\frac{u+v}{2}\Big) = d(u) + d(v). \tag{8}$$

Hence if we set $u = 2x$ and $v = 2y$ in (8) and use the equation (7), we finally obtain

$$d(x+y) = d(x) + d(y),$$

and so we can conclude that d is additive.

We claim that d is unique: Suppose that there exists another additive function $D : X \to Y$ satisfying the inequality $\|D(x) - d(x)\| \leq 2\theta$. Since $D(2^n x) = 2^n D(x)$ and $d(2^n x) = 2^n d(x)$, we see that

$$\begin{aligned}\|D(x) - d(x)\| &= \frac{1}{4^n}\|D(2^n x) - d(2^n x)\| \\ &\leq \frac{1}{2^n}[\|D(2^n x) - f(2^n x)\| + \|f(2^n x) - d(2^n x)\|] \\ &\leq \frac{1}{2^{n-2}}\theta.\end{aligned}$$

By letting $n \to \infty$ in this inequality, we have $D = d$.

If $y = z = 0$ in (1), it can be obtained that

$$\|f(\lambda x) + \lambda f(-x)\| \leq \frac{\theta}{2}. \tag{9}$$

If we also replace x with $2^n x$ and divide both sides of (9) by 2^n, then we have

$$\Big\|\frac{f(\lambda 2^n x)}{2^n} + \lambda\frac{f(-2^n x)}{2^n}\Big\| \leq \frac{\theta}{2^{n+1}}.$$

As $n \to \infty$, it is obtained that $d(\lambda x) + \lambda d(-x) = 0$ and, therefore, we have that $d(\lambda x) = \lambda d(x)$ for $x \in A$ and $\lambda \in \mathbf{T}$.

We now want to show that $d(\mu x) = \mu d(x)$ for $\mu \in \mathbf{C}$: Let $\mu \in \mathbf{C}$ be a nonzero constant and M an integer greater than $4|\mu|$. Then we get $\frac{|\mu|}{M} < \frac{1}{4} < 1 - \frac{2}{3}$. There also exist elements $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{T}$ such that $3\frac{\mu}{M} = \lambda_1 + \lambda_2 + \lambda_3$. Observe that $d(x) = d(3\frac{x}{3}) = 3d(\frac{x}{3})$. Hence we have $d(\frac{x}{3}) = \frac{1}{3}d(x)$ and

$$\begin{aligned}d(\mu x) &= d\Big(\frac{M}{3}\cdot 3\frac{\mu}{M}x\Big) = Md\Big(\frac{1}{3}\cdot 3\frac{\mu}{M}x\Big) = \frac{M}{3}d\Big(3\frac{\mu}{M}x\Big) \\ &= \frac{M}{3}d(\lambda_1 x + \lambda_3 x + \lambda_3 x) = \frac{M}{3}(d(\lambda_1 x) + d(\lambda_2 x) + d(\lambda_3 x)) \\ &= \frac{M}{3}(\lambda_1 + \lambda_2 + \lambda_3)d(x) = \frac{M}{3}3\frac{\mu}{M}d(x) = \mu d(x)\end{aligned}$$

for $x \in A$. Thus, by additivity of d, we obtain that

$$d(\alpha x + \beta y) = \alpha d(x) + \beta d(y),$$

where $\alpha\beta \neq 0$ and $\alpha, \beta \in \mathbf{C}$. In particular, $d(0x) = 0 = 0d(x)$, one can conclude that d is a $\mathbf{C}$-linear.

Let us now take $\lambda = 1$ and $x = y = 0$ in (1). Then it follows that

$$\|f(-zw) + zf(w) + f(z)w\| \leq \theta. \tag{10}$$

Define $C(z,w) = f(-zw) + zf(w) + f(z)w$. Since C is bounded,

$$\lim_{n\to\infty} \frac{C(2^n z, w)}{2^n} = 0. \tag{11}$$

Note that

$$\begin{aligned} d(-zw) &= \lim_{n\to\infty} \frac{f(-2^n zw)}{2^n} = \lim_{n\to\infty} \frac{f(-2^n z \cdot w)}{2^n} \\ &= \lim_{n\to\infty} \frac{-2^n zf(w) - f(2^n z)w + C(2^n z, w)}{2^n} \\ &= -zf(w) - d(z)w. \end{aligned}$$

Based on the fact that d is a odd function, we have

$$d(zw) = zf(w) + d(z)w. \tag{12}$$

The equation (12) now can be rewritten as

$$\begin{aligned} d(2^n z \cdot w) &= 2^n zf(w) + 2^n d(z)w, \\ d(z \cdot 2^n w) &= zf(2^n w) + 2^n d(z)w. \end{aligned}$$

Hence $zf(w) = z\frac{f(2^n w)}{2^n}$, and then we obtain $zf(w) = zd(w)$ as $n \to \infty$. Thus we have that

$$d(zw) = zd(w) + d(z)w,$$

and so the assertion follows. ///

Theorem 2.2 *Let A be a Banach algebra. Suppose that $f : A \to A$ is a function with $f(0) = 0$ and holds the following inequality*

$$\Big\|2f\Big(-\frac{\lambda(x+y)}{2} + zw\Big) + \lambda f(x) + \lambda f(y) - 2zf(w) - 2f(z)w\Big\| \le \theta, \tag{13}$$

for any $x, y, z, w \in A$ and $\lambda \in \mathbf{T} = \{\lambda : |\lambda| = 1\}$. Then there exists a unique linear derivation $d : A \to A$ such that

$$\|f(x) - d(x)\| \le \frac{3}{2}\theta, \qquad x \in A,$$

and also

$$x(f(y) - d(y)) = 0 \qquad x, y \in A.$$

Proof. If $\lambda = 1$ and $w = 0$ in (13), then one can get

$$\Big\|2f\Big(-\frac{x+y}{2}\Big) + f(x) + f(y)\Big\| \le \theta, \tag{14}$$

and also one can easily obtain that

$$\|f(-x) + f(x)\| \le \frac{\theta}{2} \tag{15}$$

if $y = x$ in (14), let's take $x = 0$ and $y = -2x$ in (14) to have

$$\|f(-2x) + 2f(x)\| \leq \theta. \tag{16}$$

Combining (15) and (16), we obtain

$$\|2f(x) - f(2x)\| \leq \|f(-2x) + 2f(x)\| + \| - [f(-2x) + f(2x)]\| \leq \frac{3}{2}\theta,$$

and so we get the following.

$$\Big\|\frac{f(2x)}{2} - f(x)\Big\| \leq \frac{3}{4}\theta.$$

An induction implies that

$$\Big\|\frac{f(2^n x)}{2^n} - f(x)\Big\| \leq \frac{3}{2}\Big(1 - \frac{1}{2^n}\Big)\theta. \tag{17}$$

For $n > m$, using the inequality in (17), one can see that

$$\begin{aligned}\Big\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\Big\| &= \frac{1}{2^m}\Big\|\frac{f(2^{n-m}\cdot 2^m x)}{2^{n-m}} - f(2^m x)\Big\| \\ &\leq \frac{3}{2^{m+1}}\Big(1 - \frac{1}{2^{n-m}}\Big)\theta.\end{aligned}$$

Since the right-hand side of the inequality approaches 0 as $m \to \infty$, the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy. Thus let

$$d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \quad x \in A.$$

It is obtained that $||f(x) - d(x)|| \leq \frac{3}{2}\theta$ for any $x \in A$ by letting $n \to \infty$ in (17).

Let us replace x and y with $2^n x$ and $2^n y$ in (14), respectively, and then divide both sides by 2^n. We have then

$$\Big\|\frac{2f(-\frac{2^n(x+y)}{2})}{2^n} + \frac{f(2^n x)}{2^n} + \frac{f(2^n y)}{2^n}\Big\| \leq \frac{\theta}{2^n}.$$

It is obtained that

$$2d(-\frac{x+y}{2}) + d(x) + d(y) = 0 \tag{18}$$

as $n \to \infty$ in the above inequality. Thus if $y = x$ in (18), we have that d is an odd function with $d(0) = 0$.

If $x := -2x$ and $y = 0$ in (18), one can get $d(2x) = 2d(x)$.

Hence we have that

$$2d\Big(\frac{x+y}{2}\Big) = d(x) + d(y). \tag{19}$$

We now set $x := 2x, y := 2y$ in (19) and use the equation (19) to find

$$d(x+y) = d(x) + d(y),$$

which shows that d is additive.

As in the proof of Theorem 2.1, we can show that d is unique.

Taking $y = x$ and $w = 0$ in (13), we have

$$\|f(-\lambda x) + \lambda f(x)\| \leq \frac{\theta}{2}. \tag{20}$$

Let us replace x with $2^n x$ and then divide both sides of (20) by 2^n. Then we have

$$\left\| \frac{f(-\lambda 2^n x)}{2^n} + \lambda \frac{f(2^n x)}{2^n} \right\| \leq \frac{\theta}{2^{n+1}}.$$

As $n \to \infty$, it yields that $d(\lambda x) = \lambda d(x)$ for any $x \in A$ and $\lambda \in \mathbf{T}$. Hence, if we utilize the same way as in the proof of Theorem 2.1, we can conclude that d is a $\mathbf{C}$-linear.

Let us now take $\lambda = 1$ and $x = y = 0$ in (13). Then we get

$$\|f(zw) - zf(w) - f(z)w\| \leq \frac{\theta}{2}. \tag{21}$$

Define $C(z,w) = f(zw) - zf(w) - f(z)w$. Then, since C is bounded,

$$\lim_{n\to\infty} \frac{C(2^n z, w)}{2^n} = 0. \tag{22}$$

Note that

$$\begin{aligned} d(zw) &= \lim_{n\to\infty} \frac{f(2^n zw)}{2^n} = \lim_{n\to\infty} \frac{f(2^n z \cdot w)}{2^n} \\ &= \lim_{n\to\infty} \frac{2^n z f(w) + f(2^n z)w + C(2^n z, w)}{2^n} \\ &= zf(w) + d(z)w. \end{aligned} \tag{23}$$

It follows from (23) that

$$\begin{aligned} d(2^n z \cdot w) &= 2^n z f(w) + 2^n d(z)w, \\ d(z \cdot 2^n w) &= z f(2^n w) + 2^n d(z)w. \end{aligned}$$

and so $zf(w) = z\frac{f(2^n w)}{2^n}$. As $n \to \infty$, we obtain $zf(w) = zd(w)$ and, therefore, we have

$$d(zw) = zd(w) + d(z)w,$$

which shows that d is the linear derivation.

The following property can be derived along the argument.

Corollary 2.3 *Let A be a Banach algebra with the unit. Suppose that $f : A \to A$ is a function with $f(0) = 0$ and holds the inequality (1) (or (13)). Then f is a linear derivation*

Proof. By Theorem 2.1(or, Theorem 2.2), $x(f(y) - d(y)) = 0$ for $x, y \in A$, where $d : A \to A$ is a derivation. Considering $x = e$(unit) and $f = d$, we can easily reach the conclusion. ///

Now in the algebra, the skew commutator $< x, y >$ is defined as follows. $< x, y >= xy + yx$. We state the following theorem [6] for the reading convenience to get the immediate property of a function f.

Theorem 2.4 *Let A be a semi-simple Banach algebra. If $d : A \to A$ is a linear derivation satisfying $< dx, x >^2 = 0$, then $d = 0$.*

Corollary 2.5 *Let A be a semi-simple Banach algebra with the unit. Suppose that $f : A \to A$ is a function with $f(0) = 0$, and holds the inequality (1) (or (13)). Suppose further that $\| < f(x), x >^2 \| \leq \varepsilon$. Then $f = 0$.*

Proof. Since A has the unit, it follows directly from Theorem 2.1 (or Theorem 2.2) that

$$f(x) = d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

We now replace x by $2^n x$ and divide both sides of the given inequality by 2^{4n}. It follows then that $< f(x), x >^2 = 0$ as $n \to \infty$. Hence $f = 0$ by the theorem 2.4. ///

3 The Generalized Derivation

Theorem 3.1 *Let A be a Banach algebra with the unit. Suppose $f : A \to A$ is a function with $f(0) = 0$ for which there exists a function $g : A \to A$ such that*

$$\|f(x + y - zw) + f(x - y) + 2f(-x) + zf(w) + g(z)w\| \leq \theta. \tag{24}$$

for x, y, z and $w \in A$. Then f is a generalized derivation and g is a derivation.

Proof. Substituting $w = 0$ in (24), we get

$$\|f(x + y) + f(x - y) + 2f(-x)\| \leq \theta.$$

Using the facts provided in the proof of Theorem 2.1, there is a uniquely determined $d : A \to A$, additive mapping, satisfying $\|f(x) - d(x)\| \leq 2\theta$, where $d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

If we take $x = y = 0$ in (24), we also have

$$\|f(-zw) + zf(w) + g(z)w\| \leq \theta. \tag{25}$$

Moreover, if we replace z and w with $2^n z$ and $2^n w$, respectively in (25) and then divide both sides by 2^{2n}, we get

$$\left\| \frac{f(-2^{2n}zw)}{2^{2n}} + z\frac{f(2^n w)}{2^n} + \frac{g(2^n z)}{2^n} w \right\| \leq \frac{\theta}{2^{2n}} \to 0,$$

as $n \to \infty$. Hence it implies that

$$\lim_{n\to\infty} \frac{g(2^n z)}{2^n} w = -d(-zw) - zd(w) = d(zw) - zd(w),$$

because d is the odd function. Suppose that $w = e(\text{unit})$ in the above equation, then it follows that

$$\lim_{n\to\infty} \frac{g(2^n z)}{2^n} = d(z) - zd(e).$$

Thus if $\delta(z) = d(z) - zd(e)$, we have

$$\begin{aligned} \delta(x+y) &= d(x) + d(y) - xd(e) - yd(e) \\ &= (d(x) - xd(e)) + (d(y) - yd(e)) \\ &= \delta(x) + \delta(y). \end{aligned}$$

Hence we showed that δ is additive.

Let $C(z,w) = f(-zw) + zf(w) + g(z)w$. Since f and g satisfies the inequality given in (25),

$$\lim_{n\to\infty} \frac{C(2^n z, w)}{2^n} = 0.$$

We note that

$$\begin{aligned} d(-zw) &= \lim_{n\to\infty} \frac{f(-2^n zw)}{2^n} = \lim_{n\to\infty} \frac{f(-2^n z \cdot w)}{2^n} \\ &= \lim_{n\to\infty} \frac{-2^n z f(w) - g(2^n z)w + C(2^n z, w)}{2^n} \\ &= -zf(w) - \delta(z)w. \end{aligned}$$

Hence by the oddness of d, we obtain that

$$d(zw) = zf(w) + \delta(z)w. \tag{26}$$

Since δ is additive, we can rewrite the equation (26) as

$$\begin{aligned} d(2^n z \cdot w) &= 2^n z f(w) + 2^n \delta(z) w, \\ d(z \cdot 2^n w) &= z f(2^n w) + 2^n \delta(z) w. \end{aligned}$$

Based on the above relations, one can have $zf(w) = z\frac{f(2^n w)}{2^n}$. Moreover, we can obtain $zf(w) = zd(w)$ as $n \to \infty$. If $z = e(\text{unit})$, we also have that $f = d$. Therefore we have

$$f(zw) = zf(w) + \delta(z)w. \tag{27}$$

We want to show now that δ is derivation using the equations developed in the previous. Indeed, using the facts that f satisfies the equation (27)

$$\begin{aligned}\delta(xy) &= f(xy) - xyf(e)\\ &= xf(y) + \delta(x)y - xyf(e)\\ &= x(f(y) - yf(e)) + \delta(x)y\\ &= x\delta(y) + \delta(x)y,\end{aligned}$$

which means that f is the generalized derivation.

We finally want to show that g is the derivation: Let us replace w by $2^n w$ in (25) and multiply by $\frac{1}{2^n}$. Then we have

$$\Big\|\frac{f(-2^n zw)}{2^n} + z\frac{f(2^n w)}{2^n} + g(z)w\Big\| \le \frac{\theta}{2^n}.$$

As $n \to \infty$, we get

$$d(-zw) + zd(w) + g(z)w = 0.$$

This implies that $d(zw) = zd(w) + g(z)w$, and thus if $w = e(\text{unit})$, we see that $g(z) + zd(e) = d(z)$. Hence we get $g(z) = d(z) - zd(e) = \delta(z)$. Since δ is the derivation, we can conclude that g is the derivation as well. ///

Theorem 3.2 *Let A be a Banach algebra with the unit. Suppose $f : A \to A$ is a function with $f(0) = 0$ for which there exists a function $g : A \to A$ such that*

$$\Big\|2f\Big(-\frac{x+y}{2} + zw\Big) + f(x) + f(y) - 2zf(w) - 2g(z)w\Big\| \le \theta. \tag{28}$$

for x, y, z and $w \in A$. Then f is a generalized derivation and g is a derivation.

Proof. Set $w = 0$ in the inequality (28) to obtain

$$\Big\|2f\Big(-\frac{x+y}{2}\Big) + f(x) + f(y)\Big\| \le \theta.$$

Similar to the the proof of Theorem 2.2, there exists exactly one additive mapping $d : A \to A$ such that $\|f(x) - d(x)\| \le \frac{3}{2}\theta$, where $d(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

If we take $x = y = 0$ in (28), then we have that

$$\|f(zw) - zf(w) - g(z)w\| \le \frac{\theta}{2}. \tag{29}$$

Replacing z and w by $2^n z$ and $2^n w$ in (29), respectively, and dividing both sides by 2^{2n}, we get

$$\Big\|\frac{f(2^{2n}zw)}{2^{2n}} - z\frac{f(2^n w)}{2^n} - \frac{g(2^n z)}{2^n}w\Big\| \le \frac{\theta}{2^{2n}} \to 0,$$

as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \frac{g(2^n z)}{2^n}w = d(zw) - zd(w).$$

Let $w = e(\text{unit})$ in the above equation. Then it follows that

$$\lim_{n\to\infty} \frac{g(2^n z)}{2^n} = d(z) - zd(e).$$

We now define $\delta(z) = d(z) - zd(e)$. By same reasoning described in the proof of Theorem 3.1, δ is additive.

Put $C(z,w) = f(zw) - zf(w) - g(z)w$ and then use (29) to get

$$\lim_{n\to\infty} \frac{C(2^n z, w)}{2^n} = 0.$$

Note that

$$\begin{aligned} d(zw) &= \lim_{n\to\infty} \frac{f(2^n zw)}{2^n} = \lim_{n\to\infty} \frac{f(2^n z \cdot w)}{2^n} \\ &= \lim_{n\to\infty} \frac{2^n z f(w) + g(2^n z)w + C(2^n z, w)}{2^n} \\ &= zf(w) + \delta(z)w. \end{aligned} \tag{30}$$

If we use the similar argument employed in the proof of Theorem 3.1, then we see that f is the generalized derivation.

It remains to prove that g is the derivation: Substituting $w := 2^n w$ in (29) and multiplying by $\frac{1}{2^n}$, we have

$$\left\| \frac{f(2^n zw)}{2^n} - z\frac{f(2^n w)}{2^n} - g(z)w \right\| \le \frac{\theta}{2^{n+1}}.$$

Taking $n \to \infty$, we get

$$d(zw) = zd(w) + g(z)w.$$

Considering $w = e(\text{unit})$, we obtain $g(z) = d(z) - zd(e) = \delta(z)$. Since δ is the derivation, so is g. ///

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Controllability for the Impulsive Semilinear Fuzzy Integrodifferential Equations with Nonlocal Conditions and Forcing Term with Memory

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Abstract

In this paper, we study the existence and uniqueness of fuzzy solutions and controllability for the impulsive semilinear fuzzy integrodifferential control system with nonlocal conditions and forcing term with memory in E_N by using the concept of fuzzy number whose values are normal, convex, upper semicontinuous and compactly supported interval in E_N.

1 Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva [3] studied the existence and uniqueness of solution for the fuzzy differential equation on E^n where E^n is normal, convex, upper semicontinuous and compactly supported

*Corresponding author

fuzzy sets in R^n. Seikkala [7] proved the existence and uniqueness of fuzzy solution for the following equation:

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number in E^1. Diamond and Kloeden [2] proved the fuzzy optimal control for the following system:

$$\dot{x}(t) = a(t)x(t) + u(t), \quad x(0) = x_0$$

where $x(\cdot), u(\cdot)$ are nonempty compact interval-valued functions on E^1. Kwun and Park [4] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E_N^1 using by Kuhn-Tucker theorems. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently Park, Park and Kwun [6] find the sufficient conditions of nonlocal controllability for the semilinear fuzzy integrodifferential equations with nonlocal initial conditions.

In this paper we prove the existence and uniqueness of fuzzy solutions and find the sufficient conditions of nonlocal controllability for the following impulsive semilinear fuzzy integrodifferential equations with nonlocal conditions and forcing term with memory:

$$\begin{aligned} \frac{dx(t)}{dt} &= A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] \qquad (1)\\ &\quad + f(t, x(t), \int_0^t q(t, s, x(s))ds) + u(t), \quad t \in J = [0, T],\\ x(0) + g(x) &= x_0 \in E_N, \qquad (2)\\ \triangle x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \qquad (3) \end{aligned}$$

where $A : J \to E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α-level intervals, $f : J \times E_N \times E_N \to E_N$ and $q : J \times J \times E_N \to E_N$ are nonlinear continuous functions, $g : E_N \to E_N$ is a nonlinear continuous function, $G(t)$ is $n \times n$ continuous matrix such that $\frac{dG(t)x}{dt}$ is continuous for $x \in E_N$ and $t \in J$ with $\|G(t)\| \leq k$, $k > 0$, $u : J \to E_N$ is control function and $I_k \in C(E_N, E_N)(k = 1, 2, \cdots, m)$ are bounded functions, $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the left and right limits of $x(t)$ at $t = t_k$, respectively.

2 Existence and uniqueness of fuzzy solution

In this section we consider the existence and uniqueness of fuzzy solutions for the impulsive semilinear fuzzy integrodifferential equation with nonlocal conditions and forcing term with memory (1)-(3)($u \equiv 0$).

We denote the suprimum metric d_∞ on E^n and the suprimum metric H_1 on $C(J : E^n)$.

Definition 2.1 Let $a, b \in E^n$.

$$d_\infty(a,b) = \sup\{d_H([a]^\alpha, [b]^\alpha) : \alpha \in (0,1]\}$$

where d_H is the Hausdorff distance.

Definition 2.2 Let $x, y \in C(J : E^n)$

$$H_1(x,y) = \sup\{d_\infty(x(t), y(t)) : t \in J\}.$$

Definition 2.3 The fuzzy process $x : J \to E_N$ is a solution of equations (1)-(2) without the inhomogeneous term if and only if

$$\begin{aligned}(\dot{x}_l^\alpha)(t) &= \min\Big\{A_l^\alpha(t)\big[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds\big],\ i,j = l,r\Big\},\\ (\dot{x}_r^\alpha)(t) &= \max\Big\{A_r^\alpha(t)\big[x_j^\alpha(t) + \int_0^t G(t-s)x_j^\alpha(s)ds\big],\ i,j = l,r\Big\},\end{aligned}$$

and

$$(x_l^\alpha)(0) = x_{0l}^\alpha - g_l^\alpha(x), \quad (x_r^\alpha)(0) = x_{0r}^\alpha - g_r^\alpha(x).$$

Now we assume the following:

(H1) The nonlinear function $f : J \times E_N \times E_N \to E_N$ satisfies a global Lipschitz condition, there exists a finite constants $k_1, k_2 > 0$ such that

$$\begin{aligned}&d_H\big([f(s,\xi_1(s),\eta_1(s))]^\alpha,\ [f(s,\xi_2(s),\eta_2(s))]^\alpha\big)\\ &\le k_1 d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) + k_2\ d_H([\eta_1(s)]^\alpha, [\eta_2(s)]^\alpha)\end{aligned}$$

for all $\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s) \in E_N$.

(H2) The nonlinear function $q : J \times J \times E_N \to E_N$ satisfies a global Lipschitz condition, there exists a finite constant $M > 0$ such that

$$d_H\big([q(t,s,\psi_1(s)]^\alpha, [q(t,s,\psi_2(s))]^\alpha\big) \le M d_H([\psi_1(s)]^\alpha, [\psi_2(s)]^\alpha)$$

for all $\psi_1(s), \psi_2(s) \in E_N$.

(H3) The nonlinear function $g : E_N \to E_N$ is a continuous function and satisfies a global Lipschitz condition

$$d_H([g(x)]^\alpha, [g(y)]^\alpha) \le L d_H([x(\cdot)]^\alpha, [y(\cdot)]^\alpha),$$

for all $x(\cdot), y(\cdot) \in E_N$, and a finite positive constant $L > 0$.

(H4) $S(t)$ is a fuzzy number satisfying for $y \in E_N$, $S'(t)y \in C^1(J : E_N) \bigcap C(J : E_N)$ the equation

$$\begin{aligned}\frac{d}{dt}S(t)y &= A\left[S(t)y + \int_0^t G(t-s)S(s)yds\right]\\ &= S(t)Ay + \int_0^t S(t-s)AG(s)yds, \quad t \in J,\end{aligned}$$

such that

$$[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)], \quad S(0) = I$$

and $S_i^\alpha(t)$ $(i = l, r)$ is continuous. That is, there exists a constant $c > 0$ such that $|S_i^\alpha(t)| \leq c$ for all $t \in J$.

In order to define the solution of (1)-(3), we shall consider the space $\Omega = \{x : J \to E_N : x_k \in C(J_k, E_N), J_k = (t_k, t_{k+1}], k = 0, 1, \cdots, m$, and there exist $x(t_k^-)$ and $x(t_k^+)(k = 1, \cdots, m)$, with $x(t_k^-) = x(t_k)\}$.

Lemma 2.4 *If x is an integral solution of (1)-(3)($u \equiv 0$), then x is given by*

$$x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s) f(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau) ds \quad (4)$$
$$+ \sum_{0 < t_k < t} S(t - t_k) I_k(x(t_k^-)), \text{ for t} \in \text{J}.$$

Proof Let x be a solution of (1)-(3). Define $\omega(s) = S(t-s)x(s)$. Then we have that

$$\begin{aligned}
\frac{d\omega(s)}{ds} &= -\frac{S(t-s)}{ds}x(s) + S(t-s)\frac{x(s)}{ds} \\
&= -A[S(t)x + \int_0^t G(t-s)S(s)x(s)ds] + S(t-s)\frac{x(s)}{ds} \\
&= S(t-s)f(s, x(s), \int_0^s q(s, \tau, x(\tau))d\tau).
\end{aligned}$$

Consider $t_k < t, k = 1, \cdots, m$. Then integrating the previous equation, we have

$$\int_0^t \frac{\omega(s)}{ds} ds = \int_0^t S(t-s) f(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau) ds.$$

For $k = 1$,

$$\omega(t) - \omega(0) = \int_0^t S(t-s) f(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau) ds$$

or

$$x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s) f(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau) ds.$$

Now for $k = 2, \cdots, m$, we have that

$$\begin{aligned}
&\int_0^{t_1} \frac{\omega(s)}{ds} ds + \int_{t_1}^{t_2} \frac{\omega(s)}{ds} ds + \cdots + \int_{t_k}^t \frac{\omega(s)}{ds} ds \\
&= \int_0^t S(t-s) f(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau) ds.
\end{aligned}$$

Then

$$\omega(t_1^-) - \omega(0) + \omega(t_2^-) - \omega(t_1^+) + \cdots - \omega(t_k^+) + \omega(t)$$
$$= \int_0^t S(t-s) f(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau) ds$$

if and only if

$$\omega(t) = \omega(0) + \int_0^t S(t-s) f(s, x(s)) ds + \sum_{0<t_k<t} [\omega(t_k^+) - \omega(t_k^-)].$$

Hence

$$\begin{aligned} x(t) = S(t)(x_0 - g(x)) \quad &+ \quad \int_0^t S(t-s) f(s, x(s), \int_0^s q(s, \tau, x(\tau)) d\tau) ds \\ &+ \quad \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-)), \end{aligned}$$

which proves the lemma.

Assume the following:
(H5) There exists $d_k, k = 1, \cdots, m$, such that

$$d_H([I_k(x(t_k^-))]^\alpha, [I_k(y(t_k^-))]^\alpha) \le d_k d_H([x(t)]^\alpha, [y(t)]^\alpha),$$

where $\sum_{k=1}^n d_k = \bar{d}$
(H6) $c\Big(L + \bar{d} + (k_1 + k_2 M \frac{T}{2})T\Big) < 1$

Theorem 2.5 *Let $T > 0$, and hypotheses (H1)-(H6) hold. Then, for every $x_0 (\in E_N)$, problem (1)-(3)($u \equiv 0$) has a unique solution $x \in \Omega$.*

Proof For each $\xi(t) \in \Omega, t \in J$ define

$$\begin{aligned} (\Phi\xi)(t) = S(t)(x_0 - g(\xi)) \quad &+ \quad \int_0^t S(t-s) f(s, \xi(s), \int_0^s q(s, \tau, \xi(\tau)) d\tau) ds \\ &+ \quad \sum_{0<t_k<t} S(t-t_k) I_k(\xi(t_k^-)). \end{aligned}$$

Thus,$(\Phi\xi)(t) : J \to \Omega$ is continuous, and $\Phi : \Omega \to \Omega$.

It is obvious that fixed points of Φ are solution for the problem (1)-(3)($u \equiv 0$). For $\xi(t), \eta(t) \in \Omega$, we have

$$\begin{aligned} &d_H([(\Phi\xi)(t)]^\alpha, [(\Phi\eta)(t)]^\alpha) \\ &\le d_H([S(t)g(\xi)]^\alpha, [S(t)g(\eta)]^\alpha) \\ &\quad + d_H\Big([\int_0^t S(t-s) f(s, \xi(s), \int_0^s q(s, \tau, \xi(\tau)) d\tau) ds]^\alpha, \end{aligned}$$

$$[\int_0^t S(t-s)f(s,\eta(s),\int_0^s q(s,\tau,\eta(\tau))d\tau)ds]^\alpha\Big)$$
$$+d_H([\sum_{0<t_k<t} S(t-t_k)I_k(\xi(t_k^-))]^\alpha,[\sum_{0<t_k<t} S(t-t_k)I_k\eta(t_k^-))]^\alpha)$$
$$\le cLd_H([\xi(\cdot)]^\alpha,[\eta(\cdot)]^\alpha)+c\int_0^t\Big(k_1d_H([\xi(s)]^\alpha,[\eta(s)]^\alpha)$$
$$+k_2M\int_0^s d_H([\xi(\tau)]^\alpha,[\eta(\tau)]^\alpha)d\tau\Big)ds+c\bar{d}\ d_H([\xi(t)]^\alpha,[\eta(t)]^\alpha).$$

Therefore,

$$d_\infty((\Phi\xi)(t),(\Phi\eta)(t))=\sup_{\alpha\in(0,1]} d_H([(\Phi\xi)(t)]^\alpha,[(\Phi\eta)(t)]^\alpha)$$
$$\le cL\sup_{\alpha\in(0,1]} d_H([\xi(\cdot)]^\alpha,[\eta(\cdot)]^\alpha)+c\int_0^t\Big(k_1\sup_{\alpha\in(0,1]} d_H([\xi(s)]^\alpha,[\eta(s)]^\alpha)$$
$$+k_2M\int_0^s\sup_{\alpha\in(0,1]} d_H([\xi(\tau)]^\alpha,[\eta(\tau)]^\alpha)d\tau\Big)ds+c\bar{d}\sup_{\alpha\in(0,1]} d_H([\xi(t)]^\alpha,[\eta(t)]^\alpha)$$
$$=cLd_\infty(\xi(\cdot),\eta(\cdot))+c\int_0^t\Big(k_1d_\infty([\xi(s)]^\alpha,[\eta(s)]^\alpha)$$
$$+k_2M\int_0^s d_\infty([\xi(\tau)]^\alpha,[\eta(\tau)]^\alpha)d\tau\Big)ds+c\bar{d}\ d_\infty([\xi(t)]^\alpha,[\eta(t)]^\alpha).$$

Hence

$$H_1(\Phi\xi,\Phi\eta)=\sup_{t\in J} d_\infty((\Phi\xi)(t),(\Phi\eta)(t))$$
$$\le c(L+\bar{d})\sup_{t\in J} d_\infty(\xi(t),\eta(t))+c\sup_{t\in J}\int_0^t\Big(k_1d_\infty([\xi(s)]^\alpha,[\eta(s)]^\alpha)$$
$$+k_2M\int_0^s d_\infty([\xi(\tau)]^\alpha,[\eta(\tau)]^\alpha)d\tau\Big)ds$$
$$=c\Big(L+\bar{d}+(k_1+k_2M\frac{T}{2})T\Big)H_1(\xi,\eta).$$

By hypotheses (H6), Φ is a contraction mapping. By the Banach fixed point theorem, (4) has a unique fixed point $x\in\Omega$.

3 Nonlocal controllability

In this section, we show the controllability for the control system (1)-(3).

The control system (1)-(3) is related to the following fuzzy integral system:

$$x(t)=S(t)(x_0-g(x))+\int_0^t S(t-s)f(s,x(s),\int_0^s q(s,\tau,x(\tau))d\tau)ds \quad (5)$$
$$+\int_0^t S(t-s)u(s)ds+\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))$$

for $t \in J, t \neq t_k (k = 1, 2, \cdots, m)$, where $S(t)$ satisfies (H3).

Definition 3.1 The equation (5) is nonlocal controllable if, there exists $u(t)$ such that the fuzzy solution $x(t)$ of (5) satisfies $x(T) = x^1 - g(x)$, i.e., $[x(T)]^\alpha = [x^1 - g(x)]^\alpha$, where x^1 is target set.

We assume that the linear control system with respect to semilinear control system (5) is nonlocal controllable. Then

$$\begin{aligned} x(T) &= S(T)(x_0 - g(x)) + \int_0^T S(T-s)u(s)ds + \sum_{0<t_k<T} S(T-t_k)I_k(x(t_k^-)) \\ &= x^1 - g(x) \end{aligned}$$

and

$$\begin{aligned} &[x(T)]^\alpha \\ &= [S(T)(x_0 - g(x)) + \int_0^T S(T-s)u(s)ds + \sum_{0<t_k<T} S(T-t_k)I_k(x(t_k^-))]^\alpha \\ &= \Bigg[S_l^\alpha(T)({x_0}_l^\alpha - g_l^\alpha(x)) + \int_0^T S_l^\alpha(T-s)u_l^\alpha(s)ds \\ &\qquad + \sum_{0<t_k<T} S_l^\alpha(T-t_k){I_k}_l^\alpha(x(t_k^-)), \\ &\quad S_r^\alpha(T)({x_0}_r^\alpha - g_r^\alpha(x)) + \int_0^T S_r^\alpha(T-s)u_r^\alpha(s)ds \\ &\qquad + \sum_{0<t_k<T} S_r^\alpha(T-t_k){I_k}_r^\alpha(x(t_k^-)) \Bigg] \\ &= [(x^1)_l^\alpha - g_l^\alpha(x), (x^1)_r^\alpha - g_r^\alpha(x)]. \end{aligned}$$

Define the α-level set of fuzzy mapping $G : \tilde{P}(R) \to E_N$ by

$$G^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-s)v(s)ds, & v \subset \overline{\Gamma_u}\ , \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

where $\overline{\Gamma_u}$ is closure of support of u. Then there exists $G_i^\alpha (i = l, r)$ such that

$$\begin{aligned} G_l^\alpha(v_l) &= \int_0^T S_l^\alpha(T-s)v_l(s)ds, v_l(s) \in [u_l^\alpha(s), u^1(s)], \\ G_r^\alpha(v_r) &= \int_0^T S_r^\alpha(T-s)v_r(s)ds, v_r(s) \in [u^1(s), u_r^\alpha(s)]. \end{aligned}$$

We assume that G_l^α, G_r^α are bijective mapping. Hence α-level set of $u(s)$ is

$$[u(s)]^\alpha = [u_l^\alpha(s), u_r^\alpha(s)]$$

$$= \Bigg[(\tilde{G}_l^{\alpha})^{-1} \Big((x^1)_l^{\alpha} - g_l^{\alpha}(x) - S_l^{\alpha}(T)({x_0}_l^{\alpha} - g_l^{\alpha}(x)) - \sum_{0<t_k<T} S_l^{\alpha}(T-t_k) {I_k}_l^{\alpha}(x(t_k^-)) \Big),$$
$$(\tilde{G}_r^{\alpha})^{-1} \Big((x^1)_r^{\alpha} - g_r^{\alpha}(x) - S_r^{\alpha}(T)({x_0}_r^{\alpha} - g_r^{\alpha}(x)) - \sum_{0<t_k<T} S_r^{\alpha}(T-t_k) {I_k}_r^{\alpha}(x(t_k^-)) \Big) \Bigg].$$

Thus we can introduce $u(s)$ of semilinear system

$$[u(s)]^{\alpha} = [u_l^{\alpha}(s), u_r^{\alpha}(s)]$$
$$= \Bigg[(\tilde{G}_l^{\alpha})^{-1} \Big((x^1)_l^{\alpha} - g_l^{\alpha}(x) - S_l^{\alpha}(T)({x_0}_l^{\alpha} - g_l^{\alpha}(x)) - \int_0^T S_l^{\alpha}(T-s) \times f_l^{\alpha}(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds - \sum_{0<t_k<T} S_l^{\alpha}(T-t_k) {I_k}_l^{\alpha}(x(t_k^-)) \Big),$$
$$(\tilde{G}_r^{\alpha})^{-1} \Big((x^1)_r^{\alpha} - g_r^{\alpha}(x) - S_r^{\alpha}(T)({x_0}_r^{\alpha} - g_r^{\alpha}(x)) - \int_0^T S_r^{\alpha}(T-s) \times f_r^{\alpha}(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds - \sum_{0<t_k<T} S_r^{\alpha}(T-t_k) {I_k}_r^{\alpha}(x(t_k^-)) \Big) \Bigg].$$

Then substituting this expression into the equation (5) yields α-level of $x(T)$.

$$[x(T)]^{\alpha}$$
$$= \Bigg[S_l^{\alpha}(T)({x_0}_l^{\alpha} - g_l^{\alpha}(x)) + \int_0^T S_l^{\alpha}(T-s) f_l^{\alpha}(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds + \sum_{0<t_k<T} S_l^{\alpha}(T-t_k) {I_k}_l^{\alpha}(x(t_k^-)) + G_l^{\alpha}(\widetilde{G}_l^{\alpha})^{-1} \Big((x^1)_l^{\alpha} - g_l^{\alpha}(x) - S_l^{\alpha}(T) \times ({x_0}_l^{\alpha} - g_l^{\alpha}(x)) - \int_0^T S_l^{\alpha}(T-s) f_l^{\alpha}(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds - \sum_{0<t_k<T} S_l^{\alpha}(T-t_k) {I_k}_l^{\alpha}(x(t_k^-)) \Big) ds,$$
$$S_r^{\alpha}(T)({x_0}_r^{\alpha} - g_r^{\alpha}(x)) + \int_0^T S_r^{\alpha}(T-s) f_r^{\alpha}(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds + \sum_{0<t_k<T} S_r^{\alpha}(T-t_k) {I_k}_r^{\alpha}(x(t_k^-)) + G_r^{\alpha}(\tilde{G}_r^{\alpha})^{-1} \Big((x^1)_r^{\alpha} - g_r^{\alpha}(x) - S_r^{\alpha}(T) \times ({x_0}_r^{\alpha} - g_r^{\alpha}(x)) - \int_0^T S_r^{\alpha}(T-s) f_r^{\alpha}(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds$$

$$- \sum_{0<t_k<T} S_r^{\alpha}(T-t_k) I_{kr}^{\alpha}(x(t_k^-)))ds\Big]$$

$$= [(x^1)_l^{\alpha} - g_l^{\alpha}(x), (x^1)_r^{\alpha} - g_r^{\alpha}(x)] = [x^1 - g(x)]^{\alpha}.$$

We now set

$$\Phi x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s) f(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds$$
$$+ \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-)) + \int_0^t S(t-s)\tilde{G}^{-1}\Big(x^1 - g(x) - S(T)(x_0 - g(x))$$
$$- \int_0^T S(T-s) f(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds - \sum_{0<t_k<T} S(T-t_k) I_k(x(t_k^-))\Big)ds$$

where the fuzzy mappings $\tilde{G}^{-1}$ satisfied above statements.

Notice that $\Phi x(T) = x^1 - g(x)$, which means that the control $u(t)$ steers the equation (5) from the origin to $x^1 - g(x)$ in time T provided we can obtain a fixed point of nonlinear operator Φ.

Assume that the following hypotheses:

(H7) Linear system of equation (5) ($f \equiv 0$) is nonlocal controllable.

(H8) $c\Big\{(L + \bar{d}) + T\Big[L(1+c) + cT(k_1 + k_2 M\frac{T}{2}) + c\bar{d}\Big]\Big\} < 1.$

Theorem 3.2 *Suppose that (H1)-(H8) are satisfied. Then the equation (5) is nonlocal controllable.*

Proof We can easily check that Φ is continuous function from Ω to itself. For $x, y \in \Omega$,

$$d_H([\Phi x(t)]^{\alpha}, [\Phi y(t)]^{\alpha})$$
$$= d_H([S(t)g(x)]^{\alpha}, [S(t)g(y)]^{\alpha}) + d_H\Bigg(\Big[\int_0^t S(t-s) f(s, x(s),$$
$$\int_0^s q(s,\tau,x(\tau))d\tau)ds\Big]^{\alpha}, \Big[\int_0^t S(t-s) f(s, y(s), \int_0^s q(s,\tau,y(\tau))d\tau)ds\Big]^{\alpha}\Bigg)$$
$$+ d_H([\sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-))]^{\alpha}, [\sum_{0<t_k<t} S(t-t_k) I_k(y(t_k^-))]^{\alpha})$$
$$+ d_H\Bigg(\Big[\int_0^t S(t-s)\widetilde{G}^{-1}\Big(x^1 - g(x) + S(T)g(x) - \int_0^T S(T-s)$$
$$\times f(s, x(s), \int_0^s q(s,\tau,x(\tau))d\tau)ds - \sum_{0<t_k<T} S(T-t_k) I_k(x(t_k^-))\Big)ds\Big]^{\alpha},$$
$$\Big[\int_0^t S(t-s)\widetilde{G}^{-1}\Big(x^1 - g(y) + S(T)g(y) - \int_0^T S(T-s)$$

$$\times f(s,y(s),\int_0^s q(s,\tau,y(\tau))d\tau)ds - \sum_{0<t_k<T} S(T-t_k)I_k(y(t_k^-))\Big)ds\Big]^{\alpha}\Big)$$

$$\leq cLd_H([x(\cdot)]^{\alpha},[y(\cdot)]^{\alpha}) + c\bar{d}d_H([x(t)]^{\alpha},[y(t)]^{\alpha})$$

$$+c\int_0^t\Bigg\{L(1+c)d_H([x(\cdot)]^{\alpha},[y(\cdot)]^{\alpha}) + c\int_0^T\Big(k_1 d_H([x(s)]^{\alpha},[y(s)]^{\alpha})$$

$$+k_2M\int_0^s d_H([x(\tau)]^{\alpha},[y(\tau)]^{\alpha})d\tau\Big)ds + c\bar{d}d_H([x(s)]^{\alpha},[y(s)]^{\alpha})\Bigg\}ds.$$

Therefore,

$$d_{\infty}(\Phi x(t),\Phi y(t)) = \sup_{\alpha\in(0,1]} d_H([\Phi x(t)]^{\alpha},[\Phi y(t)]^{\alpha})$$

$$\leq cLd_{\infty}(x(\cdot),y(\cdot)) + c\bar{d}d_{\infty}(x(t),y(t))$$

$$+c\int_0^t\Bigg\{L(1+c)d_{\infty}(x(\cdot),y(\cdot)) + c\int_0^T\Big(k_1 d_{\infty}(x(s),y(s))$$

$$+k_2M\int_0^s d_{\infty}(x(\tau),y(\tau))d\tau\Big)ds + c\bar{d}d_{\infty}(x(s),y(s))\Bigg\}ds.$$

Hence

$$H_1(\Phi x,\Phi y) = \sup_{t\in J} d_{\infty}(\Phi x(t),\Phi y(t))$$

$$\leq c\Bigg\{(L+\bar{d}) + T\Big[L(1+c) + cT(k_1+k_2M\frac{T}{2}) + c\bar{d}\Big]\Bigg\}H_1(x,y).$$

By hypotheses (H8), Φ is a contraction mapping. By the Banach fixed point theorem, (5) has a unique fixed point $x\in\Omega$.

4 Example

Consider the semilinear one dimensional heat equation on a connected domain (0,1) for a material with memory, boundary condition $x(t,0) = x(t,1) = 0$ and with initial condition $x(0,z) + \sum_{k=1}^{p} c_k x(t_k,z) = x_0(z)$, where $x_0(z)\in E_N$. Let $x(t,z)$ be the internal energy and and $f(t,x(t,z),\int_0^t q(t,s,x(t,z))ds) = \tilde{2}tx(t,z)^2 + \int_0^t (t-s)x(s)ds$ be the external heat with memory. $\Delta x(t_k,z) = x(t_k^+,z) - x(t_k^-,z)$ is impulsive effect at $t=t_k(k=1,2,\cdots,m)$.

Let $A = \tilde{2}\dfrac{\partial^2}{\partial z^2}$, $\sum_{k=1}^{p} c_k x(t_k,z) = g(x)$, $\Delta x(t_k,z) = \Delta x(t_k)$, $x(t_k^+,z) - x(t_k^-,z) = I_k(x(t_k))$ and $G(t-s) = e^{-(t-s)}$, then the balance equation becomes

$$\frac{dx(t)}{dt} = \tilde{2}[x(t) - \int_0^t e^{-(t-s)}x(s)ds] \tag{7}$$

$$+\tilde{2}tx(t)^2 + \int_0^t (t-s)x(s)ds + u(t), t \in J, t \neq t_k,$$

$$x(0) + g(x) = x_0 \in E_N, \tag{8}$$

$$\Delta x(t_k) = I_k(x(t_k)), \qquad k = 1, 2, \cdots, m. \tag{9}$$

The α-level set of fuzzy number $\tilde{2}$ is $[2]^\alpha = [\alpha + 1, 3 - \alpha]$ for all $\alpha \in [0, 1]$. Then α-level sets of $f(t, x(t), \int_0^t q(t, s, x(s))ds)$ is

$$[f(t, x(t), \int_0^t q(t, s, x(s))ds)]^\alpha$$
$$= [t(\alpha + 1)(x_l^\alpha(t))^2 + \int_0^t (t-s)x_l^\alpha(t), t(3-\alpha)(x_r^\alpha(t))^2 + \int_0^t (t-s)x_r^\alpha(t)].$$

Further, we have

$$d_H([f(t, x(t), \int_0^t q(t, s, x(s))ds)]^\alpha, [f(t, y(t), \int_0^t q(t, s, y(s))ds)]^\alpha)$$
$$= d_H\Big([t(\alpha + 1)(x_l^\alpha(t))^2 + \int_0^t (t-s)x_l^\alpha(t), t(3-\alpha)(x_r^\alpha(t))^2 + \int_0^t (t-s)x_r^\alpha(t)],$$
$$[t(\alpha + 1)(y_l^\alpha(t))^2 + \int_0^t (t-s)y_l^\alpha(t), t(3-\alpha)(y_r^\alpha(t))^2 + \int_0^t (t-s)y_r^\alpha(t)]\Big)$$
$$= t \max\{(\alpha + 1)|(x_l^\alpha(t))^2 - (y_l^\alpha(t))^2|, (3-\alpha)|(x_r^\alpha(t))^2 - (y_r^\alpha(t))^2|\}$$
$$+ \int_0^t (t-s)d_H([x_l^\alpha(s), x_r^\alpha(s)], [y_l^\alpha(s), y_r^\alpha(s)])$$
$$\leq 3T|x_r^\alpha(t) + y_r^\alpha(t)| \max\{|x_l^\alpha(t) - y_l^\alpha(t)|, |x_r^\alpha(t) - y_r^\alpha(t)|\}$$
$$+ \frac{T^2}{2} \max\{|x_l^\alpha(t) - y_l^\alpha(t)|, |x_r^\alpha(t) - y_r^\alpha(t)|\}$$
$$= k_1 d_H([x(t)]^\alpha, [y(t)]^\alpha) + k_2 d_H([x(t)]^\alpha, [y(t)]^\alpha),$$

where k_1 and k_2 are satisfies the inequality in hypotheses (H1)-(H2), and also we have

$$d_H\left([g(x)]^\alpha, [g(y)]^\alpha\right) = d_H\left(\sum_{k=1}^p c_k[x(t_k)]^\alpha, \sum_{k=1}^p c_k[y(t_k)]^\alpha\right)$$
$$\leq |\sum_{k=1}^p c_k| \max_k d_H([x(t_k)]^\alpha, [y(t_k)]^\alpha) = L d_H\left([x(\cdot)]^\alpha, [y(\cdot)]^\alpha\right),$$

where L is satisfies the inequality in hypothesis (H3). Therefore f and g satisfy the global Lipschitz condition. Then all the conditions stated in Theorem 1 are satisfied, so the problem (7)-(9) has a unique fuzzy solution.

Let initial value x_0 is $\tilde{0}$. Target set is $x^1 = \tilde{2}$. The α-level set of fuzzy number $\tilde{0}$ is $[\tilde{0}] = [\alpha - 1, 1 - \alpha], \alpha \in (0, 1]$. We introduce the α-level set of $u(s)$

of equation (7)-(9).

$$
\begin{aligned}
&[u(s)]^\alpha = [u_l^\alpha(s), u_r^\alpha(s)] \\
&= \Bigg[\tilde{G}_l^{-1}\Bigg((\alpha+1) - \sum_{k=1}^{p} c_k x_l^\alpha(t_k) - S_l^\alpha(T)\Big((\alpha-1) - \sum_{k=1}^{p} c_k x_l^\alpha(t_k)\Big) \\
&\quad - \int_0^T S_l^\alpha(T-s)\Big(s(\alpha+1)(x_l^\alpha(s))^2 + \int_0^s (s-\tau)x_l^\alpha(\tau)d\tau\Big)ds \\
&\qquad\qquad - \sum_{0<t_k<T} S_l^\alpha(T-t_k){I_k}_l^\alpha(x(t_k^-))\Bigg), \\
&\quad \tilde{G}_r^{-1}\Bigg((3-\alpha) - \sum_{k=1}^{p} c_k x_r^\alpha(t_k) - S_r^\alpha(T)\Big((1-\alpha) - \sum_{k=1}^{p} c_k x_r^\alpha(t_k)\Big) \\
&\quad - \int_0^T S_r^\alpha(T-s)\Big(s(3-\alpha)(x_r^\alpha(s))^2 + \int_0^s (s-\tau)x_r^\alpha(\tau)d\tau\Big)ds \\
&\qquad\qquad - \sum_{0<t_k<T} S_r^\alpha(T-t_k){I_k}_r^\alpha(x(t_k^-))\Bigg)\Bigg].
\end{aligned}
$$

Then substituting this expression into the integral system with respect to (7)-(9) yields α-level set of $x(T)$.

$$
\begin{aligned}
&[x(T)]^\alpha \\
&= \Bigg[S_l^\alpha(T)((\alpha-1) - \sum_{k=1}^{p} c_k x_l^\alpha(t_k)) + \int_0^T S_l^\alpha(T-s)\Big(s(\alpha+1)(x_l^\alpha(s))^2 \\
&\quad + \int_0^s (s-\tau)x_l^\alpha(\tau)d\tau\Big)ds + \sum_{0<t_k<T} S_l^\alpha(T-t_k){I_k}_l^\alpha(x(t_k^-)) \\
&\quad + \int_0^T S_l^\alpha(T-s)(\tilde{G}_l^\alpha)^{-1}\Big((\alpha+1) - \sum_{k=1}^{p} c_k x_l^\alpha(t_k) - S_l^\alpha(T) \\
&\qquad \times\Big((\alpha-1) - \sum_{k=1}^{p} c_k x_l^\alpha(t_k)\Big) - \sum_{0<t_k<T} S_l^\alpha(T-t_k){I_k}_l^\alpha(x(t_k^-)) \\
&\qquad - \int_0^T S_l^\alpha(T-s)\Big(s(\alpha+1)(x_l^\alpha(s))^2 + \int_0^s (s-\tau)x_l^\alpha(\tau)d\tau\Big)ds\Big)ds, \\
&\quad S_r^\alpha(T)((1-\alpha) - \sum_{k=1}^{p} c_k x_r^\alpha(t_k)) + \int_0^T S_r^\alpha(T-s)\Big(s(3-\alpha)(x_r^\alpha(s))^2 \\
&\quad + \int_0^s (s-\tau)x_r^\alpha(\tau)d\tau\Big)ds + \sum_{0<t_k<T} S_r^\alpha(T-t_k){I_k}_r^\alpha(x(t_k^-)) \\
&\quad + \int_0^T S_r^\alpha(T-s)(\tilde{G}_l^\alpha)^{-1}\Big((3-\alpha) - \sum_{k=1}^{p} c_k x_r^\alpha(t_k) - S_r^\alpha(T)
\end{aligned}
$$

$$\times\Big((1-\alpha)-\sum_{k=1}^{p}c_k x_r^{\alpha}(t_k)\Big)-\sum_{0<t_k<T}S_r^{\alpha}(T-t_k)I_{k\,r}^{\alpha}(x(t_k^-))$$
$$-\int_0^T S_r^{\alpha}(T-s)\Big(s(3-\alpha)(x_r^{\alpha}(s))^2+\int_0^s(s-\tau)x_r^{\alpha}(\tau)d\tau\Big)ds\Big)ds\Bigg]$$
$$=[(\alpha+1)-\sum_{k=1}^{p}c_k x_l^{\alpha}(t_k),(3-\alpha)-\sum_{k=1}^{p}c_k x_r^{\alpha}(t_k)]=[\tilde{2}-\sum_{k=1}^{p}c_k x(t_k)]^{\alpha}.$$

Then all the conditions stated in Theorem 3.2 are satisfied, so the system (7)-(9) is nonlocal controllable on $[0,T]$.

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ITERATIVE APPROXIMATION TO COMMON FIXED POINTS OF A SEQUENCE OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Let E be a Banach space, C a nonempty closed convex subset of E, $f : C \to C$ a contraction, and $T_n : C \to C$ a nonexpansive mapping with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ where $F(T_n)$ is the set of fixed points of T_n. It is proved that the iterative algorithm $x_{n+1} = \lambda_{n+1} f(x_n) + (1-\lambda_{n+1}) T_{n+1} x_n$ $(n \geq 0)$ converges strongly to a solution of certain variational inequality provided E is reflexive and has a uniformly Gâteaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings and provided the sequence $\{\lambda_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ and the sequence $\{T_n\}$ is uniformly asymptotically regular.

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1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that$\|f(x)-f(y)\| \leq k\|x-y\|, \quad x,\ y \in C$. We use Σ_C to denote the collection of all contractions on C. That is, $\Sigma_C = \{f : f : C \to C \text{ a contraction}\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in C.

Now let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ $x,\ y \in C$) and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

We consider the iteration scheme: for $N > 1$, $T_1, T_2, \cdots, T_N$ nonexpansive mappings, $u,\ x_0 \in C$ and $\lambda_n \subset (0,1)$,

$$x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0, \tag{1.1}$$

where $T_n := T_{n \mod N}$. Since Halpern [8] firstly introduced the iteration scheme (1.1) for $u = 0$, $N = 1$ (that is, he considered only one mapping) in 1967, this iteration scheme is called a Halpern type iteration. Halpern also pointed out that the control conditions

(C1) $\lim_{n\to\infty} \lambda_n = 0$;
(C2) $\sum_{n=1}^{\infty} \lambda_n = \infty$ or, equivalently, $\prod_{n=1}^{\infty}(1 - \lambda_n) = 0$

are necessary for the convergence of the iteration scheme (1.1) to a fixed point of T. Strong convergence of this type iterative sequence has been widely studied by many mathematicians: see, for example, Browder [4], Halpern [8], Lions [14], Reich [20], Shioji and Takahashi [22], Wittmann [27], Xu [28] for $N = 1$ and Bauschke [3], Jung [9], Jung et al. [11], Jung and Kim [12], O'Hara et al. [17,18], Shimizu and Takataki [21], Song and Chen [23], and Zhou et al. [30] for $N > 1$, respectively.

Very recently, Aoyama et al. [1] considered the iteration scheme with $u = x_0$ for a sequence $\{T_n\}$ of nonexpansive mappings:

$$x_{n+1} = \lambda_{n+1}u + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0,$$

under the conditions $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in C\} < \infty$ and $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$.

On the other hand, for $N = 1$, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [16]. In 2004, Xu [29] proposed the iteration scheme: for $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, $f \in \Sigma_C$ and $\lambda_n \in (0, 1)$,

$$x_{n+1} = \lambda_{n+1}f(x_n) + (1 - \lambda_{n+1})Tx_n, \quad n \geq 0,$$

in order to extend Theorem 2.2 of Moudafi [16] to a uniformly smooth Banach space. Jung [10] improved the results of Xu [29] to the case of a family of finite nonexpansive mappings. Very recently, Song and Chen [24] and Song et al. [25] considered the iteration scheme for a sequence $\{T_n\}$ of nonexpansive mappings with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$:

$$x_{n+1} = \lambda_{n+1}f(x_n) + (1 - \lambda_{n+1})T_{n+1}x_n, \quad n \geq 0, \tag{1.2}$$

with the conditions (C1) and (C2) and the uniform asymptotic regularity on the mapping sequence $\{T_n\}$ in a Banach space having a weakly sequentially continuous duality mapping and in a strictly convex Banach space with a uniformly Gâteau differentiable norm, respectively, and overcame a gap in the corresponding results of [9, 17, 18].

In this paper, motivated by above-mentioned results, we consider the iteration scheme (1.2) as the viscosity approximation method for the sequence $\{T_n\}$ of nonexpansive mappings. First, by using the uniform asymptotic regularity on the sequence $\{T_n\}$, we establish a strong convergence theorem for the sequence

$\{T_n\}$ in a reflexive Banach space having a uniformly Gâteaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Then we prove that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a common fixed point of $\{T_n\}$ under the conditions (C1) and (C2) and the uniform asymptotic regularity on the sequence $\{T_n\}$ in the same Banach space. Moreover, we show that the strong limit is a solution of certain variational inequality. The main results extend and improve the corresponding results of Aoyama et al. [1], Jung [9], O'Hara et al. [17,18], and Shimizu and Takahashi [21].

2. Preliminaries and Lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to x.

The (*normalized*) *duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x,\ y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. It is known that E is smooth if and only if each duality mapping J is single-valued. It is also well-known that if E has a uniformly Gâteaux differentiable norm, J is uniformly norm to weak* continuous on each bounded subsets of E ([5]).

Let C be a nonempty closed convex subset of E. C is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D. Let D be a subset of C. Then a mapping $Q : C \to D$ is said to be a retraction from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$ with $Qx + t(x - Qx) \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E , it is well-known [6, p. 48]) that Q is a sunny nonexpansive retraction from C onto D if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, \quad z \in D. \tag{2.2}$$

Let LIM be a continuous linear functional on l^∞. According to time and circumstances, we LIM(a_n) instead of LIM(a) for $a = \{a_n\} \in l^\infty$. LIM is said to be *Banach limit* if

$$\text{LIM}(a_n) = \text{LIM}(a_{n+1})$$

for every $a = \{a_n\} \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if LIM is a Banach limit, then

$$\liminf_{n\to\infty} a_n \le \text{LIM}(a_n) \le \limsup_{n\to\infty} a_n$$

for all $a = \{a_n\} \in l^\infty$.

Let C be a nonempty subset of E and $\{T_n\}$ a sequence of mappings from C into C. Recall [24] that the mapping sequence $\{T_n\}$ is said to be *uniformly asymptotically regular* (for short, u.a.r.) on C if for all $m \ge 1$ and any bounded sunset K of C,

$$\lim_{n\to\infty} \sup_{x\in K} \|T_m(T_n x) - T_n x\| = 0.$$

(For examples of u.a.r, see [24]).

Finally, We need the following lemmas for the proof of our main results. (Lemma 2.1 was also given in [13]. Lemma 2.2 is essentially Lemma 2 in [15] (also see [28]). Lemma 2.3 was given in [7, 26], which is essentially a variant of Lemma 1.2 in [19].)

Lemma 2.1. *Let X be a real Banach space and J be the duality mapping. Then, for any given $x,\ y \in X$, we have*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$.

Lemma 2.2. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\beta_n + \gamma_n, \quad n \ge 0,$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfying the condition:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$,
(ii) $\limsup_{n\to\infty} \beta_n \le 0$ *or* $\sum_{n=1}^{\infty} \lambda_n\beta_n < \infty$,
(iii) $\gamma_n \ge 0$ $(n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm and $\{x_n\}$ be a bounded sequence in E. Let* LIM *be a Banach limit on l^∞ and $q \in C$. Then*

$$\text{LIM}\|x_n - q\|^2 = \min_{y\in C} \text{LIM}\|x_n - y\|^2$$

if and only if

$$\text{LIM}\langle x - q, J(x_n - q)\rangle \le 0$$

for all $x \in C$, where J be the duality mapping of E.

3. Main results

First, we study the existence of solutions of certain variational inequality.

For any $m \geq 1$, $T_m : C \to C$ is nonexpansive and so, for any $t_m \in (0,1)$ and $f \in \Sigma_C$, $t_m f + (1-t_m)T_m : C \to C$ defines a strict contraction mapping. Thus, by Banach contraction mapping principle, there exists a unique fixed point $x_{t_m}^f$ satisfying

$$x_{t_m}^f = t_m f(x_{t_m}^f) + (1-t_m)T_m x_{t_m}^f. \tag{A}$$

For simplicity we will write x_m for $x_{t_m}^f$ provided no confusion occurs.

Now we show that the sequence $\{x_m\}$ defined by (A) converges strongly some common fixed point of $\{T_m\}$ $(m = 1,\ 2, \cdots)$.

Theorem 3.1. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E and $\{T_m\}$ $(m = 1, 2, \cdots)$ a u.a.r. sequence of nonexpansive mappings from C into itself with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_m\}$ be defined by* (A) *and $t_m \in (0,1)$ such that $\lim_{m\to\infty} t_m = 0$. Then as $m \to \infty$, $\{x_m\}$ converges strongly to a point in F. If we define $Q : \Sigma_C \to F$ by*

$$Q(f) := \lim_{m\to\infty} x_m, \quad f \in \Sigma_C, \tag{3.1}$$

then $Q(f)$ is the unique solution in F of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F.$$

In particular, if $f = u \in C$ is a constant, then (3.1) *is reduced to the sunny nonexpansive retraction from C onto F,*

$$\langle Q(u)-u, J(Q(u)-p)\rangle \leq 0, \quad u \in C, \quad p \in F.$$

Proof. We first show that $\{x_m\}$ is bounded. In fact, for $p \in F$, we have

$$\begin{aligned} \|x_m - p\| &\leq (1-t_m)\|T_m x_m - p\| + t_m\|f(x_m) - p\| \\ &\leq (1-t_m)\|x_m - p\| + t_m\|f(x_m) - p\| \end{aligned}$$

and so

$$\begin{aligned} \|x_m - p\| \leq \|f(x_m) - p\| &\leq \|f(x_m) - f(p)\| + \|f(p) - p\| \\ &\leq k\|x_m - p\| + t\|f(p) - p\|. \end{aligned}$$

Hence

$$\|x_m - p\| \leq \frac{1}{1-k}\|f(p) - p\|$$

and $\{x_m\}$ is bounded, so are $\{T_m x_m\}$ and $\{f(x_m)\}$. As a result, it also follows that

$$\|x_m - T_m x_m\| = t_m \|T_m x_m - f(x_m)\| \to 0 \quad (\text{as } t_m \to 0).$$

Moreover, since $\{T_m\}$ $(m = 1, 2, \cdots)$ is a u.a.r. sequence of nonexpansive mappings, we have for any $n \geq 1$,

$$\lim_{m\to\infty} \|T_n(T_m x_m) - T_m x_m\| \leq \lim_{m\to\infty} \sup_{x\in K} \|T_n(T_m x) - T_m x\| = 0,$$

where K is any bounded subset of C containing $\{x_m\}$. Hence

$$\begin{aligned}\|x_m - T_n x_m\| &\leq \|x_m - T_m x_m\| + \|T_m x_m - T_n(T_m x_m)\| \\ &\quad + \|T_n(T_m x_m) - T_n x_m\| \\ &\leq 2\|x_m - T_m x_m\| + \|T_m x_m - T_n(T_m x_m)\| \to 0 \text{ as } m \to \infty.\end{aligned}$$

That is, for any $n \geq 1$,

$$\lim_{m\to\infty} \|x_m - T_n x_m\| = 0. \tag{3.2}$$

We now show that $\{x_m\}$ converges strongly as $t_m \to 0$ a point in F. To this end, let $t_{m_k} \to 0$ and $\{x_{m_k}\} := \{x_k\}$ be subsequence of $\{x_m\}$. Define $\phi : C \to [0, \infty)$ by

$$\phi(x) = \text{LIM}\|x_k - x\|^2, \quad x \in C,$$

where LIM is a Banach limit on l^∞. Since ϕ is continuous and convex, $\phi(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, ϕ attains its infimum over C [2, p. 79]. Let

$$K = \{z \in C : \phi(z) = \min_{x\in C} \text{LIM}\|x_k - x\|^2\}.$$

It is easily seen that K is a nonempty closed convex bounded subset of E. Moreover, K is invariant under T_n for any $n \geq 1$. In fact, since $\|x_k - T_n x_k\| \to 0$ by (3.2), it follows that for each $z \in K$

$$\begin{aligned}\phi(Tz) &= \text{LIM}\|x_k - T_n z\|^2 \\ &= \text{LIM}\|T_n x_k - T_n z\|^2 \leq \text{LIM}\|x_k - z\|^2 = \phi(z).\end{aligned}$$

So, by the hypothesis, there exists a fixed point q of T_n in K, that is, $q = T_n q$. Since n is arbitrary, we have $q \in F$. By Lemma 2.3, we have for all $x \in C$

$$\text{LIM}\langle x - q, J(x_k - q)\rangle \leq 0. \tag{3.3}$$

On the other hand, since

$$\begin{aligned}x_m - q &= t_m(f(x_m) - q) + (1 - t_m)(T_m x_m - q), \\ \|x_m - q\|^2 &= t_m\langle f(x_m) - q, J(x_m - q)\rangle + (1 - t_m)\langle T_m x_m - q, J(x_m - q)\rangle \\ &\leq t_m\langle f(x_m) - q, J(x_m - q)\rangle + (1 - t_m)\|x_m - q\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\|x_m - q\|^2 &\le \langle f(x_m) - q, J(x_m - q)\rangle \\ &= \langle f(x_m) - x, J(x_m - q)\rangle + \langle x - q, J(x_m - q)\rangle.\end{aligned} \tag{3.4}$$

Hence by (3.3), for $x \in C$,

$$\begin{aligned}\mathrm{LIM}\|x_k - q\|^2 &\le \mathrm{LIM}\langle f(x_k) - x, J(x_k - q)\rangle + \mathrm{LIM}\langle x - q, J(x_k - q)\rangle \\ &\le \mathrm{LIM}\langle f(x_k) - x, J(x_k - q)\rangle \le \mathrm{LIM}\|f(x_k) - x\|\|x_k - q\|.\end{aligned}$$

In particular,

$$\mathrm{LIM}\|x_k - q\|^2 \le \mathrm{LIM}\|f(x_k) - f(q)\|\|x_k - q\| \le k\mathrm{LIM}\|x_k - q\|^2.$$

Hence

$$\mathrm{LIM}\|x_k - q\|^2 = 0$$

and there exists a subsequence, which is still denoted $\{x_k\}$, such that $x_k \to q$.

Now suppose that there is another subsequence $\{x_j\}$ of $\{x_m\}$ such that $x_j \to p$. Then p is a fixed point of T_n by (3.2) for any $n \ge 1$, that is, $p \in F$. It follows from (3.4) that

$$\|p - q\|^2 \le \langle f(p) - q, J(p - q)\rangle, \tag{3.5}$$

and

$$\|q - p\|^2 \le \langle f(q) - p, J(q - p)\rangle. \tag{3.6}$$

Adding (3.5) and (3.6) yields

$$2\|p - q\|^2 \le \|p - q\|^2 + \langle f(p) - f(q), J(p - q)\rangle \le (1 + k)\|p - q\|^2.$$

Since $k \in (0, 1)$, this implies that $p = q$. Hence $x_m \to q$ as $m \to \infty$.

Define $Q : \Sigma_C \to F$ by

$$Q(f) = \lim_{m\to\infty} x_m.$$

Since $x_m = t_m f(x_m) + (1 - t_m)T_m x_m$, we have

$$(I - f)x_m = -\frac{1 - t_m}{t_m}(I - T_m)x_m.$$

Hence for $p \in F$,

$$\begin{aligned}&\langle (I - f)x_m, J(x_m - p)\rangle \\ &= -\frac{1 - t_m}{t_m}\langle (I - T_m)x_m - (I - T_m)p, J(x_m - p)\rangle \le 0.\end{aligned}$$

Letting $m \to \infty$ yields

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \le 0.$$

This implies that $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F. \tag{3.7}$$

Since X is smooth, in F, there is the unique solution of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C, \quad p \in F.$$

In fact, suppose that $p, q \in F$ satisfy (3.7), we know that

$$\langle (I-f)q, J(q-p)\rangle \leq 0 \tag{3.8}$$

and

$$\langle (I-f)p, J(p-q)\rangle \leq 0. \tag{3.9}$$

Adding (3.8) and (3,9) up, we have

$$(1-k)\|q-p\|^2 \leq \langle (I-f)q-(I-f)p, J(q-p)\rangle \leq 0,$$

and so $q=p$.

If $f=u$ is a constant, then

$$\langle Q(u)-u, J(Q(u)-p)\rangle \leq 0, \quad u \in C, \quad p \in F.$$

Hence by (2.2), Q reduces to the sunny nonexpansive retraction from C to F. □

Now we study the strong convergence of the iteration scheme for a sequence of nonexpansive mappings.

Theorem 3.2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm such that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E and $\{T_m\}$ $(m=1,2,\cdots)$ a u.a.r. sequence of nonexpansive mappings from C into itself with $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions:*

(C1) $\lim_{n\to\infty} \lambda_n = 0$;

(C2) $\sum_{n=1}^{\infty} \lambda_n = \infty$ or, equivalently, $\prod_{n=1}^{\infty}(1-\lambda_n) = 0$.

Let $f \in \Sigma_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be generated by

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1-\lambda_{n+1}) T_{n+1} x_n, \quad n \geq 0. \tag{3.10}$$

Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ is the unique solution in F of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0 \quad f \in \Sigma_C, \quad p \in F.$$

Proof. First, we note that by Theorem 3.1, there exists the unique solution $Q(f)$ of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F,$$

where $Q : \Sigma_C \to F$ is defined by $Q(f) = \lim_{m\to\infty} z_m$ and $z_m = t_m f(z_m) + (1 - t_m)T_m z_m$ with $\lim_{m\to\infty} t_m = 0$.

We proceed with the following steps:

Step 1. $\|x_n - z\| \le \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ for all $n \ge 0$ and all $z \in F$ and so $\{x_n\}$ is bounded.

We use an inductive argument. Indeed, let $z \in F$ and

$$d = \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}.$$

Then by the nonexpansivity of T_n and $f \in \Sigma_E$,

$$\begin{aligned}\|x_1 - z\| &\le (1-\lambda_1)\|T_1x_0 - z\| + \lambda_1\|f(x_0) - z\| \\ &\le (1-\lambda_1)\|x_0 - z\| + \lambda_1(\|f(x_0) - f(z)\| + \|f(z) - z\|) \\ &\le (1-\lambda_1)\|x_0 - z\| + \lambda_1(k\|x_0 - z\| + \|f(z) - z\|) \\ &\le (1-(1-k)\lambda_1)d + \lambda_1(1-k)d \\ &= d.\end{aligned}$$

Using an induction, we obtain $\|x_{n+1} - z\| \le d$. Hence $\{x_n\}$ is bounded, and so are $\{T_{n+1}x_n\}$ and $\{f(x_n)\}$.

Step 2. $\lim_{n\to\infty} \|x_{n+1} - T_{n+1}x_n\| = 0$. Indeed, since

$$\|x_{n+1} - T_{n+1}x_n\| \le \lambda_{n+1}\|T_{n+1}x_n - f(x_n)\| \le L\lambda_{n+1}$$

for some L, by (C1), we have $\lim_{n\to\infty} \|x_{n+1} - T_{n+1}x_n\| = 0$.

Step 3. $\lim_{n\to\infty} \|x_n - T_m x_n\| = 0$ for any $m \ge 1$. Since $\{T_n\}$ $(n = 1, 2, \cdots)$ is a u.a.r. sequence of nonexpansive mappings, we have for any $m \ge 1$

$$\lim_{n\to\infty} \|T_m(T_n x_{n-1}) - T_n x_{n-1}\| \le \lim_{n\to\infty} \sup_{x\in K} \|T_m(T_n x) - T_n x\| = 0,$$

where K is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned}\|x_n - T_m x_n\| &\le \|x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_m(T_n x_{n-1})\| \\ &\quad + \|T_m(T_n x_{n-1}) - T_m x_n\| \\ &\le 2\|x_n - T_n x_{n-1}\| + \|T_n x_{n-1} - T_m(T_n x_{n-1})\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

That is, for any $m \ge 1$, $\lim_{n\to\infty} \|x_n - T_m x_n\| = 0$.

Step 4. $\limsup_{n\to\infty}\langle (I-f)Q(f), J(Q(f)-x_n)\rangle \le 0$. To prove this, let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that

$$\limsup_{n\to\infty}\langle (I-f)Q(f), J(Q(f)-x_n)\rangle = \lim_{j\to\infty}\langle (I-f)Q(f), J(Q(f)-x_{n_j})\rangle$$

and $x_{n_j} \rightharpoonup p$ for some $p \in E$. Now let $Q(f) = \lim_{m\to\infty} z_m$, where $z_m = t_m f(z_m) + (1-t_m)T_m z_m$. Then we can write

$$z_m - x_{n_j} = t_m(f(z_m) - x_{n_j}) + (1-t_m)(T_m z_m - x_{n_j}).$$

Putting

$$\begin{aligned} a_j(t_m) =& (1-t_m)^2\|T_m x_{n_j} - x_{n_j}\| \\ & \times (2\|z_m - x_{n_j}\| + \|T_m x_{n_j} - x_{n_j}\|) \to 0 \ (j\to\infty) \end{aligned}$$

by Step 3 and using Lemma 2.1, we obtain

$$\begin{aligned} \|z_m - x_{n_j}\|^2 &\le (1-t_m)^2\|T_m z_m - x_{n_j}\|^2 + 2t_m\langle f(z_m) - x_{n_j}, J(z_m - x_{n_j})\rangle \\ &\le (1-t_m)^2(\|T_m z_m - T_m x_{n_j}\| + \|T_m x_{n_j} - x_{n_j}\|)^2 \\ &\quad + 2t_m\langle f(z_m) - z_m, J(z_m - x_{n_j})\rangle + 2t_m\|z_m - x_{n_j}\|^2 \\ &\le (1-t_m)^2\|z_m - x_{n_j}\|^2 + a_j(t_m) \\ &\quad + 2t_m\langle f(z_m) - z_m, J(z_m - x_{n_j})\rangle + 2t_m\|z_m - x_{n_j}\|^2. \end{aligned}$$

The last inequality implies

$$\langle z_m - f(z_m), J(z_m - x_{n_j})\rangle \le \frac{t_m}{2}\|z_m - x_{n_j}\|^2 + \frac{1}{2t_m}a_j(t_m).$$

It follows that

$$\lim_{j\to\infty}\langle z_m - f(z_m), J(z_m - x_{n_j})\rangle \le \frac{t_m}{2}M, \tag{3.11}$$

where $M > 0$ is a constant such that $M \ge \|z_m - x_n\|^2$ for all $n \ge 0$ and $t_m \in (0,1)$. Taking the lim sup as $m\to\infty$ in (3.11) and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$\lim_{j\to\infty}\langle (I-f)Q(f), J(Q(f) - x_{n_j})\rangle \le 0.$$

Step 5. $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$. By using (3.10), we have

$$x_{n+1} - Q(f) = \lambda_{n+1}(f(x_n) - Q(f)) + (1-\lambda_{n+1})(T_{n+1}x_n - Q(f)).$$

Applying Lemma 2.1, we obtain

$$\begin{aligned}
&\|x_{n+1}-Q(f)\|^2\\
&\le (1-\lambda_{n+1})^2\|T_{n+1}x_n-Q(f)\|^2+2\lambda_{n+1}\langle f(x_n)-Q(f),J(x_{n+1}-Q(f))\rangle\\
&\le (1-\lambda_{n+1})^2\|x_n-Q(f)\|^2+2\lambda_{n+1}\langle f(x_n)-f(Q(f)),J(x_{n+1}-Q(f))\rangle\\
&\quad+2\lambda_{n+1}\langle f(Q(f))-Q(f),J(x_{n+1}-Q(f))\rangle\\
&\le (1-\lambda_{n+1})^2\|x_n-Q(f)\|^2+2k\lambda_{n+1}\|x_n-Q(f)\|\|x_{n+1}-Q(f)\|\\
&\quad+2\lambda_{n+1}\langle f(Q(f))-Q(f),J(x_{n+1}-Q(f))\rangle\\
&\le (1-\lambda_{n+1})^2\|x_n-Q(f)\|^2+k\lambda_{n+1}(\|x_n-Q(f)\|^2+\|x_{n+1}-Q(f)\|^2)\\
&\quad+2\lambda_{n+1}\langle f(Q(f))-Q(f),J(x_{n+1}-Q(f))\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1}-Q(f)\|^2 &\le \frac{1-(2-k)\lambda_{n+1}+\lambda_{n+1}^2}{1-k\lambda_{n+1}}\|x_n-Q(f)\|^2\\
&\quad+\frac{2\lambda_{n+1}}{1-k\lambda_{n+1}}\langle (I-f)Q(f),J(Q(f)-x_{n+1})\rangle\\
&\le \frac{1-(2-k)\lambda_{n+1}}{1-k\lambda_{n+1}}\|x_n-Q(f)\|^2+\frac{\lambda_{n+1}^2}{1-k\lambda_{n+1}}M^2\\
&\quad+\frac{2\lambda_{n+1}}{1-k\lambda_{n+1}}\langle (I-f)Q(f),J(Q(f)-x_{n+1})\rangle,
\end{aligned}\tag{3.12}$$

where $M=\sup_{n\ge 0}\|x_n-Q(f)\|$. Put

$$\alpha_n=\frac{2(1-k)\lambda_{n+1}}{1-k\lambda_{n+1}},\quad \beta_n=\frac{M^2\lambda_{n+1}}{2(1-k)}+\frac{1}{1-k}\langle (I-f)Q(f),J(Q(f)-x_{n+1})\rangle.$$

From (C1), (C2) and Step 4, it follows that $\alpha_n\to 0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$, and $\limsup_{n\to\infty}\beta_n\le 0$. Since (3.12) reduces to

$$\|x_{n+1}-Q(f)\|^2\le(1-\alpha_n)\|x_n-Q(f)\|^2+\alpha_n\beta_n,$$

from Lemma 3.2 with $\gamma_n=0$, we conclude that $\lim_{n\to\infty}\|x_n-Q(f)\|=0$. This completes the proof. □

As a direct consequence, we have the following:

Corollary 3.1. *Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E and $\{T_m\}$ $(m=1,2,\cdots)$ a u.a.r. sequence of nonexpansive mappings from C into itself with $F:=\bigcap_{n=1}^{\infty}F(T_n)\ne\emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions* (C1) *and* (C2) *in Theorem 3.2. Let $f\in\Sigma_C$ and $x_0\in C$ be chosen arbitrarily. Let $\{x_n\}$ be generated by*

$$x_{n+1}=\lambda_{n+1}f(x_n)+(1-\lambda_{n+1})T_{n+1}x_n,\quad n\ge 0.$$

Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ is unique solution in F of the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0 \quad f \in \Sigma_C, \quad p \in F.$$

As in [25, Corollary 3.3], by using Theorem 3.2 together with Lemma 3.1 and Lemma 3.2 of [17] (Lemma 1 of [21]), we can also obtain the following result.

Corollary 3.2. *Let E be a Banach space, C a nonempty closed convex subset of E, and T, $S : C \to C$ nonexpansive mappings with fixed points. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies the conditions* (C1) *and* (C2) *in Theorem 3.2.*
(a) *Set $T_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} T^j x$ for $n \ge 1$ and $x \in C$. For $f \in \Sigma_C$ and $x_0 \in C$, define*

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1-\lambda_{n+1}) T_{n+1} x_n, \quad n \ge 0.$$

If E is a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F(T)$ is the unique solution of the following variational inequality:

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0 \quad f \in \Sigma_C, \quad p \in F(T).$$

(b) *Set $T_n(x) = \frac{2}{n(n+1)}\sum_{k=0}^{n-1}\sum_{i+j=k} S^i T^j(x)$ for $n \ge 1$. For $f \in \Sigma_C$ and $x_0 \in C$, define*

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1-\lambda_{n+1}) T_{n+1} x_n, \quad n \ge 0.$$

Suppose that $ST = TS$ and $F(S) \cap F(T) \ne \emptyset$. If E is a Hilbert space H, then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F(ST)$ is the unique solution of the following variational inequality:

$$\langle (I-f)Q(f), Q(f)-p\rangle \le 0 \quad f \in \Sigma_C, \quad p \in F(ST),$$

where $F(ST) := F(S) \cap F(T)$.

Remark 3.1. (1) In [9,17,18], it was proved that the sequence $\{x_n\}$ generated by (1.1) for the sequence $\{T_n\}$ converges strongly to some point $Q(u)$ ($Q_{F(V)}a$ or Py). But it is not clear whether or not $Q(u)$ ($Q_{F(V)}a$ or Py) is a point in $F := \bigcap_{n=1}^{\infty} F(T_n)$. This was a gap. Theorem 3.2 (and Corollary 3.1) not only overcomes the gaps in [9,17,18] but also improves the corresponding results in [9,17,18] to the viscosity method.

(2) Theorem 3.2 generalizes Theorem 3.4 of [1] to the viscosity method in more general Banach spaces with the uniform asymptotic regularity on the sequence $\{T_n\}$ in place of the condition $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty$ without the condition $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$.

(3) Theorem 3.2 also appears to be independent of Theorem 3.2 in [25]. In fact, it easy to find examples of spaces which satisfy the fixed point property for nonexpansive mappings, which are not strictly convex. However, it appears to be unknown whether a reflexive and strictly convex space satisfies the fixed point property for nonexpansive mappings.

(4) As the viscosity method, Corollary 3.2 (a) extends Corollary 3.4 (a) in [17] to a Banach space setting.

(5) In the case of $f(x) = u$, $x \in C$, a constant, Corollary 3.2 (b) is just Theorem 1 of Shimizu and Takahashi [21].

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A NOTE ON THE p-ADIC q-TRANSFER OPERATOR

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Abstract. In this note, we consider p-adic q-transfer operator. From this operator, we derive that the eigenvalues of the p-adic q-transfer operator are q-Euler polynomials, and are associated with the eigenvalues $\frac{1}{[p]_q^n}$.

§1. Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an

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indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < 1$.

Let d be a fixed positive odd integer and let p be a fixed odd prime number. We now set

$$X = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z},$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. The basic q-numbers are defined by $[x]_q = \frac{1-q^x}{1-q}$ and $[x]_{-q} = \frac{1-(-q)^x}{1+q}$. We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x,y) = \frac{f(x)-f(y)}{x-y}, \quad \text{cf.[2,3,6]}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_{-q}} \sum_{0 \le j < p^N} (-1)^j q^j f(j) = \sum_{0 \le j < p^N} f(j)\mu_{-q}(j + p^N\mathbb{Z}_p), \text{ see [3]},$$

representing a q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ will be defined as the limit $(n \to \infty)$ of those sums, when it exists. In [3], the fermionic p-adic q-integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-q}(f) = \int_X f(x)d\mu_{-q}(x) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[dp^N]_{-q}} \sum_{x=0}^{dp^N-1} f(x)(-q)^x.$$

In previous paper[3], we constructed the q-extension of Euler polynomials by using p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$E_n(x:q) = \int_{\mathbb{Z}_p} [t+x]_q^n d\mu_{-q}(t), \quad x \in \mathbb{Z}_+, \quad \text{see [1,2,3]}. \tag{1}$$

A NOTE ON THE p-ADIC q-TRANSFER OPERATOR

In mathematics, the transfer operator encodes information about an iterated map and is frequently used to study the behavior of dynamical systems, statistical mechanics, quantum chaos and fractals. The transfer operator is sometimes called the Ruelle operator, after David Ruelle, or the Ruelle-Perron-Frobenius operator [4,5]. The iterated function to be studied is a map $f : X \to X$ for an arbitrary set X. The transfer operator is defined as an operator L acting on the space of function $\phi : X \to \mathbb{C}$ as

$$(L\phi)(x) = \sum_{y \in f^{-1}(x)} g(y)\phi(y),$$

where $g : X \to \mathbb{C}$ is an auxiliary valuation function ([4,5]). Consider a function $g : [0,1] \to [0,1]$. A shift in perspective may be gained not by considering how g acts on points or open sets, but instead by considering how g acts on distributions on the unit interval. Intuitively, one might consider a dusting of points on the unit interval, with a local density given by $\rho(x)$ at a point $x \in [0,1]$ and then consider how this dusting or density involves upon iteration by g . This verbal description may given from as

$$\rho'(y) = \int_0^1 \delta(y - g(y))\rho(x)dx,$$

where $\rho'(y)$ is the new density at point $y = g(x)$ and δ is the direct delta function.

In this viewpoint, g becomes an operator that maps densities ρ to orther densities ρ' , or notationally $L_g(\rho) = \rho'$. The operator Lg is the transfer operator(or the Rueller-Perron-Frobenius operator). In this note we consider the p-adic q-transfer operator. Finally we prove that the eigenvalues of the p-adic q-transfer operators are the q-Euler polynomials, and are associated with the eigenvalues $\frac{1}{[p]_q^n}$.

2. p-adic q-Transfer Operator

From (1), we derive

$$E_m(x : q) = \frac{[2]_q}{(1-q)^m} \sum_{i=0}^{m} \binom{m}{i} q^{xi}(-1)^i \frac{1}{1+q^{i+1}} \tag{2}$$

where $E_m(x:q)$ are q-Euler polynomials. By (1), we see that

$$\begin{aligned} E_k(x:q) &= \int_X [x+t]_q^k d\mu_{-q}(t) \\ &= \lim_{\rho\to\infty} \frac{1}{[dp^\rho]_{-q}} \sum_{n=0}^{dp^\rho-1} [x+n]_q^k (-q)^n \\ &= \lim_{\rho\to\infty} \frac{1}{[d]_{-q}} \frac{1}{[p^\rho]_{-q^d}} \sum_{i=0}^{d-1} \sum_{n=0}^{p^\rho-1} [x+i+dn]_q^k (-q)^{i+dn} \\ &= \frac{[d]_q^k}{[d]_{-q}} \sum_{i=0}^{d-1} (-q)^i \lim_{\rho\to\infty} \frac{1}{[p^\rho]_{-q^d}} \sum_{n=0}^{p^\rho-1} [\frac{x+i}{d}+n]_{q^d}^k (-q^d)^n \\ &= \frac{[d]_q^k}{[d]_{-q}} \sum_{i=0}^{d-1} (-q)^i \int_{\mathbb{Z}_p} \left[\frac{x+i}{d}+t\right]_{q^d}^k d\mu_{-q^d}(t) \end{aligned} \tag{3}$$

Now we define p-adic q-transfer operator as follows.

$$(L_{p,q}f)(x:q) = \frac{1}{[p]_{-q}} \sum_{k=0}^{p-1} (-q)^k f\left(\frac{x+i}{p} : q^p\right). \tag{4}$$

If we take $f(x:q) = E_n(x:q)$, then we have

$$\begin{aligned} (L_{p,q}E_n)(x:q) &= \frac{1}{[p]_{-q}} \sum_{k=0}^{p-1} (-q)^k E_n(\frac{x+k}{p}; q^p) \\ &= \frac{1}{[p]_q^n} \left(\frac{[p]_q^n}{[p]_{-q}} \sum_{k=0}^{p-1} (-q)^k E_n\left(\frac{x+k}{p} : q^p\right)\right) \\ &= \frac{1}{[p]_q^n} \left(\frac{[p]_q^n}{[p]_{-q}} \sum_{k=0}^{p-1} (-q)^k \int_{\mathbb{Z}_p} \left[\frac{x+k}{p}+t\right]_{q^p}^n d\mu_{-q^p}(t)\right). \end{aligned} \tag{5}$$

By (3) and (5), we have

$$(L_{p,q}E_n)(x:q) = \frac{1}{[p]_q^n} E_n(x:q). \tag{6}$$

Therefore we obtain the following theorem.

Theorem. *The eigenvalues of the p-adic q-transfer operator are the q-Euler polynomials, and are associated with the eigenvalues* $\frac{1}{[p]_q^n}$. *That is,*

$$(L_{p,q}E_n)(x:q) = \frac{1}{[p]_q^n}E_n(x:q),$$

where $E_n(x:q)$ *are q-Euler polynomials.*

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ON SOME NEW NONLINEAR DISCRETE INEQUALITIES AND THEIR APPLICATIONS

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Abstract

In this paper, we establish some new nonlinear difference inequalities in two independent variables, which can be used as handy tools in the study of qualitative properties of solutions of certain classes of difference equations. Applications are given to illustrate the usefulness of these inequalities.

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1 INTRODUCTION

The finite difference inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of finite difference equations. During past few years, many such that new inequalities have been established, which

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are motivated and inspired from the study of certain class of finite difference equations. For example, see [1-3, 5, 6, 10, 12, 17-23] and the reference therein. Our main aim here is to establish some new and more general nonlinear discrete inequalities involving functions of two independent variables, which can be used as ready and powerful tools in the analysis of certain classes of partial finite difference and sum-difference equations. Our results also generalize some recent results in [3].

2 MAIN RESULT

Throughout this paper, $I := [m_0, M) \cap Z$ and $J := [n_0, N) \cap Z$ are two fixed lattices of integral points in R, where $m_0, n_0 \in Z, M, N \in Z \cup \{\infty\}$. Let $\Omega := I \times J \subset Z^2, R_+ := [0, \infty), R_1 := [1, \infty)$ and for any $(s,t) \in \Omega$, the sub-lattice $[m_0, s] \times [n_0, t] \cap \Omega$ of Ω will be denote as $\Omega_{(s,t)}$.

If U is a lattice in $Z(\text{resp.}Z^2)$, the collection of all R-valued functions on U is denoted by $\mathcal{F}(U)$, that of all R_+-valued functions by $\mathcal{F}_+(U)$, and that of all R_1-valued function by $\mathcal{F}_1(U)$. For the sake of convenience, we extend the domain of definition of each function in $\mathcal{F}(U)$ and $\mathcal{F}_+(U)$ trivially to the ambient space $Z(\text{resp.}Z^2)$. So for example, a function in $\mathcal{F}(U)$ is regards as a function defined on $Z(\text{resp.}Z^2)$ with support in U. As usual, the collection of all continuous functions and all i-times continuously differentiable functions of a topological space X into a topological space Y will be denoted by $C(X,Y)$ and $C^i(X,Y)$, respectively.

If U is a lattice in Z^2 , the partial difference operators $\triangle_1$ and $\triangle_2$ on $u \in \mathcal{F}(Z^2)$ or $\mathcal{F}_+(Z^2)$ are defined as

$$\triangle_1 u(m,n) = u(m+1,n) - u(m,n), (m,n) \in U,$$
$$\triangle_2 u(m,n) = u(m,n+1) - u(m,n), (m,n) \in U.$$

Theorem 2.1. Suppose that u, a, and $b \in \mathcal{F}_+(\Omega)$, $p > q \geq 0$ and $k \geq 0$ are constants and $w \in C(R_+, R+)$ is nondecreasing with $w(r) > 0$ for $r > 0$. If u satisfies

$$u^p(m,n) \leq k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) u^q(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) u^q(s,t) w(u(s,t)), \qquad (1)$$

for $(m,n) \in \Omega$, then

$$u(m,n) \leq \left\{ G^{-1}\left[G\left(k^{\frac{p-q}{p}} + A(m,n) \right) + B(m,n) \right] \right\}^{\frac{1}{p-q}}, \qquad (2)$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, where

$$A(m,n) = \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t), \qquad (3)$$

$$B(m,n) = \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t), \qquad (4)$$

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(s^{\frac{1}{p-q}})}, r \geq r_0 > 0, \tag{5}$$

G^{-1} is the inverse of G, and $(m_1, n_1) \in \Omega$ is chosen such that $G(k^{\frac{p-q}{p}} + A(m,n)) + B(m,n) \in DomG^{-1}$ for all $(m,n) \in \Omega_{(m_1,n_1)}$.

Proof. It suffices to consider the case $k > 0$, for then the case $k = 0$ can be arrived at by continuity argument. Denote by $r_1(m,n)$ the right hand side of (1), then from (1) we have $r_1 > 0$,

$$u(m,n) \leq r_1^{\frac{1}{p}}(m,n), \tag{6}$$

on Ω, and r_1 is nondecreasing in each variable.

Hence for any $(x,y) \in \Omega$,

$$\triangle_1 r_1(m,n) = \sum_{t=n_0}^{n-1} a(m,t)u^q(m,t) + \sum_{t=n_0}^{n-1} b(m,t)u^q(m,t)w(u(m,t))$$

$$\leq \sum_{t=n_0}^{n-1} a(m,t)r_1^{\frac{q}{p}}(m,t) + \sum_{t=n_0}^{n-1} b(m,t)r_1^{\frac{q}{p}}(m,t)w(r_1^{\frac{q}{p}}(m,t))$$

$$\leq r_1^{\frac{q}{p}}(m,n)[\sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)w(r_1^{\frac{q}{p}}(m,t))]$$

or

$$\frac{\triangle_1 r_1(m,n)}{r_1^{\frac{q}{p}}(m,n)} \leq \sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)w(r_1^{\frac{q}{p}}(m,t))$$

Therefore, for any $(m,n) \in \Omega$,

$$\sum_{s=m_0}^{m-1} \frac{\triangle_1 r_1(s,n)}{r_1^{\frac{q}{p}}(s,n)} \leq \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)w(r_1^{\frac{q}{p}}(s,t))$$

$$= A(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)w(r_1^{\frac{q}{p}}(s,t)).$$

On the other hand, by the non-decreasing property of r_1 in each variable, we observe that

$$\sum_{s=m_0}^{m-1} \frac{\triangle_1 r_1(s,n)}{r_1^{\frac{q}{p}}(s,n)} = \sum_{s=m_0}^{m-1} \frac{r_1(s+1,n) - r_1(s,n)}{r_1^{\frac{q}{p}}(s,n)}$$

$$= \frac{r_1(m,n)}{r_1^{\frac{q}{p}}(m-1,n)} - \frac{r_1(m-1,n)}{r_1^{\frac{q}{p}}(m-1,n)} + \frac{r_1(m-1,n)}{r_1^{\frac{q}{p}}(m-2,n)} - \frac{r_1(m-2,n)}{r_1^{\frac{q}{p}}(m-2,n)}$$

$$+ \ldots + \frac{r_1(m_0+1,n)}{r_1^{\frac{q}{p}}(m_0,n)} - \frac{r_1(m_0,n)}{r_1^{\frac{q}{p}}(m_0,n)}$$

$$= \frac{r_1(m,n)}{r_1^{\frac{q}{p}}(m-1,n)} + \sum_{s=1}^{m-m_0-1} r_1(m-s,n)\left(\frac{1}{r_1^{\frac{q}{p}}(m-s-1,n)} - \frac{1}{r_1^{\frac{q}{p}}(m-s,n)}\right) - \frac{r_1(m_0,n)}{r_1^{\frac{q}{p}}(m_0,n)}$$

$$\geq \frac{r_1(m,n)}{\frac{q}{r_1^p}(m-1,n)} - \frac{r_1(m_0,n)}{\frac{q}{r_1^p}(m_0,n)} \geq \frac{r_1(m,n)}{\frac{q}{r_1^p}(m,n)} - \frac{r_1(m_0,n)}{\frac{q}{r_1^p}(m_0,n)} = r_1^{\frac{p-q}{p}}(m,n) - k^{\frac{p-q}{p}}.$$

Hence we have

$$r_1^{\frac{p-q}{p}}(m,n) \leq k^{\frac{p-q}{p}} + A(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)w(r_1^{\frac{q}{p}}(s,t)). \tag{7}$$

Setting

$$v_1(m,n) = r_1^{\frac{p-q}{p}}(m,n), \tag{8}$$

then (7) can be rewritten as

$$v_1(m,n) \leq k^{\frac{p-q}{p}} + A(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)w(v_1^{\frac{1}{p-q}}(s,t))$$

for all $(m,n) \in \Omega$. Since $k^{\frac{p-q}{p}} + A(m,n)$ is nondecreasing in each variable, for any fixed $(\overline{m}_1, \overline{n}_1) \in \Omega_{(m_1,n_1)}$, from the last inequality we have

$$v_1(m,n) \leq k^{\frac{p-q}{p}} + A(\overline{m}_1,\overline{n}_1) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)w(v_1^{\frac{1}{p-q}}(s,t)) \tag{9}$$

for $(m,n) \in \Omega_{(\overline{m}_1,\overline{n}_1)}$. Denote by $\overline{r}_1(m,n)$ the right hand side of (9). Then from (9) we observe that

$$v_1(m,n) \leq \overline{r}_1(m,n) \tag{10}$$

for $(m,n) \in \Omega_{(\overline{m}_1,\overline{n}_1)}$ and

$$\begin{aligned}
\triangle_1 \overline{r}_1(m,n) &= \sum_{t=n_0}^{n-1} b(m,t)w(v_1^{\frac{1}{p-q}}(m,t)) \\
&\leq \sum_{t=n_0}^{n-1} b(m,t)w(\overline{r}_1^{\frac{1}{p-q}}(m,t)) \\
&\leq w(\overline{r}_1^{\frac{1}{p-q}}(m,n-1)) \sum_{t=n_0}^{n-1} b(m,t) \\
&\leq w(\overline{r}_1^{\frac{1}{p-q}}(m,n)) \sum_{t=n_0}^{n-1} b(m,t),
\end{aligned}$$

i.e.,

$$\frac{\triangle_1 \overline{r}_1(m,n)}{w(\overline{r}_1^{\frac{1}{p-q}}(m,n))} \leq \sum_{t=n_0}^{n-1} b(m,t) \tag{11}$$

for $(m,n) \in \Omega_{(\overline{m}_1,\overline{n}_1)}$.

By the Mean-value Theorem for integrals, for each (m,n) the exists $\xi : \overline{r}_1(m,n) \leq \xi \leq \overline{r}_1(m+1,n)$ such that

$$\triangle_1 G(\overline{r}_1(m,n)) = G(\overline{r}_1(m+1,n)) - G(\overline{r}_1(m,n))$$

$$= \int_{\overline{r}_1(m,n)}^{\overline{r}_1(m+1,n)} \frac{ds}{w(s^{\frac{1}{p-q}})}$$

$$= \frac{\triangle_1 \overline{r}_1(m,n)}{w(\xi^{\frac{1}{p-q}})} \leq \frac{\triangle_1 \overline{r}_1(m,n)}{w(\overline{r}_1^{\frac{1}{p-q}}(m,n))},$$

and so from (11) we have

$$\triangle_1 G(\overline{r}_1(m,n)) \leq \sum_{t=n_0}^{n-1} b(m,t) \tag{12}$$

Fixing n and setting $m = s$ in (12), and then summing s from m_0 to $m-1$ we get

$$G(\overline{r}_1(m,n)) - G(\overline{r}_1(m_0,n)) \leq \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t),$$

i.e.,

$$G(\overline{r}_1(m,n)) \leq G(k^{\frac{p-q}{p}} + A(\overline{m}_1, \overline{n}_1)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)$$

or

$$\overline{r}_1(m,n) \leq G^{-1}\left[G\left(k^{\frac{p-q}{p}} + A(\overline{m}_1, \overline{n}_1)\right) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\right] \tag{13}$$

for all $(m,n) \in \Omega_{(\overline{m}_1, \overline{n}_1)}$.

Setting $m = \overline{m}_1, n = \overline{n}_1$ in (10) and (13), we have

$$v_1(\overline{m}_1, \overline{n}_1) \leq \overline{r}_1(\overline{m}_1, \overline{n}_1)$$

and

$$\overline{r}_1(\overline{m}_1, \overline{n}_1) \leq G^{-1}\left[G\left(k^{\frac{p-q}{p}} + A(\overline{m}_1, \overline{n}_1)\right) + B(\overline{m}_1, \overline{n}_1)\right],$$

respectively.

Since $(\overline{m}_1, \overline{n}_1) \in \Omega_{(m_1,n_1)}$ is arbitrary, from the last inequalities we have

$$v_1(m,n) \leq \overline{r}_1(m,n) \tag{14}$$

and

$$\overline{r}_1(m,n) \leq G^{-1}\left[G\left(k^{\frac{p-q}{p}} + A(m,n)\right) + B(m,n)\right], \tag{15}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$.

Hence from (6),(8),(14) and (15), we can arrive at the conclusion of the theorem. □

Remark 1. (i) When $a(m,n) \equiv 0, p = 1, q = 0$ in Theorem 2.1, we have the result of Cheung [3, Theorem 2.1]; (ii) When $p = 2, q = 1$, we have the other result of Cheung [3, Theorem 2.2].

Remark 2. If

$$\int_{r_0}^{\infty} \frac{ds}{w(s^{\frac{1}{p-q}})} = \infty,$$

then $G(\infty) = \infty$ and (2) is valid on Ω.

Let $q = p - 1$ in Theorem 2.1, then we have the following corollaries.

Corollary 2.1. Let the functions u, a, b, and w, and the constants p and k be defined as in Theorem 2.1. If u satisfies

$$u^p(m,n) \le k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^{p-1}(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^{p-1}(s,t)w(u(s,t)), \quad (16)$$

for $(m,n) \in \Omega$, then

$$u(m,n) \le \overline{G}^{-1}\Big[\overline{G}(k^{\frac{1}{p}} + A(m,n)) + B(m,n)\Big], \quad (17)$$

for all $(m,n) \in \Omega_{(m_2,n_2)}$, where $A(m,n)$ and $B(m,n)$ are defined as in (3) and (4) respectively,

$$\overline{G}(r) = \int_{r_0}^{r} \frac{1}{w(s)} ds, r \ge r_0 > 0, \quad (18)$$

$\overline{G}^{-1}$ denotes the inverse function of $\overline{G}$, and $(m_2, n_2) \in \Omega$ is chosen such that $\overline{G}(k^{\frac{1}{p}} + A(m,n)) + B(m,n) \in Dom\overline{G}^{-1}$ for all $(m,n) \in \Omega_{(m_2,n_2)}$.

Corollary 2.2. Let the functions u, a and b,and the constants p and k be defined as in Theorem 2.1. If u satisfies

$$u^p(m,n) \le k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^{p-1}(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^{p}(s,t), \quad (19)$$

for $(m,n) \in \Omega$, then

$$u(m,n) \le \Big(k^{\frac{1}{p}} + A(m,n)\Big) \exp B(m,n) \quad (20)$$

for all $(m,n) \in \Omega$, where $A(m,n)$ and $B(m,n)$ are defined as in (3) and (4) respectively.

Remark 3. (i) It is interesting to note here that if $k = c^p$ in (16) and (19), then the bound which appeared in (17) and (20) on the unknown function $u(m,n)$ has no relation with the parameter p, respectively;(ii) When $p = 2$ in Corollary 2.1 and 2.2, we have Cheung's results [3, Corollary 2.3 and 2.4].

Corollary 2.3. Let the functions u and a, and the constants p, q and k be defined as in Theorem 2.1. If u satisfies

$$u^p(m,n) \le k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^{q}(s,t), \quad (21)$$

for $(m,n) \in \Omega$, then

$$u(m,n) \le \Big(k^{\frac{1}{p}} + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\Big)^{\frac{1}{p-q}} \quad (22)$$

for all $(m,n) \in \Omega$.

Theorem 2.2. Suppose that u, a, and $b \in \mathcal{F}_+(\Omega)$, $p > q \geq 0$ and $k \geq 0$ are constants and $w \in C(R_+, R+)$ is nondecreasing with $w(r) > 0$ for $r > 0$. If u satisfies

$$u^p(m,n) \leq k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)u^q(s,t)w(u(s,t)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)u^q(s,t)w(u(s,t)), \tag{23}$$

for $(m,n) \in \Omega$, then

$$u(m,n) \leq \left\{ G^{-1}\left[G(k^{\frac{p-q}{p}}) + A(m,n) + B(m,n) \right] \right\}^{\frac{1}{p-q}}, \tag{24}$$

for all $(m,n) \in \Omega_{(m_3,n_3)}$, where $A(m,n), B(m,n)$ and $w(r)$ are defined as in (3),(4) and (5), respectively;$(m_3,n_3) \in \Omega$ is chosen such that $G(k^{\frac{1}{p}} + A(m,n)) + B(m,n) \in DomG^{-1}$ for all $(m,n) \in \Omega_{(m_3,n_3)}$.

Proof. Let $k > 0$, define $r_2(m,n)$ to denote the right-hand side of (23) and $v_2(m,n) = r_2^{\frac{p-q}{p}}(m,n)$, then by the same steps from (6) to (8) in the proof of Theorem 2.1, we have

$$v_2(m,n) \leq k^{\frac{p-q}{p}} + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)w(v_2^{\frac{1}{p-q}}(s,t)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)w(v_2^{\frac{1}{p-q}}(s,t))$$

for $(m,n) \in \Omega$.

Now using the same procedures form (9) to (15) in the proof of Theorem 2.1 to the last inequality we will get the desired inequality (24).□

Let $p = p - 1$ in Theorem 2.2, then we have the following corollaries.

Corollary 2.4. Let the functions u, a, b and w, and the constants p and k be defined as in Theorem 2.1. If u satisfies

$$u^p(m,n) \leq k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)u^{p-1}(s,t)w(u(s,t)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)u^{p-1}(s,t)w(u(s,t)), \tag{25}$$

for $(m,n) \in \Omega$, then

$$u(m,n) \leq \overline{G}^{-1}\left[\overline{G}(k^{\frac{1}{p}}) + A(m,n) + B(m,n) \right], \tag{26}$$

for all $(m,n) \in \Omega_{(m_4,n_4)}$, where $A(m,n), B(m,n)$ and $\overline{G}$ are defined as in (3),(4) and (18) respectively, $\overline{G}^{-1}$ denotes the inverse function of $\overline{G}$ and $(m_4,n_4) \in \Omega$ is chosen such that $\overline{G}(k^{\frac{1}{p}} + A(m,n)) + B(m,n) \in Dom\overline{G}^{-1}$ for all $(m,n) \in \Omega_{(m_4,n_4)}$.

Corollary 2.5. Let the functions u, a and b, and the constants p and k be defined as in Theorem 2.1. If u satisfies

$$u^p(m,n) \leq k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)u^p(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)u^p(s,t), \tag{27}$$

for $(m,n) \in \Omega$, then

$$u(m,n) \le k^{\frac{1}{p}} \exp\Big(A(m,n) + B(m,n)\Big), \tag{28}$$

for all $(m,n) \in \Omega$, where $A(m,n)$ and $B(m,n)$ are defined as in (3) and (4), respectively.

Remark 6. It is interesting to note here that if $k = c^p$ in (25) and (27), then the bound appeared in (26) and (28) on the unknown function $u(m,n)$ has no relation with the parameter p, respectively.

Theorem 2.3. Suppose that u, a, and $b \in \mathcal{F}_+(\Omega)$, $p > q \ge 0$ and $k \ge 0$ are constants and $w_i \in C(R_+, R_+)$ is nondecreasing with $w_i(r) > 0$ for $r > 0 (i = 1, 2)$. If u satisfies

$$u^p(m,n) \le k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) u^q(s,t) w_1(u(s,t)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) u^q(s,t) w_2(u(s,t)), \tag{29}$$

for $(m,n) \in \Omega$, then

$$u(m,n) \le \left\{ \tilde{G}^{-1}\left[\tilde{G}(k^{\frac{p-q}{p}}) + A(m,n) + B(m,n) \right] \right\}^{\frac{1}{p-q}}, \tag{30}$$

for all $(m,n) \in \Omega_{(m_5,n_5)}$, where $A(m,n)$ and $B(m,n)$ are defined as in (3) and (4) respectively,

$$\tilde{G}(r) = \int_{r_0}^{r} \frac{1}{w_1(s^{\frac{1}{p-q}}) + w_2(s^{\frac{1}{p-q}})} ds, r \ge r_0 > 0,$$

$\tilde{G}^{-1}$ is the inverse function of $\tilde{G}$ and real numbers $(m_5, n_5) \in \Omega$ are chosen so that the quantity in the square brackets of (38) is in the range of $\tilde{G}$.

Proof. Let $k > 0$, define $r_3(m,n)$ to denote the right-hand side of (29) and $v_3(m,n) = r_3^{\frac{p-q}{p}}(m,n)$, then by the same steps from (6) to (8) in the proof of Theorem 2.1, we have

$$u(m,n) \le v_3^{\frac{1}{p-q}}(m,n) \tag{31}$$

$$v_3(m,n) \le k^{\frac{p-q}{p}} + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) w_1(v_3^{\frac{1}{p-q}}(s,t)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) w_2(v_3^{\frac{1}{p-q}}(s,t)) \tag{32}$$

for $(m,n) \in \Omega$.

Setting $\overline{r}_3(m,n)$ as the right-hand side of (32), then we have $\overline{r}_3(m,n) = k^{\frac{p-q}{p}}$,

$$v_3(m,n) \le \overline{r}_3(m,n), \tag{33}$$

and

$$\frac{\triangle_1 \overline{r}_3(m,n)}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))}$$

$$
\begin{aligned}
&= \frac{\sum_{t=n_0}^{n-1} a(m,t) w_1(v_3^{\frac{1}{p-q}}(m,t))}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))} + \frac{\sum_{t=n_0}^{n-1} b(m,t) w_2(v_3^{\frac{1}{p-q}}(m,t))}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))} \\
&\le \frac{w_1(v_3^{\frac{1}{p-q}}(m,n)) \sum_{t=n_0}^{n-1} a(m,t)}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))} + \frac{w_2(v_3^{\frac{1}{p-q}}(m,n)) \sum_{t=n_0}^{n-1} b(m,t)}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))} \\
&\le \frac{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) \sum_{t=n_0}^{n-1} a(m,t)}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))} + \frac{w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n)) \sum_{t=n_0}^{n-1} b(m,t)}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))} \\
&\le \sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)
\end{aligned}
$$

i.e.,

$$
\frac{\triangle_1 \overline{r}_3(m,n)}{w_1(\overline{r}_3^{\frac{1}{p-q}}(m,n)) + w_2(\overline{r}_3^{\frac{1}{p-q}}(m,n))} \le \sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)
$$

By the definition of $\tilde{G}$ and the same steps from (11) to (15) in the proof of Theorem 2.1, we can derive from the last inequality that

$$
\overline{r}_3(m,n) \le \tilde{G}^{-1}\left[\tilde{G}(k^{\frac{p-q}{p}}) + A(m,n) + B(m,n)\right] \tag{34}
$$

for all $(m,n) \in \Omega_{(m_5,n_5)}$, where $A(m,n)$ and $B(m,n)$ are defined as in (3) and (4) respectively.

By (31),(33) and (34), we get the desired inequality (30). By continuity,(30) also holds for any $k \ge 0$. □

Corollary 2.6. Let the functions u, a and b and the constants p, q and k be defined as in Theorem 2.3. If If u satisfies

$$
u^p(m,n) \le k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) u^p(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) u^{\frac{p+q}{2}}(s,t), \tag{35}
$$

for $(m,n) \in \Omega$, then

$$
u(m,n) \le \left[(1 + k^{\frac{p-q}{2p}} \exp\Big(A(m,n) + B(m,n)\Big) - 1\right]^2 \tag{36}
$$

for $(m,n) \in \Omega$.

Proof. Let $w_1(u) = u^{p-q}, w_2(u) = u^{\frac{p-q}{2}}$, then we have

$$
\tilde{G}(r) = \int_{r_0}^{r} \frac{ds}{s + s^{\frac{1}{2}}} = 2 \ln \frac{1 + r^{\frac{1}{2}}}{1 + r_0^{\frac{1}{2}}}
$$

and

$$
\tilde{G}^{-1}(r) = \left[(1 + r_0^{\frac{1}{2}}) \exp(\frac{r}{2}) - 1\right]^2.
$$

Now by Theorem 2.3, we have the desired inequality (36). □

Theorem 2.4. Suppose that u, a, and $b \in \mathcal{F}_+(\Omega)$, $k \geq 0$ is constant and $w \in C(R_+, R+)$ is nondecreasing with $w(r) > 0$ for $r > 0$. Let $\varphi(u) \in C^1(R_+, R_+)$ with $\varphi'(u) > 0$ for $u > 0$, here $\varphi'(u)$ denotes the derivative of φ. If u satisfies

$$\varphi(u(m,n)) \leq k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\varphi'(u(s,t)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi'(u(s,t))w(u(s,t)), \tag{37}$$

for $(m,n) \in \Omega$, then

$$u(m,n) \leq \overline{G}^{-1}\left[\overline{G}\Big(\varphi^{-1}(k) + A(m,n)\Big) + B(m,n)\right], \tag{38}$$

for all $(m,n) \in \Omega_{(m_6,n_6)}$, where $A(m,n), B(m,n)$ and $\overline{G}$ are defined as in (3),(4) and (18) respectively,$\overline{G}^{-1}$ is the inverse function of $\overline{G}$ and real numbers $(m_6, n_6) \in \Omega$ are chosen so that the quantity in the square brackets of (38) is in the range of $\overline{G}$.

Proof. Let $k > 0$, define $r_4(m,n)$ to denote the right-hand side of (37), then we have

$$u(m,n) \leq \varphi^{-1}(r_4(m,n)) \tag{39}$$

and

$$\begin{aligned}\triangle_1 r_4(m,n) &= \sum_{t=n_0}^{n-1} a(m,t)\varphi'(u(m,t)) + \sum_{t=n_0}^{n-1} b(m,t)\varphi'(u(m,t))w(u(m,t)) \\ &\leq \varphi'[\varphi^{-1}(r_4(m,n))]\Big(\sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)w(\varphi^{-1}(r_4(m,t)))\Big)\end{aligned}$$

i.e.,

$$\frac{\triangle_1 r_4(m,n)}{\varphi'[\varphi^{-1}(r_4(m,n))]} \leq \sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)w(\varphi^{-1}(r_4(m,t))).$$

Using the differential mean-value theorem and the last inequality we have

$$\begin{aligned}\triangle_1[\varphi^{-1}(r_4(m,n))] &= \varphi^{-1}(r_4(m+1,n)) - \varphi^{-1}(r_4(m,n)) \\ &= \frac{1}{\varphi'(\varphi^{-1}(\theta))}\triangle_1 r_4(m,n) \leq \frac{\triangle_1 r_4(m,n)}{\varphi'[\varphi^{-1}(r_4(m,n))]} \\ &\leq \sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)w(\varphi^{-1}(r_4(m,t)))\end{aligned} \tag{40}$$

Keeping n fixed in (40) and setting $m = s$ and then summing over $s = m_0, m_0 + 1, \cdots, m-1$, we get

$$\varphi^{-1}(r_4(m,n)) \leq \varphi^{-1}(r_4(m_0,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)w(\varphi^{-1}(r_4(s,t)))$$

i.e.,

$$\varphi^{-1}(r_4(m,n)) \leq \varphi^{-1}(k) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)w(\varphi^{-1}(r_4(s,t)))$$

for all $(m,n) \in \Omega$. Now by applying Theorem 2.1 (the case when $p = 1, q = 0$) to the function $\varphi^{-1}(r_4(m,n))$ in the last inequality, we have

$$\varphi^{-1}(r_4(m,n)) \le \overline{G}^{-1}\Big[\overline{G}\Big(\varphi^{-1}(k) + A(m,n)\Big) + B(m,n)\Big], \tag{41}$$

for all $(m,n) \in \Omega_{(m_6,n_6)}$.

By (39) and (41), we get the desired inequality (38). □

Remark 7. When $\varphi(u) = u^p (p \ge 1)$, conclusion of Corollary 2.1 can be derived form Theorem 2.4. The other interesting new discrete Haraux [9] -Engler [7] type inequalities of two-variable is easily obtained by Theorem 2.4 as following.

Corollary 2.7. Let $a(m,n), b(m,n), \varphi(u)$ and $w(u)$ be as defined in Theorem 2.4. Let $u(m,n) \in \mathcal{F}_1(\Omega)$ and $p > 0, k > 1$ be real numbers. If $u(m,n)$ satisfies

$$u^p(m,n) \le k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^p(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^p(s,t)w(\log u(s,t)), \tag{42}$$

for $(m,n) \in \Omega$, then

$$u(m,n) \le \exp\left\{\overline{G}^{-1}\Big[\overline{G}\Big(\frac{1}{p}\log k + \frac{1}{p}A(m,n)\Big) + \frac{1}{p}B(m,n)\Big]\right\}, \tag{43}$$

for all $(m,n) \in \Omega_{(m_7,n_7)}$, where $A(m,n)$ and $B(m,n)$ are defined as in (3) and (4) respectively,$\overline{G}^{-1}$ is the inverse function of $\overline{G}$ and real numbers $(m_7,n_7) \in \Omega$ are chosen so that the quantity in the square brackets of (43) is in the range of $\overline{G}$.

Proof. Using the change of variable $v(m,n) = \log u(m,n)$, inequality (42) reduces to

$$\exp(pv(m,n)) \le k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)\exp(pv(s,t)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\exp(pv(s,t))w(v(s,t))$$

which is a special case of inequality (37) when $\varphi(v) = \exp(pv)$. By Theorem 2.4, the desired inequality (43) follows. □

Remark 8. It is interesting to note here that if $k = \varphi(c_1)(c_1 \ge 0$ is a constant) in (37), $k = c_2^p (c_2 > 1$ is a constant), $a(s,t) = p\bar{a}(s,t)$ and $b(s,t) = p\bar{b}(s,t)$ in (42), then the bound appeared in (38) and (43) on the unknown function $u(m,n)$ has no relation with the function φ and parameter p, respectively.

3 APPLICATIONS

(a) Consider partial difference equation

$$\triangle_2\triangle_1 u^4(m,n) = f_1(m,n,u(m,n)), \tag{3.1}$$

with the given initial boundary conditions

$$u(m,n_0)=\sigma(m), u(m_0,n)=\tau(n), \sigma(m_0)=\tau(n_0)=0, \tag{3.2}$$

where $f_1\in\mathcal{F}(\Omega\times R), \sigma\in\mathcal{F}(I)$, and $\tau\in\mathcal{F}(J)$.

Theorem 3.1.Suppose that

$$|f_1(m,n,u)|\le a(m,n)|u|+b(m,n)|u|^2 \tag{3.3}$$

and

$$\sigma^4(m)+\tau^4(n)\le k \tag{3.4}$$

for some $k\ge 0$, where $a,b\in\mathcal{F}(\Omega)$, then all solutions of (3.1)-(3.2) satisfy

$$u(m,n)\le\left[\left(k^{\frac{3}{4}}+A(m,n)\right)^{\frac{2}{3}}+\frac{2}{3}B(m,n)\right]^{\frac{1}{2}} \tag{3.5}$$

for all $(m,n)\in\Omega$, where $A(m,n)$ and $B(m,n)$ are defined as in Theorem 2.1.

Proof. If $u(m,n)$ is a solution of (3.1) with condition (3.2), then it can be written as

$$u^4(m,n)=\sigma^4(m)+\tau^4(n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_1(s,t,u(s,t)) \tag{3.5}$$

Hence by (3.3) and (3.6) we have

$$u^4(m,n)\le k+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a(s,t)|u(s,t)|+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a(s,t)|u(s,t)|^2$$

for all $(m,n)\in\Omega$. The last inequality is the special case when $p=4,q=1$ and $w(u)=u$) in Theorem 2.1, so

$$G(r)=\int_{r_0}^{r}\frac{ds}{s^{\frac{1}{3}}}=\frac{3}{2}(r^{\frac{2}{3}}-r_0^{\frac{2}{3}})$$

and

$$G^{-1}(r)=(r_0^{\frac{2}{3}}+\frac{2}{3}u)^{\frac{3}{2}}.$$

Now an application of Theorem 2.1 to the function $|u(m,n)|$ gives the assertion immediately. □

(b)Consider sum-difference equation

$$u(m,n)=k+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_2(s,t,u(s,t),\log|u(s,t)|), \tag{3.7}$$

for for all $(m,n)\in\Omega$, where $k>1$ is a constant.

Theorem 3.2. Assume that

$$|f_2(m,n,u,\log|u|)|\le a(m,n)|u|+b(m,n)|u|\log|u|, \tag{3.8}$$

for for all $(m,n) \in \Omega$, then all solutions of (3.1)-(3.2) satisfy

$$|u(m,n)| \le \exp\left\{[\log k + A(m,n)] \exp B(m,n)\right\} - 1 \tag{3.9}$$

for for all $(m,n) \in \Omega$, where $A(m,n)$ and $B(m,n)$ are defined as in Theorem 2.1.

Proof. Form (3.7) and (3.8) we observe that

$$|u(m,n)| + 1 \le k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)(|u(s,t)| + 1)$$

$$+ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)(|u(s,t)| + 1) \log(|u(s,t)| + 1)$$

for for all $(m,n) \in \Omega$.

Now an suitable application of Corollary 2.7 (in case $p = 1, w(u) = u$) to the last inequality yields (3.9) immediately. □

Remark 8. The boundedness to the solutions of (3.1)-(3.2) and (3.7)-(3.8) can't be derived by the conclusions of [3]; Under some suitable conditions, the uniqueness and continuous dependence of the solutions of (3.1)-(3.2) and (3.7)-(3.8) also can be discussed by our results, for space-saving, the details are omitted here.

In conclusion, we note that the inequalities and applications here can be extended easily to functions involving many independent variables.

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Existence and uniqueness results for nonlinear Cauchy problems of the second order

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Abstract. We use here contraction principle to establish existence and uniqueness result for the Cauchy problem

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \ , \quad y'(x_0) = z_0 \end{cases} . \tag{PC}$$

To do this, we transform first in an original way the Cauchy problem (PC) into an equivalent integral equation. Finally, a new method of succesive approximations for finding the unique solution is establish.

Keywords: contraction principle, Cauchy problem, Lipschitz condition, successive approximations sequence
MSC 2000: 34A12, 34L30

1 Introduction

Let $D = \{(x, y) \in \mathbf{R} \times \mathbf{R}^n | \ |x - x_0| \leq a, ||y - y_0|| \leq b\}$ be a rectangle and $f : D \subseteq \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ a continuous function satisfying the Lipschitz condition $||f(x, y) - f(x, z)|| \leq L \, ||y - z||$, for all $(x, y), (x, z) \in D$ and for some $L > 0$. Under these assumptions, according to the well known Picard theorem (e.g. [1], [2], [5]), the Cauchy problem

$$\begin{cases} y' & = & f(x, y) \\ y(x_0) & = & y_0 \end{cases} \tag{1.1}$$

has (locally) unique solution on $I = (x_0 - \varepsilon, x_0 + \varepsilon)$, where $\varepsilon = \min\left\{a, \dfrac{b}{M}\right\}$, $M = \sup\limits_{(x,y)\in D} ||f(x, y)||$. Moreover, the Picard theorem gives us a method to approximate the solution, usually called the *successive approximations method*.

In this sense, let us define $T : C(I) \to C(I)$ by $(Ty)(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, \mathrm{d}\, t$. Then the solution of problem (1.1) is the limit of the *successive approximations sequence* $y_0 = y(x_0)$, $y_n = Ty_{n-1}$, that is $y_n(x) = y_0 + \displaystyle\int_{x_0}^{x} f(t, y_{n-1}(t)) \, \mathrm{d}\, t, \ n \in \mathbf{N}$. We will give here results similar to the Picard theorem for local existence and uniqueness of the solution of the Cauchy problems

of the second order. The results are obtained by an original way, so these considerations lead us to a new type of approximation sequences for the solution.

2 The Results

First we give the following

Lemma *The Cauchy problem (PC) is equivalent with the integral equation*

$$z(x)=z_0+\int_{x_0}^{x} f\left(t,y_0+\int_{x_0}^{t} z(s)\,\mathrm{d}\,s,z(t)\right)\mathrm{d}\,t \tag{INT}$$

in the sense that $z=z(x)$ *is solution for the equation (INT) on* $(x_0-\varepsilon,x_0+\varepsilon)$ *if and only if* $y(x)=y_0+\int_{x_0}^{x} z(s)\,\mathrm{d}\,s$ *is solution for the Cauchy problem (PC) on* $(x_0-\varepsilon,x_0+\varepsilon)$.

Proof. First let us assume that $y=y(x)$ is solution for the Cauchy problem (PC),

$$\left\{\begin{array}{l} y''(x)=f(x,y(x),y'(x)) \\ y(x_0)=y_0\ ,\quad y'(x_0)=z_0 \end{array}\right. .$$

Let us consider $z(x)=y'(x)$. We have $y(x_0)=y_0$, so

$$y(x)=y_0+\int_{x_0}^{x} z(s)\,\mathrm{d}\,s. \tag{1}$$

By integration the equation from (PC), we obtain

$$\int_{x_0}^{x} y''(t)\,\mathrm{d}\,t=\int_{x_0}^{x} f(t,y(t),y'(t))\,\mathrm{d}\,t \quad\text{or}\quad y'(x)-y'(x_0)=\int_{x_0}^{x} f(t,y(t),y'(t))\,\mathrm{d}\,t.$$

But $y'(x_0)=z_0$ and by replacing in the right hand $y(t)$ from (1), we obtain

$$y'(x)=z_0+\int_{x_0}^{x} f\left(t,y_0+\int_{x_0}^{t} z(s)\,\mathrm{d}\,s,z(t)\right)\mathrm{d}\,t \quad\text{or}\quad z(x)=z_0+\int_{x_0}^{x} f\left(t,y_0+\int_{x_0}^{t} z(s)\,\mathrm{d}\,s,z(t)\right)\mathrm{d}\,t,$$

which is (INT). Reciprocally, let us assume that $z=z(x)$ is solution for (INT) and let us define

$$y(x)=y_0+\int_{x_0}^{x} z(s)\,\mathrm{d}\,s,$$

we obtain from (INT),

$$z(x)=z_0+\int_{x_0}^{x} f\left(t,y(t),z(t)\right)\mathrm{d}\,t \tag{2}$$

The initial conditions are satisfied, $y(x_0)=y_0+\int_{x_0}^{x_0} z(s)\,\mathrm{d}\,s=y_0$ and because $y'(x)=z(x)$, we also have $y'(x_0)=z(x_0)=z_0$, from (2). Now, by deriving (2), we obtain $z'(x)=f(x,y(x),z(x))\Leftrightarrow z'(x)=f(x,y(x),y'(x))$, so we are done.

Theorem *Let there be given a continuous function* $f:D\subseteq\mathbf{R}^3\to\mathbf{R}$ *for which exist positive constants* λ,μ *such that*

$$|f(x,y_1,z_1)-f(x,y_2,z_2)|\leq\lambda\,|y_1-y_2|+\mu\,|y_2-z_2|\,, \tag{3}$$

for all (x, y_1, z_1), $(x, y_2, z_2) \in D$. *Then for every real numbers* y_0 *and* z_0, *the Cauchy problem*

$$\begin{cases} y'' = f(x, y, y') \\ y(x_0) = y_0 \ , \quad y'(x_0) = z_0 \end{cases} \tag{PC}$$

has locally an unique solution on $(x_0 - \varepsilon, x_0 + \varepsilon)$, *where* $\varepsilon = \frac{2}{\mu + \sqrt{\mu^2 + 2\lambda}}$.

Proof. According to the Lemma, we have to prove that the integral equation (INT) is uniquely solvable on $I = (x_0 - \varepsilon, x_0 + \varepsilon)$. The equation (INT) can be written as a fixed point problem. To do this let us consider the Banach space $C(I)$ of continuous functions defined on I, endowed with the uniform convergence norm $||w|| = \sup_{x \in I} |w(x)|$. Let us consider the operator $T : C(I) \to C(I)$ given by the formula

$$Tz(x) = z_0 + \int_{x_0}^{x} f\left(t, y_0 + \int_{x_0}^{t} z(s)\,\mathrm{d}\,s, z(t)\right)\mathrm{d}\,t \quad , \qquad z \in C(I).$$

Now the integral equation (INT), can be equivalently written as $z = Tz$, $z \in C(I)$, which is a fixed point problem. The theorem is proved if we show that T is a contraction. In this sense, for every z_1, $z_2 \in C(I)$ and for all $x \in (x_0, x_0 + \varepsilon)$, we have $|Tz_1(x) - Tz_2(x)| =$

$$= \left| \int_{x_0}^{x} f\left(t, y_0 + \int_{x_0}^{t} z_1(s)\,\mathrm{d}\,s, z_1(t)\right)\mathrm{d}\,t - \int_{x_0}^{x} f\left(t, y_0 + \int_{x_0}^{t} z_2(s)\,\mathrm{d}\,s, z_2(t)\right)\mathrm{d}\,t \right| =$$

$$= \left| \int_{x_0}^{x} \left[f\left(t, y_0 + \int_{x_0}^{t} z_1(s)\,\mathrm{d}\,s, z_1(t)\right) - f\left(t, y_0 + \int_{x_0}^{t} z_2(s)\,\mathrm{d}\,s, z_2(t)\right)\right]\mathrm{d}\,t \right| \leq$$

$$\leq \int_{x_0}^{x} \left| f\left(t, y_0 + \int_{x_0}^{t} z_1(s)\,\mathrm{d}\,s, z_1(t)\right) - f\left(t, y_0 + \int_{x_0}^{t} z_2(s)\,\mathrm{d}\,s, z_2(t)\right)\right| \mathrm{d}\,t \leq$$

$$\leq \int_{x_0}^{x} \left(\lambda \left| \int_{x_0}^{t} z_1(s)\,\mathrm{d}\,s - \int_{x_0}^{t} z_2(s)\,\mathrm{d}\,s \right| + \mu\,|z_1(t) - z_2(t)| \right)\mathrm{d}\,t =$$

$$= \int_{x_0}^{x} \left(\lambda \left| \int_{x_0}^{t} [z_1(s) - z_2(s)]\,\mathrm{d}\,s \right| + \mu\,|z_1(t) - z_2(t)| \right)\mathrm{d}\,t \leq$$

$$\leq \int_{x_0}^{x} \left(\lambda \int_{x_0}^{t} |z_1(s) - z_2(s)|\,\mathrm{d}\,s + \mu\,|z_1(t) - z_2(t)| \right)\mathrm{d}\,t \leq$$

$$\leq \int_{x_0}^{x} \left(\lambda \int_{x_0}^{t} ||z_1 - z_2||\,\mathrm{d}\,s + \mu\,||z_1 - z_2|| \right)\mathrm{d}\,t =$$

$$= \int_{x_0}^{x} \left[\lambda\,||z_1 - z_2||\,(t - x_0) + \mu\,||z_1 - z_2|| \right]\mathrm{d}\,t =$$

$$= \left[\lambda \cdot \frac{(x - x_0)^2}{2} + \mu(x - x_0) \right] \cdot ||z_1 - z_2|| \leq \left(\frac{\lambda \varepsilon^2}{2} + \mu\varepsilon \right) \cdot ||z_1 - z_2||\,,$$

so $|Tz_1(x) - Tz_2(x)| \leq \left(\frac{\lambda \varepsilon^2}{2} + \mu\varepsilon \right) \cdot ||z_1 - z_2||$, for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ (case $x \in (x_0 - \varepsilon, x_0)$ is similar). By taking the supremum with respect to $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ in the last inequality, it follows that $||Tz_1 - Tz_2|| \leq \left(\frac{\lambda \varepsilon^2}{2} + \mu\varepsilon \right) \cdot ||z_1 - z_2||$, for all z_1, $z_2 \in C(I)$. Finally, T is a

q-contraction, where $q=\frac{\lambda\varepsilon^2}{2}+\mu\varepsilon<1$. Indeed, the last inequality is equivalent with a quadratic inequation, $\frac{\lambda\varepsilon^2}{2}+\mu\varepsilon-1<0\Leftrightarrow 0<\varepsilon<\frac{-\mu+\sqrt{\mu^2+2\lambda}}{\lambda}=\frac{2}{\mu+\sqrt{\mu^2+2\lambda}}$, true. □

3. Applications

We give existence and uniqueness results under some hypoteses which are approximative lipschitzianity conditions. We use the following theorem of Lagrange type:

Lemma *Let $D\subseteq\mathbf{R}^n$ be a connected set and let $\phi=\phi(x_1,...,x_n):D\to\mathbf{R}$ be a function with partial derivatives $\dfrac{\partial\phi}{\partial x_k}$, $1\leq k\leq n$. Then for every $a,x\in D$, $a=(a_1,...,a_n)$, $x=(x_1,...,x_n)$ there exist ξ_k between a_k and x_k, $1\leq k\leq n$, such that*

$$\phi(x)-\phi(a)=\sum_{k=1}^{n}\frac{\partial\phi}{\partial x_k}(a_1,...,a_{k-1},\xi_k,x_{k+1},...,x_n)(x_k-a_k).$$

For proof and other comments, see [4].

Theorem *Let there be given a continuous function $f=f(x,y,z):D\subseteq\mathbf{R}^3\to\mathbf{R}$ with bounded partial derivatives $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$, i.e. $\left|\frac{\partial f}{\partial y}(x,y,z)\right|\leq\lambda$, $\left|\frac{\partial f}{\partial z}(x,y,z)\right|\leq\mu$, for all points $(x,y,z)\in D$ and for some $\lambda,\mu>0$ Then for every y_0 and z_0, the Cauchy problem*

$$\begin{cases} y''=f(x,y,y') \\ y(x_0)=y_0\ ,\quad y'(x_0)=z_0 \end{cases} \tag{PC}$$

has locally an unique solution, defined at least on $(x_0-\varepsilon,x_0+\varepsilon)$, where $\varepsilon=\frac{2}{\mu+\sqrt{\mu^2+2\lambda}}$.

Proof. We are in the hypoteses of the theorem. Indeed, under the assumptions of our theorem the function f satisfies the inequality (3). In this sense, let us define for arbitrary fixed x, the application $\phi(y,z)=f(x,y,z)$. From Lemma, for all (y_1,z_1) and (y_2,z_2), there exists ξ between y_1 and y_2 and η between z_1 and z_2 such that

$$\phi(y_1,z_1)-\phi(y_2,z_2)=\frac{\partial\phi}{\partial y}(\xi,z_1)(y_1-y_2)+\frac{\partial\phi}{\partial z}(y_1,\eta)(z_1-z_2).$$

Therefore, $|f(x,y_1,z_1)-f(x,y_2,z_2)|=\left|\frac{\partial\phi}{\partial y}(\xi,z_1)(y_1-y_2)+\frac{\partial\phi}{\partial z}(y_1,\eta)(z_1-z_2)\right|\leq$

$$\leq\left|\frac{\partial\phi}{\partial y}(\xi,z_1)\right|\cdot|y_1-y_2|+\left|\frac{\partial\phi}{\partial z}(y_1,\eta)\right|\cdot|z_1-z_2|\leq\lambda|y_1-y_2|+\mu|z_1-z_2|.\square$$

Let us consider the particular case when f is linear in y and z, in sense that $f(x,y,z)=a(x)+b(x)y+c(x)z$, where $a,b,c:J\subseteq\mathbf{R}\to\mathbf{R}$ are continuous. Obviously, $\frac{\partial f}{\partial y}(x,y,z)=b(x)$, $\frac{\partial f}{\partial z}(x,y,z)=c(x)$, so we can state the following results regarding linear Cauchy problems:

Theorem *Let there be given continuous functions $a,b,c:J\subseteq\mathbf{R}\to\mathbf{R}$, b,c bounded, with $|b(x)|\leq\lambda$, $|c(x)|\leq\mu$, for all $x\in J$ and some positive reals λ,μ. Then for every real numbers x_0, y_0, z_0, the linear Cauchy problem $\begin{cases} y''=a(x)+b(x)y+c(x)y'' \\ y(x_0)=y_0\ ,\quad y'(x_0)=z_0 \end{cases}$ has an unique solution, defined at least on $(x_0-\varepsilon,x_0+\varepsilon)$, where $\varepsilon=\frac{2}{\mu+\sqrt{\mu^2+2\lambda}}$.*

4. Numerical Examples

The contraction principle allows us to develop a method for finding numerical approximations of the solution. The approximation sequence $(z_n)_{n\in\mathbf{N}}$ associated to the problem

$$z = Tz \quad , \quad z \in C(I)$$

is given by the recurrence

$$z_{n+1} = Tz_n \quad , \quad n \in \mathbf{N},$$

where z_0 is arbitrary chosen in $C(I)$. We can establish the following method of succesive approximations:

Theorem *Assume that the hypoteses of the theorem are fulled. Then the sequence* $(z_n)_{n\in\mathbf{N}}$ *given by the recurrence*

$$z_{n+1}(x) = z_0 + \int_{x_0}^{x} f\left(t, y_0 + \int_{x_0}^{t} z_n(s)\,\mathrm{d}\,s, z_n(t)\right)\mathrm{d}\,t \quad , \quad n \in \mathbf{N},$$

with $z_0(x) = z_0$, *converges uniformly to the solution* $z = z(x)$ *of the integral equation (INT).*

In particular case when

$$f(x, y, z) = a(x) + b(x)y + c(x)z,$$

with $a, b, c : J \subseteq \mathbf{R} \to \mathbf{R}$ are given continuous function, the approximation sequence acquires a nice form, given by the following corollary, as a method of succesive approximations for linear Cauchy problem of the second order.

Corollary *Let there be given* $a, b, c : J \subseteq \mathbf{R} \to \mathbf{R}$ *be continuous,* b, c *bounded, with*

$$|b(x)| \leq \lambda \quad , \quad |c(x)| \leq \mu,$$

for all $x \in J$ *and some positive reals* λ, μ. *Denote by* A, B *the antiderivatives of the functions* a, *respective* b *with* $A(x_0) = B(x_0) = 0$. *Then the sequence* $(z_n)_{n\in\mathbf{N}}$ *given by the recurrence*

$$z_{n+1}(x) = z_0 + A(x) + B(x)y_0 + \int_{x_0}^{x}\left(\int_{x_0}^{t} z_n(s)\,\mathrm{d}\,s\right)dt + \int_{x_0}^{x} c(t)z_n(t)\,\mathrm{d}\,t,$$

with $z_0(x) = z_0$ *converges uniformly on* $I = (x_0 - \varepsilon, x_0 + \varepsilon)$ *to a function* $z = z(x)$ *so that the function* $y(x) = y_0 + \int_{x_0}^{x} z(s)\,\mathrm{d}\,s$ *is the unique solution of the linear Cauchy problem*

$$\begin{cases} y'' = a(x) + b(x)y + c(x)y'' \\ y(x_0) = y_0 \quad , \quad y'(x_0) = z_0 \end{cases}.$$

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A Kind of Steffensen Method and Its Third-order Variant *

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Abstract: For solving nonlinear equations, we suggest a kind of Steffensen method which still only uses two evaluations of the function and maintains second-order convergence. We also suggest a variant of the kind of Steffensen method which is still derivative free and uses four evaluations of the function to achieve cubic convergence. Their error equations and asymptotic convergence constants are deduced. The numerical examples support the suggested methods.

Key words: Nonlinear equations; Newton's method; Steffensen method; Convergence.

1 Introduction

It is well known that Newton's method has second-order convergence for solving the simple root of a nonlinear equation $f(x) = 0$ with the iteration (See [2]):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where x_0 is an initial guess of the root. Recently, a variant of Newton's method has been suggested in [4] as follows:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where x^*_{n+1} is the intermediate result by using a Newton's iteration (1.1). This method only uses one evaluation of the function and two evaluations of the first derivatives, and obtains accelerated third-order convergence. Newton-type methods with cubic convergence in [1] have been generalized from iteration (1.2). A lot of cubically convergent methods which need evaluations of the function and its first derivatives have been compiled in §12.5 in [3].

It's also well known that Steffensen method as the following (see §7.2.8 in [2]):

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

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is a noticeable improvement of Newton's method, since it is derivative free and maintains quadratic convergence. The variant of Steffensen method in [5] is as follows:

$$\begin{cases} x_{n+1} = x_n - \dfrac{2f^2(x_n)}{[f(x_n + f(x_n)) - f(x_n)] - [f(x^*_{n+1} - f(x_n)) - f(x^*_{n+1})]}, \\ x^*_{n+1} = x_n - \dfrac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)}, \qquad n = 0, 1, 2, \ldots, \end{cases} \tag{1.4}$$

In this paper, we suggest a kind of Steffensen method as follows:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n) - f(x_n - f(x_n))}, \quad n = 0, 1, 2, \ldots. \tag{1.5}$$

where x_0 is an initial guess of the root. The method still uses two evaluations of the function and maintains second-order convergence. We also suggest a variant of the kind of Steffensen method as follows:

$$\begin{cases} x_{n+1} = x_n - \dfrac{2f^2(x_n)}{[f(x^*_{n+1} + f(x_n)) - f(x^*_{n+1})] - [f(x_n - f(x_n)) - f(x_n)]}, \\ x^*_{n+1} = x_n - \dfrac{f^2(x_n)}{f(x_n) - f(x_n - f(x_n))}, \qquad n = 0, 1, 2, \ldots, \end{cases} \tag{1.6}$$

Iteration (1.4), as well as iteration (1.6), only uses four evaluations of the function, but achieves cubic convergence. Other variants of Steffensen method (1.3) can be found in [5].

We deduce error equations and asymptotic convergence constants of (**??**) and (**??**), and present numerical examples in the following sections.

2 Convergence of the Kind of Steffensen Method

Theorem 2.1. *Let $f : D \to R$ be a twice differentiable function with a simple root $a \in D$, $D \subset R$ be an open set, x_0 be close enough to a, then the kind of Steffensen method (1.5) is at least quadratically convergent, and satisfies the following error equation*

$$e_{n+1} = \frac{(1 - f'(a))f''(a)}{2f'(a)} e_n^2 + o(e_n^2), \tag{2.1}$$

where $e_n = x_n - a, n = 0, 1, 2, \ldots$.

Proof. By Taylor's expansion, we have

$$\begin{aligned} f(x_n) &= f'(a)e_n + f''(a)e_n^2/2 + o(e_n^2), \\ f(x_n - f(x_n)) &= f'(a)[e_n - f'(a)e_n - f''(a)e_n^2/2] + f''(a)(1 - f'(a))^2 e_n^2/2 + o(e_n^2), \\ f(x_n) - f(x_n - f(x_n)) &= f'(a)^2 e_n + 3f'(a)f''(a)e_n^2/2 - f'(a)^2 f''(a)e_n^2/2 + o(e_n^2), \end{aligned}$$

$$\begin{aligned}
f^2(x_n) &= f'(a)^2e_n^2 + f'(a)f''(a)e_n^3 + o(e_n^3),\\
\frac{f^2(x_n)}{f(x_n)-f(x_n-f(x_n))} &= \frac{f'(a)e_n + f''(a)e_n^2 + o(e_n^2)}{f'(a)+3f''(a)e_n/2 - f'(a)f''(a)e_n/2 + o(e_n)},\\
e_{n+1} &= e_n - \frac{f'(a)e_n + f''(a)e_n^2 + o(e_n^2)}{f'(a)+3f''(a)e_n/2 - f'(a)f''(a)e_n/2 + o(e_n)}\\
&= \frac{f''(a)e_n^2/2 - f'(a)f''(a)e_n^2/2 + o(e_n^2)}{f'(a)+3f''(a)e_n/2 - f'(a)f''(a)e_n/2 + o(e_n)}\\
&= \frac{(1-f'(a))f''(a)}{2f'(a)}e_n^2 + o(e_n^2). \quad \square
\end{aligned}$$

Remark 2.1. The asymptotic convergence constants of Newton's method (1.1), Steffensen method (1.3) and the kind of Steffensen method (1.5) are $C = \frac{f''(a)}{2f'(a)}$, $\frac{(1+f'(a))f''(a)}{2f'(a)}$ and $\frac{(1-f'(a))f''(a)}{2f'(a)}$ respectively.

3 Convergence of the Variant of the Kind of Steffensen Method

Theorem 3.1. *Let $f : D \to R$ be a triple differentiable function with a simple root $a \in D$, $D \subset R$ be an open set, x_0 be close enough to a, then the variant of the kind of Steffensen method (1.6) is at least cubically convergent, and satisfies the following error equation*

$$e_{n+1} = \frac{(1-f'(a))(f'(a)f'''(a) - 2f'(a)^2f'''(a) + 3f''(a)^2)}{12f'(a)^2}e_n^3 + o(e_n^3), \tag{3.1}$$

where $e_n = x_n - a, n = 1, 2, \ldots$.

Proof. By Taylor's expansion and Theorem 2.1, we have

$$\begin{aligned}
f(x_n) &= f'(a)e_n + \tfrac{1}{2!}f''(a)e_n^2 + \tfrac{1}{3!}f'''(a)e_n^3 + o(e_n^3),\\
x_{n+1}^* - a &= \frac{(1-f'(a))f''(a)}{2f'(a)}e_n^2 + o(e_n^2),\\
f(x_{n+1}^*) &= f'(a)(x_{n+1}^* - a) + \tfrac{1}{2!}f''(a)(x_{n+1}^* - a)^2 + o(e_n^3),\\
f(x_{n+1}^* + f(x_n)) &= f'(a)((x_{n+1}^* - a) + f(x_n)) + \tfrac{1}{2!}f''(a)((x_{n+1}^* - a) + f(x_n))^2\\
&\quad + \tfrac{1}{3!}f'''(a)((x_{n+1}^* - a) + f(x_n))^3 + o(e_n^3),\\
\tfrac{f(x_{n+1}^*+f(x_n))-f(x_{n+1}^*)}{f(x_n)} &= f'(a) + \tfrac{1}{2!}f''(a)(2(x_{n+1}^* - a) + f(x_n)) + \tfrac{1}{3!}f'''(a)f'(a)^2e_n^2 + o(e_n^2).
\end{aligned}$$

Similarly,

$$\frac{f(x_n - f(x_n)) - f(x_n)}{-f(x_n)} = f'(a) + \frac{f''(a)}{2!}(2(x_n - a) - f(x_n)) + \frac{f'''(a)}{3!}[f'(a)^2 - 3f'(a) + 3]e_n^2 + o(e_n^2).$$

Thus, iteration (1.6) satisfies

$$\begin{aligned} e_{n+1} &= e_n - \frac{f'(a)e_n + \frac{1}{2!}f''(a)e_n^2 + \frac{1}{3!}f'''(a)e_n^3 + o(e_n^3)}{f'(a) + \frac{1}{2!}f''(a)[e_n + (x^*_{n+1} - a)] + \frac{1}{3!}f'''(a)[f'(a)^2 - \frac{3}{2}f'(a) + \frac{3}{2}]e_n^2 + o(e_n^2)} \\ &= \frac{[\frac{(1-f'(a))f''(a)^2}{4f'(a)} + \frac{1}{3!}f'''(a)(f'(a)^2 - \frac{3}{2}f'(a) + \frac{1}{2})]e_n^3 + o(e_n^3)}{f'(a) + \frac{f''(a)}{2!}e_n + \frac{(1-f'(a))f''(a)^2}{4f'(a)}e_n^2 + \frac{1}{3!}f'''(a)[f'(a)^2 - \frac{3}{2}f'(a) + \frac{3}{2}]e_n^2 + o(e_n^2)} \\ &= \frac{(1-f'(a))(f'(a)f'''(a) - 2f'(a)^2f'''(a) + 3f''(a)^2)}{12f'(a)^2}e_n^3 + o(e_n^3). \quad \square \end{aligned}$$

Remark 3.1. The asymptotic cubic convergence constants of Weerakoon and Fernando's variant of Newton's method (1.2) and the variant of the kind of Steffensen method (1.6) are $C = \frac{3f''(a)^2+f'(a)f'''(a)}{12f'(a)^2}$ and $\frac{(1-f'(a))(f'(a)f'''(a)-2f'(a)^2f'''(a)+3f''(a)^2)}{12f'(a)^2}$ respectively.

Remark 3.2. The asymptotic cubic convergence constant of the variant of Steffensen method (1.4) should be corrected as $\frac{(1+f'(a))(f'(a)f'''(a)+2f'(a)^2f'''(a)+3f''(a)^2)}{12f'(a)^2}$ in [5].

4 Numerical Examples

We show the numerical examples for the related methods as follows, where NM—Newton's method (1.1); WFM—Weerakoon and Fernando's variant of Newton's method (1.2); SM—Steffensen Method (1.3); VSM—the kind of Steffensen method (1.4); KSM—the kind of Steffensen method (1.5); VKSM—the variant of the kind of Steffensen method (1.6).

Table 1. $f(x) = \frac{1}{3}(x^3 - 8), a = 2, x_0 = 2.6$

Method	n	C	1	2	3	4	5	6
NM	$\|e_n\|$		1.4881e-01	1.0543e-02	5.5383e-05	1.5336e-09		
	e_n/e_{n-1}^2	0.5	4.1337e-01	4.7609e-01	4.9825e-01	4.9999e-01		
WFM	$\|e_n\|$		3.4423e-02	1.1466e-05	4.4409e-16			
	e_n/e_{n-1}^3	0.2917	1.5936e-01	2.8112e-01	2.9457e-01			
SM	$\|e_n\|$		4.2704e-01	2.5850e-01	1.1558e-01	2.8021e-02	1.8780e-03	8.7904e-06
	e_n/e_{n-1}^2	2.5	1.1862e+00	1.4175e+00	1.7296e+00	2.0978e+00	2.3918e+00	2.4925e+00
VSM	$\|e_n\|$		2.8072e-01	4.7540e-02	3.1862e-04	1.0105e-10		
	e_n/e_{n-1}^3	3.125	1.2996e+00	2.1489e+00	2.9655e+00	3.1240e+00		
KSM	$\|e_n\|$		1.1188e+00	6.2191e-01	2.9238e-01	8.9923e-02	1.0795e-02	1.7232e-04
	e_n/e_{n-1}^2	-1.5	-3.1078e+00	-4.9683e-01	-7.5594e-01	-1.0519e+00	-1.3350e+00	-1.4786e+00
VKSM	$\|e_n\|$		1.2196e-01	3.5687e-04	5.6930e-12			
	e_n/e_{n-1}^3	0.125	-5.6464e-01	1.9672e-01	1.2526e-01			

Table 2. $f(x) = 3e^{x-2} - 3, a = 2, x_0 = 2.5$

Method	n	C	1	2	3	4	5	6
NM	$\|e_n\|$		1.0653e-01	5.4781e-03	1.4978e-05	1.1216e-10		
	e_n/e_{n-1}^2	0.5	4.2612e-01	4.8271e-01	4.9909e-01	5.0000e-01		
WFM	$\|e_n\|$		3.0105e-02	8.9246e-06	4.4409e-16			
	e_n/e_{n-1}^3	0.3333	2.4084e-01	3.2709e-01	6.2474e-01			
SM	$\|e_n\|$		3.7241e-01	2.2584e-01	9.0878e-02	1.5815e-02	4.9658e-04	4.9307e-07
	e_n/e_{n-1}^2	2	1.4896e+00	1.6284e+00	1.7817e+00	1.9149e+00	1.9855e+00	1.9995e+00
VSM	$\|e_n\|$		2.7332e-01	5.8169e-02	6.4596e-04	8.9836e-10		
	e_n/e_{n-1}^3	3.3333	2.1866e+00	2.8489e+00	3.2819e+00	3.3330e+00		
KSM	$\|e_n\|$		3.9334e-01	1.0887e-01	1.0733e-02	1.1407e-04	1.3010e-08	2.2204e-16
	e_n/e_{n-1}^2	-1	-1.5734e+00	-7.0366e-01	-9.0551e-01	-9.9021e-01	-9.9990e-01	-1.3119e+00
VKSM	$\|e_n\|$		3.7812e-02	1.8333e-05	2.2204e-15			
	e_n/e_{n-1}^3	0.3333	3.0250e-01	3.3911e-01	3.6034e-01			

Table 3. $f(x) = e^{x^2} + \sin x + x - 1, a = 0, x_0 = 0.5$

Method	n	C	1	2	3	4	5	6
NM	$\|e_n\|$		1.0038e-01	4.5009e-03	1.0069e-05	5.0691e-11		
	e_n/e_{n-1}^2	0.5	4.0151e-01	4.4672e-01	4.9703e-01	4.9999e-01		
WFM	$\|e_n\|$		2.8507e-02	4.7171e-06	3.0322e-17			
	e_n/e_{n-1}^3	0.2083	2.2806e-01	2.0362e-01	2.8889e-01			
SM	$\|e_n\|$		4.3028e-01	3.1274e-01	1.4391e-01	2.6263e-02	9.7885e-04	1.4339e-06
	e_n/e_{n-1}^2	1.5	1.7211e+00	1.6892e+00	1.4713e+00	1.2682e+00	1.4191e+00	1.4966e+00
VSM	$\|e_n\|$		3.6979e-01	1.2397e-01	1.0302e-03	1.3876e-10		
	e_n/e_{n-1}^3	0.125	2.9583e+00	2.4517e+00	5.4064e-01	1.2692e-01		
KSM	$\|e_n\|$		3.2832e-01	3.9190e-02	7.3903e-04	2.7288e-07	3.7152e-14	
	e_n/e_{n-1}^2	-0.5	-1.3133e+00	-3.6358e-01	-4.8117e-01	-4.9963e-01	-4.9893e-01	
VKSM	$\|e_n\|$		6.9952e-02	1.4344e-04	1.1065e-12			
	e_n/e_{n-1}^3	-0.375	-5.5962e-01	-4.1906e-01	-3.7490e-01			

The examples confirm that NM, SM, KSM are second-order methods, WFM and VKSM are third-order methods, illustrate the asymptotic convergence, and suggest KSM and VKSM for their derivative-free property and good numerical results corresponding to well-known SM and VSM.

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Holomorphic Functions on the Mixed Norm Spaces on the Polydisc II

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Abstract

The paper continues the investigation of holomorphic mixed norm spaces $\mathcal{A}^{p,q}_{\vec{\omega}}$ in the unit polydisc of $\mathbb{C}^n$. We prove that a mixed norm is equivalent to a "derivative norm" for all $0 < p \leq \infty, 0 < q < \infty$ and a large class of weights $\vec{\omega}$. As an application, we prove that pluriharmonic conjugation is bounded in these mixed norm spaces.

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1 Introduction

Let $U^1 = U$ be the unit disc in the complex plane, U^n the unit polydisc in $\mathbb{C}^n$, and $H(U^n)$ the set of all holomorphic functions on U^n.

For the integral means of a function f given in U^n, we write

$$M_p(f,r) = \left(\frac{1}{(2\pi)^n} \int_{[0,2\pi)^n} |f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p d\theta \right)^{1/p},$$

$r = (r_1, \dots, r_n), 0 \leq r_j < 1, j \in \{1, \dots, n\}, 0 < p < \infty$, $\theta = (\theta_1, \dots, \theta_n)$, $d\theta = d\theta_1 \cdots d\theta_n$ and

$$M_\infty(f,r) = \sup_{\theta \in [0,2\pi)^n} |f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|.$$

Let $\omega(x), 0 \leq x < 1$, be a weight function which is positive and integrable on (0,1). We extend ω on U by setting $\omega(z) = \omega(|z|)$, and also on U^n by $\vec{\omega} = (\omega_1, \dots, \omega_n)$.

Let $\mathcal{L}^{p,q}_{\vec{\omega}} = \mathcal{L}^{p,q}_{\vec{\omega}}(U^n), 0 < p \le \infty, 0 < q < \infty$, denote the mixed norm space, the class of all measurable functions defined on U^n such that

$$\|f\|^q_{p,q,\vec{\omega}} = \int_{(0,1)^n} M^q_p(f,r) \prod_{j=1}^n \omega_j(r_j) dr_j < \infty,$$

and $\mathcal{A}^{p,q}_{\vec{\omega}} = \mathcal{A}^{p,q}_{\vec{\omega}}(U^n)$ be the intersection of $\mathcal{L}^{p,q}_{\vec{\omega}}$ and $H(U^n)$. When $p = q$ we come to weighted Bergman spaces $\mathcal{A}^{p,p}_{\vec{\omega}} = \mathcal{A}^p_{\vec{\omega}}$ with general weights $\vec{\omega}$. Mixed norm, weighted Bergman and closely related spaces have been studied, for example, in [1, 2, 3, 4, 6, 7, 8, 10, 11, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

Following [12], for a given weight ω on U, define the *distortion function* of ω by

$$\psi(r) = \psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(t) dt, \qquad 0 \le r < 1.$$

We put $\psi(z) = \psi(|z|)$ for $z \in U$. Also, a class of *admissible weights*, a large class of weight functions ω in U is defined in [12]. For a list of examples of admissible weights, see [12, pp. 660-663].

In [20, Theorem 1] the second author, among others, proved the following result.

Theorem A. *Let $f \in H(U^n)$ and $\omega_j(z_j),\ j = 1, \dots, n$ are admissible weights on the unit disc U, with distortion functions $\psi_j(z_j)$. If $0 < p, q < \infty$, and $f \in \mathcal{A}^{p,q}_{\vec{\omega}}$, then for all $j = 1, \dots, n,\ \psi_j(z_j)\frac{\partial f}{\partial z_j}(z) \in \mathcal{L}^{p,q}_{\vec{\omega}}$, and there is a positive constant $C = C(p, q, \vec{\omega}, n)$ such that*

$$\|f\|_{p,q,\vec{\omega}} \ge C|f(0)| + C \sum_{j=1}^n \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{p,q,\vec{\omega}}. \tag{1.1}$$

For $1 \le p, q < \infty$ the reverse inequality holds as well.

Remark 1. For all $0 < p, q < \infty$ the equivalence between the left-hand and right-hand sides of (1.1) is established in [15, 20] for standard weights $\omega_j(z_j) = (1 - |z_j|)^{\alpha_j}$, $\alpha_j > -1$. See also [13] and [14].

In [9] the authors solved an open problem posed by S. Stević ([13, 14]) regarding the reverse inequality in (1.1) for the case of the unit disk, by proving the following result:

Theorem B. *Assume $0 < p \le \infty, 0 < q < \infty$, and that ω is a differentiable weight function on U satisfying the following condition*

$$\frac{\omega'(r)}{\omega^2(r)} \int_r^1 \omega(s) ds \le L < \infty, \qquad r \in (0, 1), \tag{1.2}$$

for a positive constant L. Then

$$\int_0^1 M^q_p(f,r)\omega(r) dr \asymp |f(0)|^q + \int_0^1 M^q_p(f', r)(\psi_\omega(r))^q \omega(r) dr \tag{1.3}$$

for all $f \in H(U)$.

We write $a \asymp b$ if the ratio a/b is bounded from above and below by two positive constants when the variable varies, and say that a and b are comparable. Note that condition (1.2) is weaker than that of admissible weights, see [9].

An interesting problem is to extend Theorem B to the polydisc case. This will be done by proving the next theorem.

Theorem 1. *Let $f \in H(U^n), 0 < p \leq \infty, 0 < q < \infty$, and the weights $\omega_j(z_j)$, $j = 1, \dots, n$, satisfy condition (1.2), with distortion functions $\psi_j(z_j)$, $j = 1, \dots, n$. Then $f \in \mathcal{A}^{p,q}_{\vec{\omega}}$ if and only if $\psi_j(z_j)\frac{\partial f}{\partial z_j}(z) \in \mathcal{L}^{p,q}_{\vec{\omega}}$ for all $j = 1, \dots, n$. Moreover,*

$$\|f\|_{p,q,\vec{\omega}} \asymp |f(0)| + \sum_{j=1}^{n} \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{p,q,\vec{\omega}}. \tag{1.4}$$

Theorem 1 generalizes both Theorems A and B. In Section 2 we present several auxiliary results which will be used in the proofs of the main results of this paper. A proof of Theorem 1 is given in Section 3. In Section 4 we turn to pluriharmonic functions in U^n, that is, the real parts of holomorphic functions. As an application of Theorem 1, we prove that the operator of pluriharmonic conjugation is bounded in mixed norm spaces $\mathcal{L}^{p,q}_{\vec{\omega}}(U^n)$ for all $0 < p \leq \infty, 0 < q < \infty$.

2 Auxiliary results

In this section we collect and prove several auxiliary lemmas which we use in the proof of the main result. Throughout the paper, the letters $C(p, q, \alpha, \beta, \dots), C_\alpha$ etc. stand for positive constants depending only on the parameters indicated and which may vary from line to line.

Lemma 1. **([9])** *Let $\{A_k\}_{k=0}^{\infty}$ be a sequence of complex numbers, $\alpha, \gamma > 0$. Then the quantities*

$$Q_1 = \sum_{k=0}^{\infty} e^{-k\alpha} |A_k|^\gamma, \quad Q_2 = |A_0|^\gamma + \sum_{k=0}^{\infty} e^{-k\alpha} |A_{k+1} - A_k|^\gamma$$

are comparable.

Lemma 2. **([9])** *Given a function φ on $[0, 1)$ define the sequence $\{r_k\}_{k=0}^{\infty} \subset [0, 1)$ by $\varphi(r_k) = e^k$, $k \geq 0$.*
(a) If the function φ satisfies $\varphi(0) = 1$ and

$$\sup_{0<r<1} \frac{\varphi''(r)\varphi(r)}{\varphi'(r)^2} \leq M < \infty, \tag{2.1}$$

then for every $k \geq 0$,

$$\frac{\varphi'(y)}{\varphi'(x)} \leq e^{2M}, \qquad r_k < x < y < r_{k+2}.$$

(b) If the function φ satisfies

$$\sup_{0<r<1} \frac{|\varphi''(r)|\varphi(r)}{\varphi'(r)^2} \leq M < \infty, \tag{2.2}$$

then for every $k \geq 0$,

$$e^{-2M} \leq \frac{\varphi'(y)}{\varphi'(x)} \leq e^{2M}, \qquad x, y \in [r_k, r_{k+2}].$$

Lemma 3. *Let $f \in H(U^n), 0 < p \leq \infty, \ell = \min\{1, p\}$. Then for any r_j, ρ_j, $0 < r_j < \rho_j < 1$, $j = 1, \ldots, n$,*

$$M_p^\ell(f, \rho_1, \ldots, \rho_n) - M_p^\ell(f, r_1, \ldots, r_n) \leq C \sum_{j=1}^{n} (\rho_j - r_j)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_j}, \rho_1, \ldots, \rho_n \right),$$

where the positive constant C depends only on p and n.

Proof. First assume that $n = 2$. Then by [20, Lemma 3] and the monotonicity of the integral means, we have that

$$\begin{aligned} &M_p^\ell(f, \rho_1,\rho_2) - M_p^\ell(f, r_1, r_2) \\ &\quad = \Big(M_p^\ell(f, \rho_1, \rho_2) - M_p^\ell(f, r_1, \rho_2) \Big) + \Big(M_p^\ell(f, r_1, \rho_2) - M_p^\ell(f, r_1, r_2) \Big) \\ &\quad \leq C(\rho_1 - r_1)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_1}, \rho_1, \rho_2 \right) + C(\rho_2 - r_2)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_2}, r_1, \rho_2 \right) \\ &\quad \leq C(\rho_1 - r_1)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_1}, \rho_1, \rho_2 \right) + C(\rho_2 - r_2)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_2}, \rho_1, \rho_2 \right). \end{aligned}$$

For $n > 2$ the proof is similar and will be omitted.

Lemma 4. *Let $f \in H(U^n)$ and $0 < p \leq \infty$.*
(a) Then for any $0 < r_j < \rho_j < 1$, $j, k \in \{1, \ldots, n\}$

$$M_p \left(\frac{\partial f}{\partial z_k}, r_1, \ldots, r_n \right) \leq C \frac{M_p(f, \rho_1, \ldots, \rho_n)}{\rho_k - r_k},$$

where the positive constant C depends only on p and n.
(b) If $u = Re\, f$ in U^n and $1 \leq p \leq \infty$, then for any $0 < r_j < \rho_j < 1$, $j, k \in \{1, \ldots, n\}$

$$M_p \left(\frac{\partial f}{\partial z_k}, r_1, \ldots, r_n \right) \leq C \frac{M_p(u, \rho_1, \ldots, \rho_n)}{\rho_k - r_k},$$

where the positive constant C depends only on p and n.

Proof. *(a)* We may assume that $k = 1$. Applying the corresponding inequality for the case $n = 1$ (with fixed $r_2, \ldots, r_n$), which holds for $0 < p \leq \infty$, then the monotonicity of the integral means in arguments $r_2, \ldots, r_n$, we obtain

$$M_p \left(\frac{\partial f}{\partial z_1}, r_1, r_2, \ldots, r_n \right) \leq C \frac{M_p(f, \rho_1, r_2 \ldots, r_n)}{\rho_1 - r_1} \leq C \frac{M_p(f, \rho_1, \rho_2 \ldots, \rho_n)}{\rho_1 - r_1}.$$

(b) The proof of this statement is similar to the proof of *(a)*, with the difference that the corresponding one-dimensional inequality holds true for $1 \leq p \leq \infty$.

Lemma 5. *Let $0 < p, q < \infty$. Then for any $r_j \in (0,1)$, $j,k \in \{1,\dots,n\}$,*

$$M_p^q\left(\frac{\partial f}{\partial z_k}, r_1, \dots, r_n\right) \leq \frac{C(p,q)}{R^{1+q}} \int_{r_k-R}^{r_k+R} M_p^q(u, r_1, \dots, r_{k-1}, t, r_{k+1}, \dots, r_n)dt,$$

for all $f \in H(U^n)$, $u = Re\, f$, and $r_k \in (0,1)$ such that $0 < R < r_k < R + r_k < 1$.

Proof. It suffices to apply the corresponding one variable inequality, see [9, Lemma 7].

Let $Ph(U^n)$ denote the set of all (real-valued) pluriharmonic functions on U^n. For the subspace of $\mathcal{L}^{p,q}_{\vec{\omega}}(U^n)$ consisting of pluriharmonic functions let $Ph^{p,q}_{\vec{\omega}}(U^n) = Ph(U^n) \cap \mathcal{L}^{p,q}_{\vec{\omega}}(U^n)$.

Lemma 6. *For any $a \in U^n$, the point evaluation $u \mapsto u(a)$ is a bounded linear functional on $Ph^{p,q}_{\vec{\omega}}(U^n)$ for all $0 < p, q < \infty$.*

Proof. The result follows from the Hardy–Littlewood inequality (HL-property) on $|u|^p$ analogously to [20, Lemma 2] or [14, Lemma 3].

3 Proof of Theorem 1

In order to prove the main theorem, we need some more auxiliary functions.

Suppose that the weights $\omega_j(r_j)$ are differentiable on $(0,1)$ and satisfy

$$\frac{\omega_j'(r_j)}{\omega_j^2(r_j)} \int_{r_j}^1 \omega_j(t)\, dt \leq C, \qquad 0 < r_j < 1, \quad j = 1, \dots, n. \tag{3.1}$$

Their distortion functions are defined by

$$\psi_j(r_j) = \psi_{\omega_j}(r_j) = \frac{1}{\omega_j(r_j)} \int_{r_j}^1 \omega_j(t)\, dt, \qquad 0 < r_j < 1, \quad j = 1, \dots, n.$$

Given a weight ω_j, and $0 < q < \infty$, define the function φ_j on $(0,1)$ by

$$\varphi_j(r_j) \equiv \varphi_{q,\omega_j}(r_j) = \left(q \int_{r_j}^1 \omega_j(t)\, dt\right)^{-1/q}, \quad 0 < r_j < 1, \quad j = 1, \dots, n. \tag{3.2}$$

Note that each of the functions φ_j is strictly increasing on $(0,1)$. Let $\psi_\omega(r) = \prod_{j=1}^n \psi_j(r_j)$ and $\varphi_\omega(r) = \prod_{j=1}^n \varphi_j(r_j)$. It is easy to check that

$$\frac{\varphi_j(r_j)}{\varphi_j'(r_j)} = q\, \psi_j(r_j), \qquad \omega_j(r_j) = \frac{\varphi_j'(r_j)}{\varphi_j(r_j)^{1+q}}, \qquad j = 1, \dots, n, \tag{3.3}$$

and that condition (3.1) is equivalent to (2.1) with $\varphi = \varphi_j$.

Define also the measures on $(0,1)$ by

$$dm_{\varphi_j}(r_j) = \frac{\varphi_j'(r_j)}{\varphi_j(r_j)} dr_j, \qquad j = 1, \dots, n, \qquad dm_{\varphi}(r) = \prod_{j=1}^{n} dm_{\varphi_j}(r_j).$$

We may assume that $n = 2$. The proof for the case $n > 2$ is only technically complicated. We have to prove the inequality

$$\begin{aligned}\int_{(0,1)^2} M_p^q(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 &\le C|f(0,0)|^q \\ &+ C\int_{(0,1)^2} M_p^q\left(\frac{\partial f}{\partial z_1}, r_1, r_2\right)\psi_1^q(r_1)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 \\ &+ C\int_{(0,1)^2} M_p^q\left(\frac{\partial f}{\partial z_2}, r_1, r_2\right)\psi_2^q(r_2)\omega_1(r_1)\omega_2(r_2)dr_1dr_2. \qquad (3.4)\end{aligned}$$

Denoting

$$\begin{aligned}F_0(r_1, r_2) &= \frac{M_p(f, r_1, r_2)}{\varphi_1(r_1)\varphi_2(r_2)}, \\ F_1(r_1, r_2) &= \frac{M_p\left(\frac{\partial f}{\partial z_1}, r_1, r_2\right)}{\varphi_1'(r_1)\varphi_2(r_2)}, \qquad F_2(r_1, r_2) = \frac{M_p\left(\frac{\partial f}{\partial z_2}, r_1, r_2\right)}{\varphi_1(r_1)\varphi_2'(r_2)}, \qquad (3.5)\end{aligned}$$

and taking into account (3.3) and (3.5), we can rewrite (3.4) in the form

$$\|F_0\|^q_{L^q(dm_\varphi)} \le C|f(0,0)|^q + C\|F_1\|^q_{L^q(dm_\varphi)} + C\|F_2\|^q_{L^q(dm_\varphi)}. \qquad (3.6)$$

Without loss of generality we may assume that $\varphi_j(0) = 1,\ j = 1, 2$.

We prove (3.6) only for $0 < p < 1$. The proof for the case $1 \le p \le \infty$ is similar and is omitted. Assuming that $F_1, F_2 \in L^q(dm_\varphi)$ and choosing two sequences $\{r_k\}_{k=0}^{\infty}$, $\{\rho_k\}_{k=0}^{\infty}$ as in Lemma 2, $\varphi_1(r_k) = e^k$, $\varphi_2(\rho_k) = e^k$, we obtain by Lemmas 1 and 3

$$\begin{aligned}\|F_0\|^q_{L^q(dm_\varphi)} &= \int_0^1\int_0^1 M_p^q(f, r, \rho)\frac{\varphi_1'(r)\,\varphi_2'(\rho)}{\varphi_1(r)^{1+q}\,\varphi_2(\rho)^{1+q}}drd\rho \\ &\le C\sum_{k=0}^{\infty} M_p^q(f, r_{k+1}, \rho_{k+1})\int_{r_k}^{r_{k+1}}\int_{\rho_k}^{\rho_{k+1}}\frac{\varphi_1'(r)\,\varphi_2'(\rho)}{\varphi_1(r)^{1+q}\,\varphi_2(\rho)^{1+q}}drd\rho \\ &= C\sum_{k=0}^{\infty} M_p^q(f, r_{k+1}, \rho_{k+1})\left(e^{-qk} - e^{-q(k+1)}\right)^2\frac{1}{q^2} \\ &\le C\sum_{k=0}^{\infty} e^{-2qk}\left(M_p^p(f, r_k, \rho_k)\right)^{q/p} \\ &\le C(M_p^p(f, 0, 0))^{q/p} \\ &\qquad + C\sum_{k=0}^{\infty} e^{-2qk}\left(M_p^p(f, r_{k+1}, \rho_{k+1}) - M_p^p(f, r_k, \rho_k)\right)^{q/p}\end{aligned}$$

$$\leq C|f(0,0)|^q + C\sum_{k=0}^{\infty} e^{-2qk}\Bigg[(r_{k+1}-r_k)^p M_p^p\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right) + (\rho_{k+1}-\rho_k)^p M_p^p\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right)\Bigg]^{q/p}$$

$$\leq C|f(0,0)|^q + C\sum_{k=0}^{\infty} e^{-2qk}(r_{k+1}-r_k)^q M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right) + C\sum_{k=0}^{\infty} e^{-2qk}(\rho_{k+1}-\rho_k)^q M_p^q\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right),$$

where the involved constants $C = C(p, q, \varphi_1, \varphi_2) > 0$ depend only on p, q and the functions φ_1, φ_2. By Lagrange's theorem

$$r_{k+1} - r_k = (e-1)e^k\big(\varphi_1'(x_k)\big)^{-1}, \qquad \text{where} \qquad r_k < x_k < r_{k+1},$$
$$\rho_{k+1} - \rho_k = (e-1)e^k\big(\varphi_2'(y_k)\big)^{-1}, \qquad \text{where} \qquad \rho_k < y_k < \rho_{k+1}.$$

Hence

$$\|F_0\|^q_{L^q(dm_\varphi)} \leq C|f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_1'(x_k)\big)^{-q}e^{-qk} + C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_2'(y_k)\big)^{-q}e^{-qk}. \tag{3.7}$$

On the other hand,

$$\|F_1\|^q_{L^q(dm_\varphi)} = \int_0^1\int_0^1 M_p^q\left(\frac{\partial f}{\partial z_1}, r, \rho\right)\frac{\big(\varphi_1'(r)\big)^{1-q}\varphi_2'(\rho)}{\varphi_1(r)\big(\varphi_2(\rho)\big)^{1+q}}drd\rho$$

$$\geq \sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\int_{r_{k+1}}^{r_{k+2}}\frac{\big(\varphi_1'(r)\big)^{1-q}}{\varphi_1(r)}dr\right)\left(\int_{\rho_{k+1}}^{\rho_{k+2}}\frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}}d\rho\right).$$

Since the function $\varphi_2(\rho)$ is increasing, and

$$\int_{r_{k+1}}^{r_{k+2}}\frac{\varphi_1'(r)}{\varphi_1(r)}dr = 1, \qquad \int_{\rho_{k+1}}^{\rho_{k+2}}\frac{\varphi_2'(\rho)}{\varphi_2(\rho)}d\rho = 1,$$

by the mean value theorem for integrals, there exist numbers ξ_k, $r_{k+1} < \xi_k < r_{k+2}$, such that

$$\|F_1\|^q_{L^q(dm_\varphi)} \geq \sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_1'(\xi_k)\big)^{-q}\big(\varphi_2(\rho_{k+2})\big)^{-q} \geq C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_1'(\xi_k)\big)^{-q}e^{-qk}. \tag{3.8}$$

Similarly, there exist numbers η_k, $\rho_{k+1} < \eta_k < \rho_{k+2}$, such that

$$\|F_2\|^q_{L^q(dm_\varphi)} \geq C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1} \right) \left(\varphi_2'(\eta_k) \right)^{-q} e^{-qk}. \tag{3.9}$$

Combining inequalities (3.7)-(3.9), and using Lemma 2(a), we get

$$\begin{aligned}
\|F_0\|^q_{L^q(dm_\varphi)} &\leq C|f(0,0)|^q + C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1} \right) \left(\varphi_1'(x_k) \right)^{-q} e^{-qk} \\
&\quad + C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1} \right) \left(\varphi_2'(y_k) \right)^{-q} e^{-qk} \\
&\leq C|f(0,0)|^q + C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1} \right) \left(\varphi_1'(\xi_k) \right)^{-q} e^{-qk} \\
&\quad + C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1} \right) \left(\varphi_2'(\eta_k) \right)^{-q} e^{-qk} \\
&\leq C|f(0,0)|^q + C\|F_1\|^q_{L^q(dm_\varphi)} + C\|F_2\|^q_{L^q(dm_\varphi)}.
\end{aligned} \tag{3.10}$$

In order to obtain the reverse inequality first note that

$$\begin{aligned}
\|F_0\|^q_{L^q(dm_\varphi)} &= \int_0^1 \int_0^1 M_p^q(f, r, \rho) \frac{\varphi_1'(r)\, \varphi_2'(\rho)}{\varphi_1(r)^{1+q}\, \varphi_2(\rho)^{1+q}} dr d\rho \\
&\geq \sum_{k=0}^{\infty} M_p^q(f, r_k, \rho_k) \int_{r_k}^{r_{k+1}} \int_{\rho_k}^{\rho_{k+1}} \frac{\varphi_1'(r)\, \varphi_2'(\rho)}{\varphi_1(r)^{1+q}\, \varphi_2(\rho)^{1+q}} dr d\rho \\
&= \frac{1}{q^2} \sum_{k=0}^{\infty} M_p^q(f, r_k, \rho_k) \left(e^{-qk} - e^{-q(k+1)} \right)^2 \\
&\geq C_q \sum_{k=0}^{\infty} e^{-2qk} M_p^q(f, r_k, \rho_k).
\end{aligned} \tag{3.11}$$

On the other hand, employing Lemma 4, we have that

$$\begin{aligned}
&\|F_1\|^q_{L^q(dm_\varphi)} = \int_0^1 \int_0^1 M_p^q \left(\frac{\partial f}{\partial z_1}, r, \rho \right) \frac{\left(\varphi_1'(r) \right)^{1-q} \varphi_2'(\rho)}{\varphi_1(r) \left(\varphi_2(\rho) \right)^{1+q}} dr d\rho \\
&\leq C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1} \right) \left(\int_{r_k}^{r_{k+1}} \frac{\left(\varphi_1'(r) \right)^{1-q}}{\varphi_1(r)} dr \right) \left(\int_{\rho_k}^{\rho_{k+1}} \frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho) \right)^{1+q}} d\rho \right) \\
&\leq C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1} \right) \left(\varphi_1'(x_k) \right)^{-q} \left(\varphi_2(\rho_k) \right)^{-q} \\
&= C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1} \right) \left(\varphi_1'(x_k) \right)^{-q} e^{-kq} \\
&\leq C \sum_{k=0}^{\infty} M_p^q \left(f, r_{k+2}, \rho_{k+2} \right) \left(r_{k+2} - r_{k+1} \right)^{-q} \left(\varphi_1'(x_k) \right)^{-q} e^{-kq}
\end{aligned}$$

for some $x_k \in (r_k, r_{k+1})$. By Lagrange's theorem we have that

$$e^{k+2}(1-e^{-1}) = \varphi_1(r_{k+2}) - \varphi_1(r_{k+1}) = \varphi_1'(z_k)(r_{k+2} - r_{k+1}),$$

for some $z_k \in (r_{k+1}, r_{k+2})$. Hence by Lemma 2(a)

$$\begin{aligned} |f(0,0)|^q + \|F_1\|^q_{L^q(dm_\varphi)} & \\ &\leq |f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(f, r_{k+2}, \rho_{k+2}\right)\left(\frac{\varphi_1'(z_k)}{\varphi_1'(x_k)}\right)^q e^{-q(k+2)}e^{-qk} \\ &\leq |f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(f, r_{k+2}, \rho_{k+2}\right) e^{2Mq}e^{-2q(k+1)} \\ &\leq C\sum_{k=0}^{\infty} M_p^q\left(f, r_k, \rho_k\right) e^{-2qk} \end{aligned} \tag{3.12}$$

Similarly it can be proved that

$$|f(0,0)|^q + \|F_2\|^q_{L^q(dm_\varphi)} \leq C\sum_{k=0}^{\infty} M_p^q\left(f, r_k, \rho_k\right) e^{-2qk}. \tag{3.13}$$

From (3.11)-(3.13) the inequality follows.

4 Pluriharmonic conjugates

In this section we discuss pluriharmonic functions in mixed norm spaces $Ph^{p,q}_{\vec{\omega}}(U^n)$. The problem of harmonic conjugation in mixed norm and Bergman spaces is classical and goes back to Hardy and Littlewood [5]. For pluriharmonic conjugation on the unit ball, unit polydisc and more general bounded symmetric domains in $\mathbb{C}^n$, see [8, 10, 11, 21], where standard weight functions were considered. For harmonic conjugation in mixed norm spaces on the unit disc, with general weights see [9, 14].

Theorem 2. *Let $1 \leq p \leq \infty, 0 < q < \infty$, and each of the weight functions $\omega_j(z_j)$, $j = 1, \ldots, n$, satisfies* (3.1). *Then $Ph^{p,q}_{\vec{\omega}}(U^n)$ is a self-conjugate space. Moreover, if $f \in H(U^n)$, $f = u + iv$, $u \in Ph^{p,q}_{\vec{\omega}}(U^n)$, and v is the pluriharmonic conjugate of u normalized so that $v(0) = 0$, then*

$$\|f\|_{p,q,\vec{\omega}} \leq C(p, q, \vec{\omega}, n)\|u\|_{p,q,\vec{\omega}}. \tag{4.1}$$

Proof. Denoting

$$F_0(r_1, r_2) = \frac{M_p(f, r_1, r_2)}{\varphi_1(r_1)\varphi_2(r_2)} \qquad \text{and} \qquad F_3(r_1, r_2) = \frac{M_p(u, r_1, r_2)}{\varphi_1(r_1)\varphi_2(r_2)}, \tag{4.2}$$

we can easily see that (4.1) is equivalent to

$$\|F_0\|_{L^q(dm_\varphi)} \leq C(p,q,\vec{\omega},n)\|F_3\|_{L^q(dm_\varphi)}. \tag{4.3}$$

Since $1 \leq p \leq \infty$, the method of the proof of Theorem 1 works for this case as well. Indeed, similar to (3.11), we obtain

$$\|F_3\|^q_{L^q(dm_\varphi)} \geq C_q \sum_{k=0}^{\infty} e^{-2qk} M_p^q(u, r_k, \rho_k). \tag{4.4}$$

On the other hand, employing Lemma 4(b), we have that

$$\begin{aligned}\|F_1\|^q_{L^q(dm_\varphi)} &\leq C \sum_{k=0}^{\infty} M_p^q \left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right) \left(\varphi_1'(x_k)\right)^{-q} \left(\varphi_2(\rho_k)\right)^{-q} \\ &\leq C \sum_{k=0}^{\infty} M_p^q \left(u, r_{k+2}, \rho_{k+2}\right) (r_{k+2} - r_{k+1})^{-q} \left(\varphi_1'(x_k)\right)^{-q} e^{-kq}\end{aligned}$$

for some $x_k \in (r_k, r_{k+1})$. By Lagrange's theorem and Lemma 2(a) we obtain

$$|f(0,0)|^q + \|F_1\|^q_{L^q(dm_\varphi)} \leq C \sum_{k=0}^{\infty} M_p^q \left(u, r_k, \rho_k\right) e^{-2qk} \tag{4.5}$$

Similarly, (4.5) can be stated for F_2 instead of F_1. Thus,

$$\|F_0\|_{L^q(dm_\varphi)} \leq C|f(0,0)| + C\|F_1\|_{L^q(dm_\varphi)} + C\|F_2\|_{L^q(dm_\varphi)} \leq C\|F_3\|_{L^q(dm_\varphi)},$$

as desired.

An interesting question is whether Theorem 2 holds true for $0 < p < 1$. In this case we are able to prove a slightly weaker result.

Theorem 3. *Let $0 < p \leq \infty, 0 < q < \infty$, and the weight functions $\omega_j(z_j)$, $j = 1, \ldots, n$, together with their corresponding functions $\varphi_j = \varphi_{\omega_j}$ defined by (3.2), satisfy (2.2). Then $Ph^{p,q}_{\vec{\omega}}(U^n)$ is a self-conjugate space. Moreover, if $f \in H(U^n), f = u + iv$, $u \in Ph^{p,q}_{\vec{\omega}}(U^n)$, and v is the pluriharmonic conjugate of u normalized so that $v(0) = 0$, then*

$$\|f\|_{p,q,\vec{\omega}} \leq C(p,q,\vec{\omega},n)\|u\|_{p,q,\vec{\omega}}. \tag{4.6}$$

Proof. Again we have to prove the inequality (4.3). The proof is now based on Lemmas 2(b), 5 and 6. Note that in view of (3.10) it suffices to prove the inequality

$$|f(0,0)| + \|F_1\|_{L^q(dm_\varphi)} + \|F_2\|_{L^q(dm_\varphi)} \leq C\|F_3\|_{L^q(dm_\varphi)}.$$

By the monotonicity of the integral means and the mean value theorem for integrals, we deduce that

$$\begin{aligned}\|F_1\|^q_{L^q(dm_\varphi)} &= \int_0^1 \left[\int_0^1 M_p^q\left(\frac{\partial f}{\partial z_1}, r, \rho\right)\frac{\left(\varphi_1'(r)\right)^{1-q}}{\varphi_1(r)}dr\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\\ &\le C\int_0^1 \left[\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho\right)\int_{r_k}^{r_{k+1}}\frac{\left(\varphi_1'(r)\right)^{1-q}}{\varphi_1(r)}dr\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\\ &= C\int_0^1 \left[\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho\right)\left(\varphi_1'(x_k)\right)^{-q}\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\\ &\le C\int_0^1 \left[\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, \frac{r_{k+1}+r_{k+2}}{2}, \rho\right)\left(\varphi_1'(x_k)\right)^{-q}\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\end{aligned}$$

for some $x_k \in (r_k, r_{k+1})$. An application of Lemma 5 with $R = \frac{1}{2}(r_{k+2} - r_{k+1})$ and $r_1 \mapsto \frac{1}{2}(r_{k+1} + r_{k+2})$, $k \ge 0$, yields

$$\|F_1\|^q_{L^q(dm_\varphi)} \le C\int_0^1 \left[\sum_{k=0}^{\infty}\frac{\left(\varphi_1'(x_k)\right)^{-q}}{(r_{k+2}-r_{k+1})^{1+q}}\int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)dt\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho.$$

Next, we apply Lagrange's theorem and Lemma 2(b) to obtain

$$\begin{aligned}&\|F_1\|^q_{L^q(dm_\varphi)}\\ &\le C\int_0^1 \left[\sum_{k=0}^{\infty}\frac{\left(\varphi_1'(x_k)\right)^{-q}\left(\varphi_1'(y_k)\right)^{q}}{(r_{k+2}-r_{k+1})\,e^{q(k+2)}}\int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)dt\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\\ &\le C\int_0^1 \left[\sum_{k=0}^{\infty}\frac{e^{-q(k+2)}}{r_{k+2}-r_{k+1}}\int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)dt\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\\ &\le C\int_0^1 \left[\sum_{k=0}^{\infty}(r_{k+2}-r_{k+1})^{-1}\int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)\left(\varphi_1(t)\right)^{-q}dt\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\\ &\le C\int_0^1 \left[\sum_{k=0}^{\infty}\frac{\varphi_1'(y_k)}{\varphi_1(r_{k+2})-\varphi_1(r_{k+1})}\int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)\left(\varphi_1(t)\right)^{-q}dt\right]\frac{\varphi_2'(\rho)\,d\rho}{\left(\varphi_2(\rho)\right)^{1+q}},\end{aligned}$$

where $r_{k+1} < y_k < r_{k+2}$, $\varphi_1(r_k) = e^k$. Since the function $\varphi_1(t)$ is increasing, we get by Lemma 2(b)

$$\begin{aligned}\|F_1\|^q_{L^q(dm_\varphi)} &\le C\int_0^1 \left[\sum_{k=0}^{\infty}\varphi_1'(y_k)\int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)\left(\varphi_1(t)\right)^{-1-q}dt\right]\frac{\varphi_2'(\rho)\,d\rho}{\left(\varphi_2(\rho)\right)^{1+q}}\\ &\le C\int_0^1 \left[\sum_{k=0}^{\infty}\int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)\frac{\varphi_1'(t)}{\left(\varphi_1(t)\right)^{1+q}}dt\right]\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\\ &\le C\|F_3\|^q_{L^q(dm_\varphi)}.\end{aligned}$$

Similarly it can be proved that

$$\|F_2\|_{L^q(dm_\varphi)} \le C\|F_3\|_{L^q(dm_\varphi)}.$$

Finally, by Lemma 6,

$$|f(0,0)| = |u(0,0)| \le C\|F_3\|_{L^q(dm_\varphi)}.$$

This completes the proof of Theorem 3.

Note that although condition (2.2) is stronger than (2.1), the class of weight functions $\omega(z)$ satisfying (2.2) is still rather wide. For example,

$$\omega(r) = \left(\log\frac{1}{1-r}\right)^\gamma (1-r)^\beta \exp\left(\frac{-c}{(1-r)^\alpha}\right), \quad \alpha>0, c>0, \beta\in\mathbb{R}, \gamma\in\mathbb{R},$$

is a typical weight function satisfying (2.2), see [9].

Pluriharmonic conjugation makes it possible to extend Theorem 1 to pluriharmonic functions. The partial differential operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_j}$ are defined by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \qquad z_j = x_j + iy_j.$$

Theorem 4. *Let $u \in Ph(U^n)$ and one of the following two conditions holds:*

(a) $1 \le p \le \infty, 0 < q < \infty$, and the weights $\omega_j(z_j)$, $j = 1,\dots,n$, satisfy condition (3.1), with distortion functions $\psi_j(z_j)$, $j = 1,\dots,n$.

(b) $0 < p \le \infty, 0 < q < \infty$, and the weight functions $\omega_j(z_j)$, $j = 1,\dots,n$, together with their corresponding functions $\varphi_j = \varphi_{\omega_j}$ defined by (3.2), satisfy (2.2). Then

$$\|u\|_{p,q,\vec{\omega}} \asymp |u(0)| + \sum_{j=1}^n \left\|\psi_j \frac{\partial u}{\partial z_j}\right\|_{p,q,\vec{\omega}} \asymp |u(0)| + \sum_{j=1}^n \left\|\psi_j \frac{\partial u}{\partial \overline{z}_j}\right\|_{p,q,\vec{\omega}}. \tag{4.7}$$

Proof. Since the function u is real–valued, the second equivalence in (4.7) is obvious. Let now $f \in H(U^n), f = u + iv$, and v be the pluriharmonic conjugate of u normalized so that $v(0) = 0$. Then by Theorems 1-3 and Cauchy-Riemann equations

$$|u(0)| + \sum_{j=1}^n \left\|\psi_j \frac{\partial u}{\partial z_j}\right\|_{p,q,\vec{\omega}} = |f(0)| + C\sum_{j=1}^n \left\|\psi_j \frac{\partial f}{\partial z_j}\right\|_{p,q,\vec{\omega}} \asymp \|f\|_{p,q,\vec{\omega}} \asymp \|u\|_{p,q,\vec{\omega}},$$

as desired.

Remark 2. It is not difficult to see that Theorem B holds for the case of holomorphic functions on the unit ball $B \subset \mathbb{C}^n$, where ∇f appears instead of f' in (1.3). Note that by the maximal theorem the inequality in Lemma 3 becomes

$$M_p^\ell(f,\rho) - M_p^\ell(f,r) \le C(\rho - r)^\ell M_p^\ell(\nabla f, \rho),$$

$0 < r < \rho < 1$, $f \in H(B)$, where $\ell = \min\{1, p\}$, $p \in (0, \infty]$.

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Criteria for functions to be weighted Bloch

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Abstract

In this paper we give the definition of (α,β)-Bloch space of analytic functions, then we investigate the relation between this (α,β)-Bloch space and the Dirichlet space in the unit disc. Besides, we obtain characterizations for this (α,β)-Bloch space by $\mathcal{Q}_p$ spaces of analytic functions. Moreover, we extend the result due to Yamashita [27] by our (α,β)-Bloch functions. Finally, we characterize our (α,β)-Bloch space by $\mathcal{Q}_p$ spaces of harmonic functions.

1 Introduction

In this section we define all function spaces and classes which will be studied later, as well as certain concepts, and fix the notation.
Let $\Delta=\{z:|z|<1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (see e.g. [3] , [9], [16] and [23]) is defined as follows:

$$\mathcal{B}=\{f: f \text{ analytic in } \Delta \text{ and } \sup_{z\in\Delta}(1-|z|^2)|f'(z)|<\infty\}. \tag{1}$$

So, Bloch functions on the unit disc may be defined as those analytic functions f on Δ for which the radii of the schlicht disks in the range of f are bounded above. The Bloch functions are somewhat analogous to functions in the disc algebra and Bloch functions can be characterized as those analytic functions which are uniformly continuous when Δ is given the hyperbolic metric.
The little Bloch space $\mathcal{B}_0$ is a subspace of $\mathcal{B}_0$ consisting of all $f\in\mathcal{B}$ such that

$$\mathcal{B}_0=\{f: f \text{ analytic in } \Delta \text{ and } \lim_{|z|\to 1^-}(1-|z|^2)|f'(z)|=0\}.$$

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There are some interesting studies of Bloch space in several complex variables (see e.g. [4, 17, 18, 19, 20, 22, 21, 28] and others).
The Dirichlet space (see e.g. [2] and [29]) is defined by

$$\mathcal{D} = \left\{ f : f \text{ analytic in } \Delta \text{ and } \iint_{\Delta} \left|f'(z)\right|^2 d\sigma_z < \infty \right\}, \tag{2}$$

where $d\sigma_z$ is the Euclidean area element $dxdy$.
In 1994, Aulaskari and Lappan [6] introduced a class of holomorphic functions, the so called $\mathcal{Q}_p$-spaces as follows:

$$\mathcal{Q}_p = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta} \iint_{\Delta} \left|f'(z)\right|^2 g^p(z,a) d\sigma_z < \infty \right\}, \tag{3}$$

where the weight function $g(z,a) = \ln\left|\frac{1-\bar{a}z}{a-z}\right|$ is defined as the composition of the Möbius transformation $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ since $a, z \in \Delta$ and the fundamental solution of the two-dimensional real Laplacian. One idea of this work was to "close" the gap between the Dirichlet space and the Bloch space. Main results are :

- $\mathcal{D} \subset \mathcal{Q}_p \subset \mathcal{Q}_q \subset \text{BMOA}, \quad 0 < p < q < 1$ (see [8]) where, BMOA is the space of analytic functions of bounded mean oscillation,
- $\mathcal{Q}_1 = \text{BMOA}$ (see [6]), $\mathcal{Q}_p = \mathcal{B}$, for $p > 1$ (see [6]).

For more information about the study of $\mathcal{Q}_p$ spaces of analytic functions we refer to [5], [6], [7], [11], [26] and others. It should be mentioned here also that several authors (see e.g. [10], [12] and [22]) tried to generalize the idea of these spaces to higher dimensions using several real or complex variables. Also, there are some generalizations in Clifford analysis (see [1], [13], [14] and [15]).

Now, we will give the following definitions:

Definition 1.1 *Let α, β be real numbers ≥ 0. For an analytic function in Δ, we define the (α,β)-Bloch space $\mathcal{B}_{\alpha,\beta}$ as follows:*

$$\mathcal{B}_{\alpha,\beta} = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{a,z\in\Delta} \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}} |f'(z)| < \infty \right\}. \tag{4}$$

Also, we set

$$\mathcal{B}_{\alpha,\beta}(f) = \sup_{a,z\in\Delta} \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}} |f'(z)| \, .$$

Our new definition in this paper gives us the ability to close the gap between the Dirichlet space and the Bloch space.
If $\beta = 0$, then we will get the well known α-Bloch space. If $\alpha = 1$ and $\beta = 0$, then we will get the Bloch space as given by (1).
Also, the little (α,β)-Bloch space $\mathcal{B}_{\alpha,\beta,0}$, is a subspace of $\mathcal{B}_{\alpha,\beta}$ consisting of all $f \in \mathcal{B}_{\alpha,\beta}$ such that

$$\lim_{|z|\to 1^-} \lim_{|a|\to 1^-} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}} |f'(z)| = 0$$

Remark 1.1 *The expression $\mathcal{B}_{\alpha,\beta}(f)$ defines a seminorm while the natural norm is given by*

$$\|f\|_{\mathcal{B}_{\alpha,\beta}} = |f(0)| + \mathcal{B}_{\alpha,\beta}(f).$$

With this norm the space $\mathcal{B}_{\alpha,\beta}$ is a Banach space.

We will need the following lemma in the sequel:

Lemma 1.1 [29] *Let $0 < q < \infty$, $|a| < 1$. Then*

$$\int_{\Gamma_z} \frac{1}{|1-\bar{a}z|^{2q}}\, d\Gamma_z \leq \frac{\lambda}{\big(1-|a|\big)^q},$$

where λ is a constant not depending on a and Γ_z is the boundary of the unit disk Δ.

Remark 1.2 *Two quantities A_f and B_f, both depending on an analytic function f on Δ, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on Δ we have:*

$$\frac{1}{C}B_f \leq A_f \leq CB_f.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.

2 (α,β)-Bloch space and Dirichlet space

In this section we will give characterizations between the Dirichlet space and (α,β)-Bloch space.

Proposition 2.1 *Let f be an analytic function in Δ; $|a| < 1$ and $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$. Then we have that*

$$(1-|a|^2)^{2(\beta+\alpha)}|f'(a)|^2 \leq \frac{1}{\pi R^2}\iint_{\Delta} \big|f'(z)\big|^2 d\sigma_z, \tag{5}$$

where $0 < R < 1$.

Proof: Let $U(a,R) = \{z : \rho(z,a) = |\varphi_a(z)| = \big|\frac{z-a}{1-\bar{a}z}\big| < R\}$ be a pseudohyperbolic disc with center a and radius R. Then,

$$\begin{aligned}\iint_{\Delta} \big|f'(z)\big|^2 d\sigma_z &\geq \iint_{U(a,R)} \big|f'(z)\big|^2 d\sigma_z \\ = \quad \pi R^2(1-|a|^2)^2\big|f'(a)\big|^2 &\geq \pi R^2(1-|a|^2)^{2(\beta+\alpha)}\big|f'(a)\big|^2\end{aligned}$$

Since, $\varphi_a(a) = 0$, then

$$\iint_{\Delta} \big|f'(z)\big|^2 d\sigma_z \geq \pi R^2 \frac{(1-|a|^2)^{2(\beta+\alpha)}}{(1-|\varphi_a(a)|^2)^{2\beta}}\big|f'(a)\big|^2.$$

Our proposition is therefore established.

Inequality (5) yields the following result.

Corollary 2.1 *From proposition 2.1, for $\alpha, \beta \geq 0$, with $\alpha + \beta \geq 1$ and $|a| < 1$ we obtain that*

$$\mathcal{D} \subset \mathcal{B}_{\alpha,\beta}.$$

This corollary means, that the Dirichlet space is a subspace of the Bloch spaces $\mathcal{B}_{\alpha,\beta}$ for $\alpha, \beta \geq 0$, with $\alpha + \beta \geq 1$

Remark 2.1 *It is not strange to obtain our results in Proposition 2.1 and hence in Corollary 2.1 by the term $\frac{(1-|a|^2)^{2(\beta+\alpha)}}{(1-|\varphi_a(a)|^2)^{2\beta}}\left|f'(a)\right|^2$ not the term $\frac{(1-|z|^2)^{2(\beta+\alpha)}}{(1-|\varphi_a(z)|^2)^{2\beta}}\left|f'(z)\right|^2$, since we have used the same technique as in [6, 24] in both articles the authors replaces z by a to obtain characterizations. The same will be done in Proposition 3.1 and Corollary 3.1 in this paper.*

Proposition 2.2 *Let f be an analytic function in Δ, and $f \in \mathcal{B}_{\alpha,\beta}$. Then for $0 < \beta < \infty$, and $0 \leq \alpha < \frac{1}{2}$ we have that*

$$\sup_{z\in\Delta} \iint_{\Delta} \left|f'(z)\right|^2 d\sigma_z \leq (2)^{2\beta}\pi\lambda J(\alpha)\ \mathcal{B}^2_{\alpha,\beta}(f), \tag{6}$$

where $J(\alpha) = \int_0^1 (1-r^2)^{-2\alpha}\ r\ dr$ and λ is a constant not depending on a.

Proof: Since,

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}|f'(z)| \leq \mathcal{B}^2_{\alpha,\beta}(f).$$

Then,

$$\iint_{\Delta} \left|f'(z)\right|^2 d\sigma_z \leq \mathcal{B}^2_{\alpha,\beta}(f) \iint_{\Delta} \frac{(1-|\varphi_a(z)|^2)^{2\beta}}{(1-|z|^2)^{2(\beta+\alpha)}} d\sigma_z$$

Since,

$$\left(1-|\varphi_a(z)|^2\right) = \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\bar{a}z|^2} \tag{7}$$

Then,

$$\begin{aligned} \iint_{\Delta} \left|f'(z)\right|^2 d\sigma_z &\leq \mathcal{B}^2_{\alpha,\beta}(f) \iint_{\Delta} \frac{(1-|a|^2)^{2\beta}(1-|z|^2)^{-2\alpha}}{|1-\bar{a}z|^{4\beta}}\, d\sigma_z \\ &= \pi\, \mathcal{B}^2_{\alpha,\beta}(f) \int_0^1 (1-r^2)^{-2\alpha}(1-|a|^2)^{2\beta} \int_{\Gamma_z} \frac{1}{|1-\bar{a}z|^{4\beta}} d\Gamma_z\, r dr. \end{aligned}$$

Using lemma 1.1, we obtain that

$$\iint_{\Delta} \left|f'(z)\right|^2 d\sigma_z \leq (2)^{2\beta}\pi\lambda J(\alpha)\, \mathcal{B}^2_{\alpha,\beta}(f).$$

Our proposition is therefore proved.

Theorem 2.1 *Let f be an analytic function in Δ. Then, for $0 < R < 1$, the following statements are equivalent:*

1. *$f \in \mathcal{B}_{\alpha,\beta}$ for all $0 \leq \beta < \infty$ and $0 \leq \alpha < \frac{1}{2}$.*
2. *$f \in \mathcal{D} < \infty$.*

Proof: The implication $(1 \Rightarrow 2)$ follows from proposition 2.2. From 2.1, we have that $(2 \Rightarrow 1)$.
The importance of the above theorem is to give us a characterization for the (α, β)-Bloch spaces by the help of integral norms of the Dirichlet space $\mathcal{D}$.

3 Analytic $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{Q}_p$ spaces

Proposition 3.1 *Let f be an analytic function in Δ; $|a| < 1$. Then we have that*

$$\frac{(1-|a|^2)^{2(\beta+\alpha)}}{(1-|\varphi_a(a)|^2)^{2\beta}}|f'(a)|^2 \leq \frac{16}{R^2(1-R^2)^p}\iint_{\Delta}\left|f'(z)\right|^2(1-|\varphi_a(z)|^2)^p\, d\sigma_z \quad (8)$$

where $0 < R < 1$ and $p > 0$.

Proof: By subharmonicity, we have for an analytic function g on $U(a,R)$ that

$$|g(0)|^2 \leq \frac{1}{R^2}\iint_{U(0,R)}|g(w)|^2 d\sigma_w. \quad (9)$$

Let f be an analytic function on $U(a,R)$ and applying inequality (9) to the function $g = f' \circ \varphi_a$, and using change of variables, we get that

$$\begin{aligned} |f'(a)|^2 &\leq \frac{1}{R^2}\iint_{U(0,R)}|f'(\varphi_a(w)|^2 d\sigma_w \\ &= \frac{1}{R^2}\iint_{U(a,R)}|f'(z)|^2\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 d\sigma_z. \end{aligned}$$

Since,

$$\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 \leq \frac{16}{(1-|a|^2)^2} \quad \text{(see [24])}.$$

Therefore,

$$|f'(a)|^2 \leq \frac{16}{R^2(1-|a|^2)^2}\iint_{U(a,R)}|f'(z)|^2 d\sigma_z$$

Since,

$$(1-|\varphi_a(z)|^2)^p \geq (1-R^2)^p \quad \text{and} \quad \varphi_a(a) = 0.$$

Then,

$$\begin{aligned} &\iint_{U(a,R)}\left|f'(z)\right|^2(1-|\varphi_a(z)|^2)^p dxdy \geq \frac{R^2(1-R^2)^p}{16}(1-|a|^2)^2|f'(a)|^2 \\ \geq\ &\frac{R^2(1-R^2)^p}{16}\frac{(1-|a|^2)^{2(\beta+\alpha)}}{(1-|\varphi_a(a)|^2)^{2\beta}}|f'(a)|^2. \end{aligned}$$

Consider the inequality

$$\iint_{\Delta} \left|f'(z)\right|^2 (1-|\varphi_a(z)|^2)^p d\sigma_z \geq \iint_{U(a,R)} \left|f'(z)\right|^2 (1-|\varphi_a(z)|^2)^p d\sigma_z.$$

We obtain that

$$\frac{(1-|a|^2)^{2(\beta+\alpha)}}{(1-|\varphi_a(a)|^2)^{2\beta}} |f'(a)|^2 \leq \frac{16}{R^2(1-R^2)^p} \iint_{\Delta} \left|f'(z)\right|^2 (1-|\varphi_a(z)|^2)^p d\sigma_z$$

Our proposition is therefore established.

Last inequality yields the following result.

Corollary 3.1 *From proposition 2.1, we get for $p>0$, $0<\alpha<\infty$, $0<\beta<\infty$; $\alpha+\beta \geq 1$ and $|a|<1$ that*

$$\mathcal{B}^2_{\alpha,\beta}(f) \leq \frac{16}{R^2(1-R^2)^p} \mathcal{Q}_p(f)\,,$$

where ,

$$\mathcal{Q}_p(f) = \sup_{a\in\Delta} \iint_{\Delta} \left|f'(z)\right|^2 (1-|\varphi_a(z)|^2)^p d\sigma_z \approx \sup_{a\in\Delta} \iint_{\Delta} \left|f'(z)\right|^2 (g(z,a))^p \; d\sigma_z < \infty.$$

Proposition 3.2 *Let f be an analytic function in Δ, $0 \leq \alpha < \infty$ and $0 \leq \beta < \infty$ with $\alpha+\beta \geq 1$ and $2(\alpha+\beta)-1 < p < \infty$. Then we have that*

$$\iint_{\Delta} \left|f'(z)\right|^2 (g(z,a))^p \; d\sigma_z \leq J(p,\alpha,\beta)\; \mathcal{B}^2_{\alpha,\beta}(f), \tag{10}$$

where $J(p,\alpha,\beta) = 2\pi \int_0^1 \frac{(\log \frac{1}{r})^p}{(1-r^2)^{2(\alpha+\beta)}}\; r\; d\,r$.

Proof: Since,

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}} |f'(z)| < \mathcal{B}_{\alpha,\beta}(f),$$

then we obtain

$$\iint_{\Delta} \left|f'(z)|^2 g(z,a))^p \, d\sigma_z \leq \mathcal{B}^2_{\alpha,\beta}(f) \iint_{\Delta} \frac{(\log \frac{1}{|z|})^p}{(1-|z|^2)^{2(\alpha+\beta)}} \; d\sigma_z$$

$$= \quad 2\pi \mathcal{B}^2_{\alpha,\beta}(f) \int_0^1 \frac{(\log \frac{1}{r})^p}{(1-r^2)^{2(\alpha+\beta)}}\; r\; d\,r = J(p,\alpha,\beta)\mathcal{B}^2_{\alpha,\beta}(f).$$

Combining Corollary 3.1 and Proposition 3.2, we have the following theorem:

Theorem 3.1 *Let f be an analytic function in Δ, $0 \leq \alpha < \infty$ and $0 \leq \beta < \infty$ with $\alpha+\beta \geq 1$ and $2(\alpha+\beta)-1 < p < \infty$. Then the following statements are equivalent:*

1. *$f \in \mathcal{B}_{\alpha,\beta}$ for all $0 \leq \beta < \infty$ and $0 \leq \alpha < \infty$,*
2. *$f \in \mathcal{Q}_p$ for all $2(\alpha+\beta)-1 < p < \infty$.*
3. *$f \in \mathcal{Q}_p$ for some $2(\alpha+\beta)-1 < p < \infty$.*

Proof: The implication $(1 \Rightarrow 2)$ follows from Proposition 3.2. $(2 \Rightarrow 3)$ is very clear. $(3 \Rightarrow 1)$ this follows from Corollary 3.1 .

4 Yamashita's theorem for (α, β)-Bloch functions

In [27], Yamashita obtained a criteria for holomorphic Bloch functions in terms of the area and the length of the images of non-Euclidean disks and non-Euclidean circles, respectively. In this section we will extend the results of Yamashita [27] for (α, β)-Bloch functions.
Now, we let

$$\sigma(a,z) = \frac{1}{2}\log\frac{|1-\bar{a}z|+|a-z|}{|1-\bar{a}z|-|a-z|} \quad \text{(see [27])}$$

be the non-Euclidean hyperbolic distance between a and z in Δ. for $0<\rho<\infty$ and $a\in\Delta$, we set

$$H(z,\rho) = \{a\in\Delta \ ; \ \sigma(a,z)<\rho\}$$

and

$$\Gamma(z,\rho) = \{a\in\Delta \ ; \ \sigma(a,z)=\rho\}.$$

Let f be non-constant and holomorphic in Δ and let $A_f(z,\rho)$ be the area of the Riemannian image $F(z,\rho)$ of $H(z,\rho)$ by f, and let $\mathcal{A}_f(z,\rho)$ be the area of the image $F(z,\rho)$ of $H(z,\rho)$ by f, we note that $F(z,\rho)$ is the projection of $F(z,\rho)$ to $\mathbb{C}$. Suppose that $L_f(z,\rho)$ be the length of the Riemannian image of $\Gamma(z,\rho)$ by f, and let $\mathcal{L}_f(z,\rho)$ be the length of the outer boundary of $F(z,\rho)$. Here, the outer boundary of a bounded domain G in $\mathbb{C}$ means the boundary of $\mathbb{C}\backslash E$, where E is the unbounded component of the complement $\mathbb{C}\backslash G$ of G. It is easy to observe that

$$A_f(z,\rho)\geq\mathcal{A}_f(z,\rho) \quad \text{and} \quad L_f(z,\rho)\geq\mathcal{L}_f(z,\rho)$$

for each $0<\rho<\infty$ and each $z\in\Delta$.

Proposition 4.1 *Let f be non-constant and holomorphic in Δ. Then for each $0<\rho<\infty$, $\alpha,\beta\geq 0$, where $1\leq\alpha+\beta<\infty$ and for each $z\in\Delta$, we have that*

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}|f'(z)| \leq \left(\frac{\mathcal{A}_f(z,\rho)}{(1-R^2)^{\beta}\pi t^2}\right)^{\frac{1}{2}},$$

and

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}|f'(z)| \leq \frac{\mathcal{L}_f(z,\rho)}{2(1-R^2)^{\beta}\pi t},$$

where $t=\frac{e^{2\rho}-1}{e^{2\rho}+1}$ *and R is a constant.*

Proof: For the proof of this proposition we may assume that $f'(z)\neq 0$. Now set

$$g(a) = f(\frac{a+z}{1+\bar{z}a}) = c_0+c_1a+c_2a^2+\dots$$

for $|a|<1$, where

$$(1-|z|^2)^{\alpha+\beta}f'(z)\leq c_1=(1-|z|^2)\,f'(z)\neq 0, \ \ \alpha,\beta\geq 0 \text{ and } 1\leq\alpha+\beta<\infty.$$

So, it follows from theorems 1 and 2 in [25] that

$$\pi t^2|c_1|^2 \leq \mathcal{A}_f(z,\rho) \quad \text{and} \quad 2\pi\, t\,|c_1| \leq \mathcal{L}_f(z,\rho).$$

Since, $|\varphi_a(z)| < R$, it follows that

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}|f'(z)| \leq \left(\frac{\mathcal{A}_f(z,\rho)}{(1-R^2)^{\beta}\pi t^2}\right)^{\frac{1}{2}} \text{ and } \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}|f'(z)| \leq \frac{\mathcal{L}_f(z,\rho)}{2(1-R^2)^{\beta}\pi t}$$

Theorem 4.1 *Let f be non-constant and holomorphic function in Δ and let $1 \leq \alpha+\beta < \infty$. Then the following are mutually equivalent:*

(i) f is (α,β)-Bloch,

(ii) there exists $0 < \rho < \infty$ such that

$$\sup_{a,z\in\Delta} A_f(z,\rho) < \infty,$$

(iii) there exists $0 < \rho < \infty$ such that

$$\sup_{a,z\in\Delta} \mathcal{A}_f(z,\rho) < \infty,$$

(iv) there exists $0 < \rho < \infty$ such that

$$\sup_{a,z\in\Delta} L_f(z,\rho) < \infty,$$

(v) there exists $0 < \rho < \infty$ such that

$$\sup_{a,z\in\Delta} \mathcal{L}_f(z,\rho) < \infty.$$

Proof: $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$ are obvious. The implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iv)$ are not difficult to prove. Now, we assume (i) with

$$\sup_{a,z\in\Delta} \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}|f'(z)| = k < \infty, \;\; a \in [0,1)\ .$$

For each fixed $0 < \rho < \infty$, we set

$$t = \frac{e^{2\rho}-1}{e^{2\rho}+1}\,.$$

Then

$$A_f(z,\rho) = \iint_{H(z,\rho)} \left|f'(a)\right|^2 d\sigma_a \leq k^2 \iint_{|a|<t} (1-|a|^2)^{-2(\alpha+\beta)}\, d\sigma_a$$

Similarly, we obtain

$$= \pi \frac{k^2 t^2}{(1-t^2)^{2(\alpha+\beta)}} . L_f(z,\rho) = \iint_{\Gamma(z,\rho)} \left| f'(a) \right| d\sigma_a$$
$$\leq k \iint_{\Gamma(z,\rho)} (1-|a|^2)^{-(\alpha+\beta)} \, d\sigma_a$$
$$= 2\pi \frac{kt}{(1-t^2)^{(\alpha+\beta)}}.$$

Also, $(iii) \Rightarrow (i)$ and $(v) \Rightarrow (i)$ are immediate consequences of Proposition 4.1.

5 Harmonic (α, β)-Bloch spaces

Now, we consider a similar criterion for the normality of harmonic functions. Let h be a real harmonic function in Δ. A harmonic function h is said to be a normal function in Δ if

$$U(h,\alpha,\beta) = \sup_{a,z\in\Delta} \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}} \frac{|grad\, h(z)|}{1+|h(z)|^2} < \infty \, .$$

We denote

$$\mathcal{Q}_p(h) = \sup_{a\in\Delta} \iint_{\Delta} \left(\frac{|grad\, h(z)|}{1+|h(z)|^2} \right)^2 (1-|\varphi_a(z)|^2)^p dxdy$$

Proposition 5.1 *Let h be a real harmonic function in Δ, $p > 2(\alpha+\beta)-1$ and $|a| < 1$. Then we have that*

$$\iint_{\Delta} \left(\frac{|grad\, h(z)|}{1+|h(z)|^2} \right)^2 (1-|\varphi_a(z)|^2)^p dxdy \leq J(p,\alpha,\beta) U^2(h,\alpha,\beta). \qquad (11)$$

Proof: The proof of Proposition 5.1 is much akin that of Proposition 3.2, so it will be omitted.

Theorem 5.1 *Let h be a real harmonic function in Δ and let $1 \leq \alpha+\beta < \infty$, then the following conditions are equivalent:*

(i) h is a normal function,

(ii) $\mathcal{Q}_p(h) < \infty$ for all $p > 2(\alpha+\beta)-1$

(iii) $\mathcal{Q}_p(h) < \infty$ for some $p > 2(\alpha+\beta)-1$.

Proof: $(i) \Rightarrow (ii)$ This follows from Proposition 5.1, $(ii) \Rightarrow (iii)$ is trivial. $(iii) \Rightarrow (i)$ is very similar to Theorem 3 in [6].

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A RELATED FIXED POINT THEOREM ON TWO METRIC SPACES SATISFYING A GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

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Abstract. In this paper, we prove a related fixed point theorem on two complete metric spaces satisfying a general contractive condition of integral type.

1. Introduction

Jungck [9] proved a fixed point theorem for commuting maps generalizing the Banach's fixed point and further he [10] introduced more generalizing commutativity, so called compatibility, which is more general than that of weak commutativity defined by Sessa [13]. Lately, Branciari [4] obtained a fixed point results for a single mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. Rhoades [12] proved two fixed point theorems involving more general contractive conditions. Vijayaraju et al. [14] established a general principle, which maked it possible to proved many fixed point theorems for a pair of maps of integral type. It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. It has been observed in Hicks and Rhoades [8] that some of the defining properties of the metric are not needed in the proofs of certain metric theorems. Hicks and Rhoades [8] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Aliouche [1] gave a common fixed point theorem for selfmappings of a symmetric space under a contractive condition of integral type. Altun and Turkoglu [2] proved a fixed point theorem for mappings satisfying a general contractive of operator type. Motivated by this fact, Fisher [6] and Fisher and Murthy [7] proved related fixed point theorems for two pairs of mappings on two complete metric spaces. Popa [11] proved a generalization of Theorem 2 is proved by Fisher and Murthy in [7] for pairs of mappings satisfying two implicit relations on two metric spaces.

The main purpose of this paper is to give a related fixed point theorem for two pairs of mappings on two metric spaces satisfying a general contractive condition of integral type. These theorems generalizes of Fisher [6] and Fisher and Murthy [7] results in two metric spaces satisfying a general contractive condition of integral type.

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2. Preliminaries

Let X be nonempty set and $f : X \to X$ be a mapping. A point $x \in X$ is said to be a fixed point of f if it solves the (fixed-point) equation: $f(x) = x$.

Theorem 1. *Let (X, d) be a metric space, and $f : X \to X$ a mapping. f is said to be Lipschitz if there exists a real number $c \geq 0$ such that for all x, $y \in X$, we have*

$$d(f(x), f(y)) \leq cd(x, y).$$

f is said to be a contraction mapping if, in the above inequality, $c < 1$, and it is nonexpansive if $c = 1$.

The next result is known as Banach's contraction principle, being due to Banach (1922) (see [3] and [5]).

Theorem 2. *Let (X, d) be a complete metric space, and $f : X \to X$ a contraction mapping. Then, f has a unique fixed point $x^* \in X$ such that for each $x \in X$ $\lim_{n\to\infty} f^n x = x^*$.*

The following theorem was given by Branciari [4] was to analyze the existence of fixed points for mappings of f defined on a complete metric space (X, d) satisfing a contractive condition of integral type.

Theorem 3. *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that for each x, $y \in X$ one has*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, +\infty)$, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

3. A Fixed point theorem and results

Now, we prove the following related fixed point theorem for two pairs of mappings on two complete metric spaces.

Theorem 4. *Let (X, d) and (Y, ρ) be complete metric spaces; A, B be mappings of X into Y and S, T be mappings of Y into X satisfying, for all x, $x' \in X$ and y, $y' \in Y$,*

$$\int_0^{d(SAx,TBx')} \varphi_1(t)dt \leq c \int_0^{m(x,y)} \varphi_1(t)dt \tag{3.1}$$

$$\int_0^{\rho(BSy,ATy')} \varphi_2(t)dt \leq c \int_0^{m'(x,y)} \varphi_2(t)dt, \tag{3.2}$$

where $c \in (0,1)$, $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ $(i = 1, 2)$ *are Lebesgue-integrable mappings which are summable, non-negative and such that*

$$\int_0^{\varepsilon} \varphi_i(t)dt > 0 \quad \textit{for each } \varepsilon > 0 \tag{3.3}$$

and

$$m(x,y) = \max\{d(x,x'), d(x,SAx), d(x',TBx'), \rho(Ax,Bx')\}, \tag{3.4}$$

$$m'(x,y) = \max\{\rho(y,y'), \rho(y,BSy), \rho(y',ATy'), d(Sy,Ty')\}. \tag{3.5}$$

If one of the mappings A, B, S, *and* T *is continuous, then* SA *and* TB *have a unique common fixed point* z *in* X *and* BS *and* AT *have a unique common fixed point* w *in* Y. *Further,* $Az = Bz = w$ *and* $Sw = Tw = z$.

Proof. Let x be an arbitrary point in X, let

$$Ax = y_1,\ Sy_1 = x_1,\ Bx_1 = y_2,\ Ty_2 = x_2,\ Ax_2 = y_3$$

and in general let

$$Sy_{2n-1} = x_{2n-1},\ Bx_{2n-1} = y_{2n},\ Ty_{2n} = x_{2n},\ Ax_{2n} = y_{2n+1}$$

for all $n = 1, 2, \ldots$. Using inequality (3.1) and (3.4) we get

$$\begin{aligned}
\int_0^{d(x_{2n+1},x_{2n})} \varphi_1(t)dt &= \int_0^{d(SAx_{2n},TBx_{2n-1})} \varphi_1(t)dt \\
&\le c \int_0^{m(x_{2n},x_{2n-1})} \varphi_1(t)dt \\
&= c \int_0^{\max\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1}),\rho(y_{2n+1},y_{2n})\}} \varphi_1(t)dt.
\end{aligned}$$

Since $c < 1$, it follows that

$$\int_0^{d(x_{2n+1},x_{2n})} \varphi_1(t)dt \le c \int_0^{\max\{d(x_{2n},x_{2n-1}),\rho(y_{2n},y_{2n+1})\}} \varphi_1(t)dt. \tag{3.6}$$

Using inequality (3.1) again, it follows similarly that

$$\int_0^{d(x_{2n},x_{2n-1})} \varphi_1(t)dt \le c \int_0^{\max\{d(x_{2n-1},x_{2n-2}),\rho(y_{2n},y_{2n-1})\}} \varphi_1(t)dt. \tag{3.7}$$

Similarly, using inequality (3.2) and (3.5) we get

$$\int_0^{\rho(y_{2n},y_{2n-1})} \varphi_2(t)dt \le c \int_0^{\max\{d(x_{2n-1},x_{2n}),\rho(y_{2n-1},y_{2n})\}} \varphi_2(t)dt \tag{3.8}$$

and

$$\int_0^{\rho(y_{2n-1},y_{2n})} \varphi_2(t)dt \leq c \int_0^{\max\{d(x_{2n-2},x_{2n-1}),\rho(y_{2n-2},y_{2n-1})\}} \varphi_2(t)dt. \tag{3.9}$$

Using inequalities (3.6) and (3.8), we have

$$\begin{aligned} \int_0^{d(x_{2n+1},x_{2n})} \varphi_1(t)dt &\leq c \int_0^{\max\{d(x_{2n},x_{2n-1}),\rho(y_{2n},y_{2n+1})\}} \varphi_1(t)dt \\ &\leq c \int_0^{\max\{d(x_{2n},x_{2n-1}),cd(x_{2n-1},x_{2n}),c\rho(y_{2n-1},y_{2n})\}} \varphi_1(t)dt \\ &\leq c \int_0^{\max\{d(x_{2n},x_{2n-1}),\rho(y_{2n-1},y_{2n})\}} \varphi_1(t)dt \end{aligned} \tag{3.10}$$

and similarly from inequalities (3.7) and (3.9), we have

$$\int_0^{d(x_{2n},x_{2n-1})} \varphi_1(t)dt \leq c \int_0^{\max\{d(x_{2n-1},x_{2n-2}),\rho(y_{2n-1},y_{2n-2})\}} \varphi_1(t)dt. \tag{3.11}$$

It now follows from inequalities (3.8), (3.9), (3.10) and (3.11) that

$$\begin{aligned} \int_0^{d(x_{n+1},x_n)} \varphi_1(t)dt &\leq c \int_0^{\max\{d(x_n,x_{n-1}),\rho(y_n,y_{n-1})\}} \varphi_1(t)dt, \\ \int_0^{\rho(y_{n+1},y_n)} \varphi_2(t)dt &\leq c \int_0^{\max\{d(x_n,x_{n-1}),\rho(y_n,y_{n-1})\}} \varphi_2(t)dt \end{aligned}$$

and an induction argument shows that

$$\begin{aligned} \int_0^{d(x_{n+1},x_n)} \varphi_1(t)dt &\leq c^{n-1} \int_0^{\max\{d(x_1,x_2),\rho(y_1,y_2)\}} \varphi_1(t)dt, \\ \int_0^{\rho(y_{n+1},y_n)} \varphi_2(t)dt &\leq c^{n-1} \int_0^{\max\{d(x_1,x_2),\rho(y_1,y_2)\}} \varphi_2(t)dt \end{aligned}$$

for all $n = 1, 2,....$ which implies that $(c < 1)$, $\lim_n \int_0^{d(x_{n+1},x_n)} \varphi_1(t)dt = 0$ and $\lim_n \int_0^{\rho(y_{n+1},y_n)} \varphi_2(t)dt = 0$ which, from (3.3), implies that $\lim_n d(x_{n+1}, x_n) = 0$ and $\lim_n \rho(y_{n+1}, y_n) = 0$ respectively. It follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y.

Now suppose that A is continuous. Then

$$\lim_n Ax_{2n} = Az = \lim_n y_{2n+1} = w$$

and so $Az = w$. Using inequality (3.1) and (3.4) we have

$$\begin{aligned}\int_0^{d(Sw,x_{2n})} \varphi_1(t)dt &= \int_0^{d(SAz,TBx_{2n-1})} \varphi_1(t)dt \\ &\leq c \int_0^{m(z,x_{2n-1})} \varphi_1(t)dt.\end{aligned}$$

Letting n tend to infinity, we have

$$\int_0^{d(Sw,z)} \varphi_1(t)dt \leq c \int_0^{d(z,SAz)} \varphi_1(t)dt = c \int_0^{d(z,Sw)} \varphi_1(t)dt$$

which is a contradiction. Therefore, $\int_0^{d(Sw,z)} \varphi_1(t)dt = 0$ and (3.3) implies that

$$Sw = z = SAz.$$

Now using inequality (3.2) and (3.5) we have

$$\begin{aligned}\int_0^{\rho(Bz,y_{2n+1})} \varphi_2(t)dt &= \int_0^{\rho(BSw,ATy_{2n})} \varphi_2(t)dt \\ &\leq c \int_0^{m'(w,y_{2n})} \varphi_2(t)dt.\end{aligned}$$

Letting n tend to infinity, we have

$$\int_0^{\rho(Bz,w)} \varphi_2(t)dt \leq c \int_0^{\max\{\rho(w,BSw),d(Sw,z)\}} \varphi_2(t)dt = c \int_0^{\rho(w,Bz)} \varphi_2(t)dt$$

which is a contradiction. Therefore, $\int_0^{\rho(Bz,w)} \varphi_2(t)dt = 0$ and (3.3) implies that

$$Bz = w = BSw.$$

Using inequality (3.1) and (3.4) we have

$$\begin{aligned}\int_0^{d(z,Tw)} \varphi_1(t)dt &= \int_0^{d(SAz,TBz)} \varphi_1(t)dt \\ &\leq c \int_0^{m(z,z)} \varphi_1(t)dt \\ &= c \int_0^{d(z,Tw)} \varphi_1(t)dt\end{aligned}$$

which is a contradiction. Therefore, $\int_0^{d(z,Tw)} \varphi_1(t)dt = 0$ and (3.3) implies that

$$Tw = z = TBz.$$

The same results of course hold if one of the mappings B, S, T is continuous insted of A. To prove uniquness, suppose that TB has a second fixed point z'. Then using inequalities (3.1), (3.2), (3.4) and (3.5) we have

$$\begin{aligned}\int_0^{d(z,z')} \varphi_1(t)dt &= \int_0^{d(SAz,TBz')} \varphi_1(t)dt \\ &\leq c \int_0^{m(z,z')} \varphi_1(t)dt \\ &= c \int_0^{\rho(Az,Bz')} \varphi_2(t)dt = c \int_0^{\rho(ASw,BTBz')} \varphi_2(t)dt \\ &\leq c^2 \int_0^{m'(w,Bz')} \varphi_2(t)dt \\ &= c^2 \int_0^{\max\{\rho(Az,Bz'),d(z,z')\}} \varphi_2(t)dt \\ &= c^2 \int_0^{d(z,z')} \varphi_1(t)dt\end{aligned}$$

which implies that $\int_0^{d(z,z')} \varphi_1(t)dt = 0$, which, from (3.3), implies that $d(z, z') = 0$, or $z = z'$, proving that z is the unique fixed point of TB. It follows similarly that z is the unique fixed point of SA and w is the unique fixed point of BS and AT. This completes the proof of the theorem. □

Remark 1. *If $\varphi_i(t) = 1$ for $i = 1, 2$ in Theorem 4, we obtain Theorem 2 of [7].*

Remark 2. *If $\varphi_i(t) = 1$ for $i = 1, 2$ and $Tx = Ax = Bx$ for all $x \in X$ in Theorem 4, we obtain main theorem of [6].*

Putting $X = Y$ in Theorem 4 gives us the following corollary.

Corollary 1. *Let (X, d) be a complete metric space, $c \in [0, 1)$, let A, B, S, T be mappings of X into itself such that, for all x, y in X,*

$$\int_0^{d(SAx,TBy)} \varphi(t)dt \leq c \int_0^{m(x,y)} \varphi(t)dt$$

$$\int_0^{d(BSx,ATy)} \varphi(t)dt \leq c \int_0^{m'(x,y)} \varphi(t)dt$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-integrable mapping which is summable, non-negative and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$. If one of the mappings A, B, S, and T is continuous then SA and TB have a unique common fixed point z and BS and AT have a unique common fixed point w. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Remark 3. *If $\varphi(t) = 1$ in above corollary, we obtain corollary in page 346 of [7].*

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On iterates of Cheney-Sharma operator

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Abstract

The fixed points of the Cheney-Sharma operators of uniform approximation are pointed out. Some properties of the iterates of these operators are investigated.

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Keywords and Phrases: Cheney-Sharma operator, uniform approximation, weakly Picard operators, fixed points, monotone iteration.

1 Introduction

Here we investigate some properties of the iterates of Cheney-Sharma linear operators of uniform approximation, from fixed point theory point of view. These operators firstly appear in [3] and was investigated in [1] (where was studied the rate of convergence and preservation of monotonicity and global smoothness). Other results in the approximation with these operators was obtained in [6].

Firstly, we recall the notion of weakly Picard operator (shortly WPO) introduced in the Fixed Point Theory by I.A. Rus (see [4]). Some related properties of WPO's was investigated in [5].

Let (X,d) be a metric space and $A: X \to X$ an operator. Will be used the following notations:
$F_A = \{x \in X | \, A(x) = x\}$ - the fixed point set of A,
$I(A) = \{Y \in P(X) | \, A(Y) \subset Y\}$ - the family of the nonempty invariant subsets of A,
$A^0 = 1_X, \ A^1 = A, ..., \ A^{n+1} = A \circ A^n, \ n \in \mathbb{N}$.

Definition 1 *(I.A. Rus, [5]) Let (X,d) be a metric space.*
1) An operator $A: X \to X$ is weakly Picard operator (briefly WPO) if the sequence of successive approximations $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.
2) If the operator $A: X \to X$ is WPO and $F_A = \{x^*\}$, then by definition the operator A is Picard operator.
3) If the operator $A: X \to X$ is WPO, then we can consider the operator A^∞ defined by $A^\infty: X \to X, \ A^\infty(x) = \lim\limits_{n\to\infty} A^n(x)$.

Theorem 1 *(I.A. Rus, [5]) Let (X,d) be a metric space. An operator $A: X \to X$ is WPO if and only if there exits a partition of X, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, such that:*
(i) $X_\lambda \in I(A), \ \forall \lambda \in \Lambda$;
(ii) $A|_{X_\lambda}: X_\lambda \to X_\lambda$ is PO, $\forall \lambda \in \Lambda$.

The above theorem characterize the WPO's. Now, let us consider the Cheney-Sharma operators.

Definition 2 *(in [3], [2]) Let $(t_m)_{m\in\mathbb{N}^*}$ be a sequence of positive real numbers. The operator $G_m: C[0,1] \to C[0,1]$ given by*

$$(G_m f)(x) = (1+mt_m)^{-m} \cdot \sum_{k=0}^{m} C_m^k x (x+kt_m)^{k-1} \cdot (1-x+(m-k)t_m)^{m-k} \cdot f\left(\frac{k}{m}\right) \tag{1}$$

for all $x \in [0,1]$ and $m \in \mathbb{N}^*$, is named the Cheney-Sharma operator.

We see that for $t_m = 0,\ \forall m \in \mathbb{N}^*$, G_m becomes the well known Bernstein operator. Let $e_0(x) = 1$ and $e_1(x) = x$ for $x \in [0,1]$. It is proved that $(G_m e_0)(x) = 1$, for all $x \in [0,1]$ and for $(t_m)_{m\in\mathbb{N}^*}$ such that $\lim\limits_{m\to\infty} mt_m = 0$ we have

$$\frac{x}{1+mt_m} \leq (G_m e_1)(x) \leq \frac{x}{1+t_m} \tag{2}$$

for all $x \in [0,1]$ (see [2]). The uniform approximation properties of G_m (using the Bohman-Korovkin theorem) can be summarized in:

Theorem 2 *(in [2]) If $(t_m)_{m\in\mathbb{N}^*}$ is a sequence of positive real numbers such that $\lim\limits_{m\to\infty} mt_m = 0$, then for any $f \in C[0,1]$ we have $\lim\limits_{m\to\infty} G_m f = f$, uniform on $C[0,1]$.*

2 Main results

We denote

$$p_{m,k}(x) = (1+mt_m)^{-m} C_m^k x (x+kt_m)^{k-1} \cdot (1-x+(m-k)t_m)^{m-k} \tag{3}$$

and since $(G_m e_0)(x) = 1$, we infer that $p_{m,k}(x) \geq 0$ and $\sum\limits_{k=0}^{m} p_{m,k}(x) = 1$ for all $x \in [0,1]$.

For $\gamma \in \mathbb{R}$, let $X_\gamma = \{f \in C[0,1] : f(0) = \gamma\}$. It is easy to prove that $C[0,1] = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$ and because $(G_m e_0)(x) = 1$, $X_\gamma \in I(G_m)$ for all $\gamma \in \mathbb{R}$ and $m \in \mathbb{N}^*$.

Theorem 3 *The operators G_m, $m \in \mathbb{N}^*$ are WPO's on $C[0,1]$. For any $\gamma \in \mathbb{R}$ and $m \in \mathbb{N}^*$, G_m is Picard operator on X_γ having the fixed point $\gamma^* \in X_\gamma$, $\gamma^*(x) = \gamma$, $\forall x \in [0,1]$.*

Proof. We see that $(C[0,1], \|\cdot\|_C)$ is Banach space with,

$$\|f\|_C = \max\{|f(x)| : x \in C[0,1]\}.$$

Let $\gamma \in \mathbb{R}$ and $f, g \in X_\gamma$. Since X_γ is closed in $C[0,1]$, we infer that it is Banach space too. For all $x \in [0,1]$ and $m \in \mathbb{N}^*$, we have,

$$|G_m f(x) - G_m g(x)| \leq \left|(1+mt_m)^{-m} C_m^0 \cdot (1-x+mt_m)^m [f(0) - g(0)]\right| +$$

$$+\left|\sum_{k=1}^{m} (1+mt_m)^{-m} C_m^k x (x+kt_m)^{k-1} \cdot (1-x+(m-k)t_m)^{m-k} \cdot [f\left(\frac{k}{m}\right) - g\left(\frac{k}{m}\right)]\right| \leq$$

$$\leq \sum_{k=1}^{m} (1+mt_m)^{-m} C_m^k x (x+kt_m)^{k-1} \cdot (1-x+(m-k)t_m)^{m-k} \cdot \left|f\left(\frac{k}{m}\right) - g\left(\frac{k}{m}\right)\right| \leq$$

$$\leq \sum_{k=1}^{m} p_{m,k}(x) \cdot \|f-g\|_C = \left[\sum_{k=0}^{m} p_{m,k}(x) - p_{m,0}(x)\right] \cdot \|f-g\|_C =$$

$$= \left[1 - \left(1 - \frac{x}{1+mt_m}\right)^m\right] \cdot \|f-g\|_C \leq \left[1 - \left(1 - \frac{1}{1+mt_m}\right)^m\right] \cdot \|f-g\|_C .$$

Consequently,

$$\|G_m f - G_m g\|_C \leq \left[1 - \left(1 - \frac{1}{1+mt_m}\right)^m\right] \cdot \|f-g\|_C . \tag{4}$$

Then, according to Banach's fixed point principle, G_m is contraction on X_γ, that is Picard operator. According to Theorem 1 we infer that G_m is WPO on $C[0,1]$.

Because, $G_m\gamma^*(x) = \sum\limits_{k=0}^{m} p_{m,k}(x) \cdot \gamma = \gamma$ for all $x \in [0,1]$, follows that $G_m\gamma^* = \gamma^*$. So, γ^* is the unique

fixed point of G_m in X_γ.
Considering the sequence of successive approximations, $(G_m^n)_{n\in\mathbb{N}}$ we obtain,

$$G_m^\infty f = \lim_{n\to\infty} G_m^n f = \gamma^* = f(0) \quad \text{for all} \quad f \in X_\gamma. \tag{5}$$

Moreover, by induction, we obtain the error estimate in the above convergence:

$$\|G_m^n f - \gamma^*\|_C \leq \left[1 - \left(1 - \frac{1}{1+mt_m}\right)^m\right]^n \cdot diam(f) \tag{6}$$

for all $f \in X_\gamma$, where $diam(f) = \max\{|f(x) - f(y)| : x, y \in [0,1]\}$. ■

Now, consider a sequence $(t_m)_{m\in\mathbb{N}^*}$ such that $\lim_{m\to\infty} mt_m = 0$ and $(G_m f)_{m\in\mathbb{N}^*}$ is increasing sequence on $(C[0,1], \leq)$ for f concave and increasing, $f \in C[0,1]$. Such sequence exists, for instance, $t_m = 0$ get the Bernstein operator and it is well known that $(B_m f)_m$ is increasing for f concave.

Theorem 4 *If $(t_m)_{m\in\mathbb{N}^*}$ is a sequence of positive real numbers such that $\lim_{m\to\infty} mt_m = 0$ and $(G_m f)_{m\in\mathbb{N}^*}$ is increasing sequence for f concave and increasing, then the sequences $(G_m f)_{m\in\mathbb{N}^*}$ and $(G_k^n f)_{n\in\mathbb{N}}$ for $k \in \mathbb{N}^*$ fixed, have monotone iteration, presenting the following duality in $(C[0,1], \leq)$:*

$$f(0) = G_k^\infty f = \lim_{n\to\infty} G_k^n f \leq ... \leq G_k^{n+1} f \leq G_k^n f \leq ... \leq G_k^1 f \leq G_k^0 f = f \tag{7}$$

$$f(0) = G_0 f \leq G_1 f \leq ... \leq G_m f \leq G_{m+1} f \leq ... \leq \lim_{m\to\infty} G_m f = f. \tag{8}$$

Proof. From inequality (2) follows that $(G_m e_1)(x) \leq x$, for all $x \in [0,1]$. Since f is concave and increasing, using the Jensen's inequality we obtain,

$$(G_m f)(x) = \sum_{k=0}^m p_{m,k}(x) \cdot f\left(\frac{k}{m}\right) \leq f\left(\sum_{k=0}^m p_{m,k}(x) \cdot \frac{k}{m}\right) = f((G_m e_1)(x)) \leq f(x).$$

Because the operator G_m is monotone, by induction we have

$$G_m^{n+1} f \leq G_m^n f \leq ... \leq G_m^1 f \leq G_m^0 f = f$$

which lead to (7). On the other hand, $f(0) = G_0 f(x) \leq G_1 f(x) = f(0) + (1+t_1)^{-1} \cdot [f(1) - f(0)]x \leq f(x)$, for all $x \in [0,1]$, which lead to (8). ■

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B-spline solution for a singularly perturbed convection-dominated diffusion equation

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Abstract

In this study, B-spline method is applied to the convection-dominated diffusion problems. The numerical solution of the equations are discussed and illustrated with an example. Computational results are provided to demonstrate the viability of the present new method.

Keywords: B-spline; Perturbed convection-dominated problem.

1. Introduction

Convection-dominated diffusion problems take place in fluid mechanics [1] and can be applied in various physical and engineering process such as weather prediction, ocean circulation, petroleum reservoir [2-5], etc. On the other hand, singularly perturbed problems are also important in many branches of science and engineering too. Extensive information about such as problems can be found in [6,7]. There are many publications dealing with the convection-dominated diffusion equations. They introduced various numerical methods. For instance, a finite element method has been proposed for the problems within the frameworks Galerkin formulation and Eulerian-Lagrangian localized adjoint methods (ELLAM) in a recent work [8]. Also a Wavelet-Galerkin method has been studied for the numerical treatment of singularly perturbed convection-dominated diffusion problems in ref. [9]. In this work, they showed that the Wavelet-Galerkin method is a very effective tools such problems. A more detailed and extensive review of different numerical methods for convection-dominated diffusion equations can be found in [2,10]. Following previous work [11], we present and analyze the B-spline method for numerical analysis of the singularly perturbed convection-dominated diffusion equations. For the numerical tests, we consider the singularly perturbed convection-dominated diffusion problems given in [9]

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + \nu(x,t) \ , \ 0 \leq x \leq 1, t \geq 0 \ , \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, \qquad t \geq 0 \ , \tag{2}$$

$$u(1,t) = 0, \qquad t \geq 0, \tag{3}$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 \leq x \leq 1\ , \tag{4}$$

2. The third-degree B-splines

A detailed description of B-spline functions generated by subdivision can be found in [13]. Consider equally-spaced knots of a partition $\pi : a = x_0 < x_1 < ... < x_n = b$ on [a,b]. Let $S_3[\pi]$ be the space of continuously-differentiable, piecewise, third-degree polynomials on π. That is, $S_3[\pi]$ is the space of third-degree splines on π. Consider the B-splines basis in $S_3[\pi]$. The third-degree B-splines are defined as

$$B_0(x) = \tfrac{1}{6h^3} \begin{cases} x^3 & 0 \leq x < h \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3 & h \leq x < 2h \\ 3x^3 - 24hx^2 + 60h^2x - 44h^3 & 2h \leq x < 3h \\ -x^3 + 12hx^2 - 48h^2x + 64h^3 & 3h \leq x < 4h \end{cases} \tag{5}$$

$$B_{i-1}(x) = B_0(x - (i-1)h),\ i = 2, 3, ...,$$

To solve singularly perturbed convection-dominated diffusion equation, B_i , B_i' *and* B_i'' evaluated at the nodal points are needed. Their coefficients are summarized in Table 1.

Table 1: Values of B_i, B_i' *and* B_i''

	$\mathbf{x}_i$	$\mathbf{x}_{i+1}$	$\mathbf{x}_{i+2}$	$\mathbf{x}_{i+3}$	$\mathbf{x}_{i+4}$
B_i	0	1/6	4/6	1/6	0
B_i'	0	-3/6h	0/6h	3/6h	0
B_i''	0	$6/6h^2$	$-12/6h^2$	$6/6h^2$	0

3. B-spline solutions for singularly perturbed convection-dominated diffusion equation

In this section a spline method for solving singularly perturbed convection-dominated diffusion equation is outlined, which is based on the collocation approach [12]. Let

$$S(x) = \sum_{j=-3}^{n-1} C_j B_j(x) \tag{6}$$

be an approximate solution of Eq.(1), where C_i are unknown real coefficients and $B_j(x)$ are third-degree B-spline functions. Let $x_0, x_1, ..., x_n$ be $n+1$ grid points in the interval $[a, b]$, so that

$x_i = a + ih, i = 0, 1, ..., n; x_0 = a, x_n = b, h = (b-a)/n.$

Difference schemes for this problem considered as following:

$$\frac{u_{i+1}-u_i}{\Delta t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + \nu(x,t) \tag{7}$$

where $\Delta t = k$

$$-k\varepsilon u''_{i+1} + k u'_{i+1} + u_{i+1} = u_i + k\nu(x,t) \tag{8}$$

and the initial condition is given in (4)

$$u(x,0) = f(x) = u_0, \tag{9}$$

Subsituting (9) in (8) then is obtained as follows

$$t = 0 + \Delta t \qquad -k\varepsilon u''_1 + k u'_1 + u_1 = u_0 + k\nu(x,t) \tag{10}$$

$$t = 0 + 2\Delta t \qquad -k\varepsilon u''_2 + k u'_2 + u_2 = u_1 + k\nu(x,t) \tag{11}$$

$$\begin{array}{cc} . & . \\ . & . \\ . & . \end{array}$$

$$t = 0 + n\Delta t \qquad -k\varepsilon u_n'' + k u_n' + u_n = u_{n-1} + k\nu(x,t) \tag{12}$$

The approximate solution of the equation (10)-(12) are sought in the form of the B-spline functions $S(x)$, it follows that

$$t = 0 + \Delta t \qquad -k\varepsilon S_1'' + k S_1' + S_1 = u_0 + k\nu(x,t) \tag{13}$$

$$t = 0 + 2\Delta t \qquad -k\varepsilon S_2'' + k S_2' + S_2 = u_1 + k\nu(x,t) \tag{14}$$

$$\begin{array}{cc} . & . \\ . & . \\ . & . \end{array}$$

$$t = 0 + n\Delta t \qquad -k\varepsilon S_n'' + k S_n' + S_n = u_{n-1} + k\nu(x,t) \tag{15}$$

and boundary conditions (2)-(3)

$$\sum_{j=-3}^{n-1} C_j B_j(0) = 0 \text{ for } x = 0, \tag{16}$$

$$\sum_{j=-3}^{n-1} C_j B_j(1) = 0 \text{ for } x = 1 \ , \tag{17}$$

The spline solution of eq.(13) with the boundary conditions is obtained by solving to the following matrix equation. The value of spline functions at the knots $\{x_i\}_{i=0}^{n}$ are determined using Table 1. Then we can write in matrix-vector form as follows

$AC = F$

where

$C = [\ \mathrm{C}_{-3}\ ,\ C_{-2}\ ,\ C_{-1}\ ,\ .\ .\ .\ ,\ C_{n-3}\ ,\ C_{n-2}\ ,C_{n-1}\]^T$

$F = [0\ ,\ f(0) + \nu(0,k)\ ,\ f(h) + \nu(h,k)\ ,\ f(2h) + \nu(2h,k),\ .\ .\ .\ ,$

$f((n-1)h) + \nu((n-1)h,k)\ ,\ 0]^T$

T denoting transpose.

The matrix A can be writen as

$$A = \begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & 0 & ... & 0 \\ \varphi_1 & \varphi_2 & \varphi_3 & 0 & 0 & ... & 0 \\ 0 & \varphi_1 & \varphi_2 & \varphi_3 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & ... & \varphi_1 & \varphi_2 & \varphi_3 & 0 \\ 0 & 0 & ... & 0 & \varphi_1 & \varphi_2 & \varphi_3 \\ 0 & 0 & ... & 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{bmatrix}$$

where

$\varphi_1 = -k\varepsilon\beta(\frac{6}{6h^2}) + k\alpha(\frac{3}{6h}) + \frac{1}{6}$

$\varphi_2 = -k\varepsilon\beta(\frac{-12}{6h^2}) + k\alpha(\frac{0}{6h}) + \frac{4}{6}$

$\varphi_3 = -k\varepsilon\beta(\frac{6}{6h^2}) + k\alpha(\frac{-3}{6h}) + \frac{1}{6}$

It is easy to see that, the same approximation is applied the other equations (14)-(15).

4. Numerical results

In this section, the method discussed in section 2 and 3 is tested on the following problems from the literature[9]. All computations were carried out using MATLAB 6.5.

Example

Consider the diffusion problem(1) with $\nu(x,t)=\frac{(t^2-t+(1-x))}{\varepsilon(1-e^{-1/\varepsilon})}e^{-(1-x)t/\varepsilon}$, initial condition

$$u(x,0)=1, \qquad 0\le x\le 1\ ,$$

and boundary conditions

$$u(0,t)=1+\frac{1-e^{-t/\varepsilon}}{1-e^{-1/\varepsilon}}, \qquad t\ge 0,$$

$$u(1,t)=1, \qquad t\ge 0.$$

The exact solution of this problem is $u(x,t)=1+\frac{1-e^{-(1-x)t/\varepsilon}}{1-e^{-1/\varepsilon}}$. The observed maximum absolute errors for various values of k and for different ε values are given in Table 2. The numerical results are illustrated in Figures 1,2 and 3.

Table 2: The maximum absolute errors, n=91

k	$\varepsilon=1$	$\varepsilon=0.1$	$\varepsilon=0.01$
0.01	2.573543668e-004	0.024312386	0.374292263
0.001	2.590668455e-005	0.002496068	0.044766988

4. Conclusions

In this paper, we focused on the solutions of the convection-dominated diffusion problems by using B-spline method. The results illustrated in Figs. 1, 2 and 3 showed that when ε was increased, the maximum absolute error increased. Analyzing the curves inclination it was possible to conclude that when the diffusion coefficient increased the inclination was larger. Use of B-splines has shown that it is an applicable method for determining the diffusion. The method shows promise for small diffusion coefficients (ε) while it is sensitive to higher values of diffusion coefficients and becomes unstable. That's why further work is required to adapt it with the boundary treatment.

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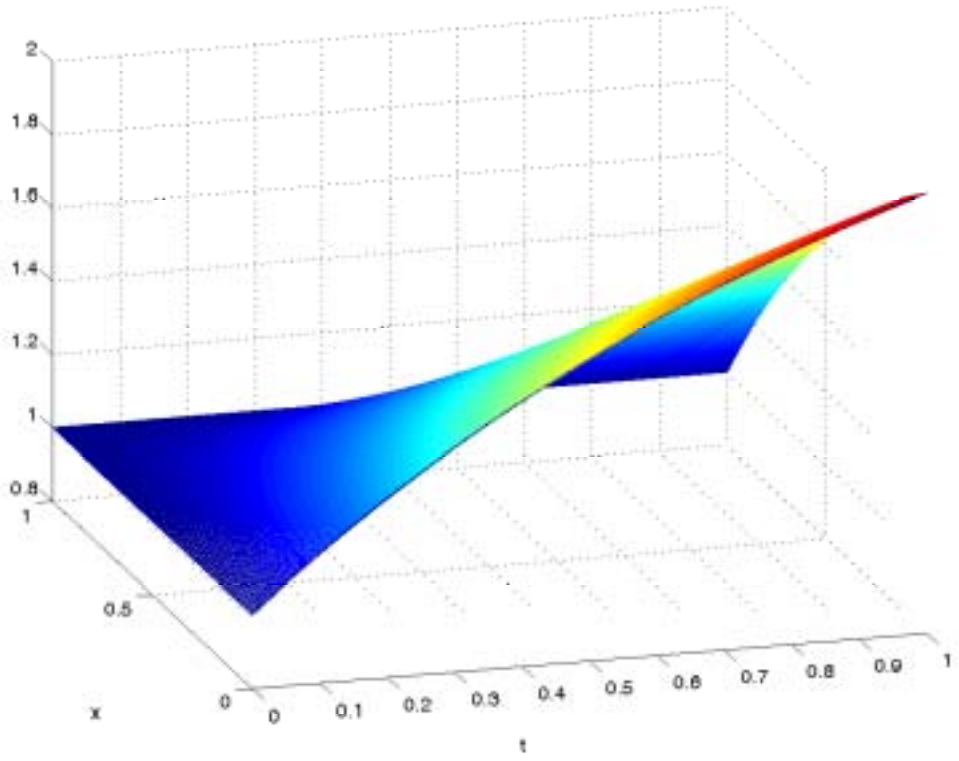


Figure 1: Results for $\varepsilon = 1, n = 91, k = 0.001$

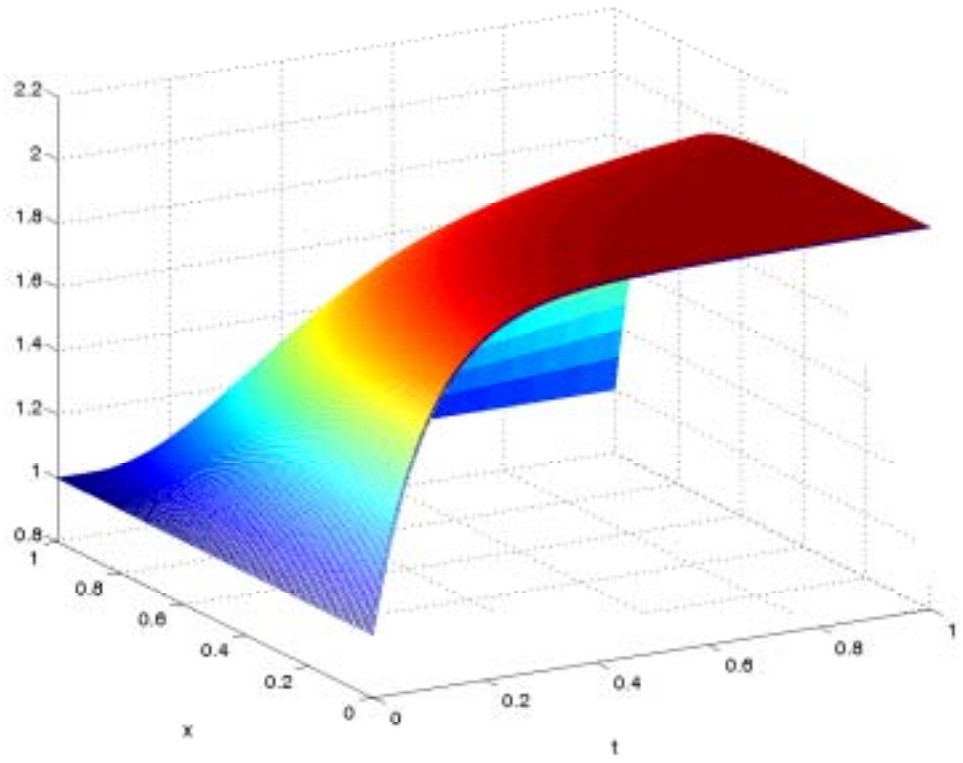


Figure 2: Results for $\varepsilon = 0.1, n = 91, k = 0.001$

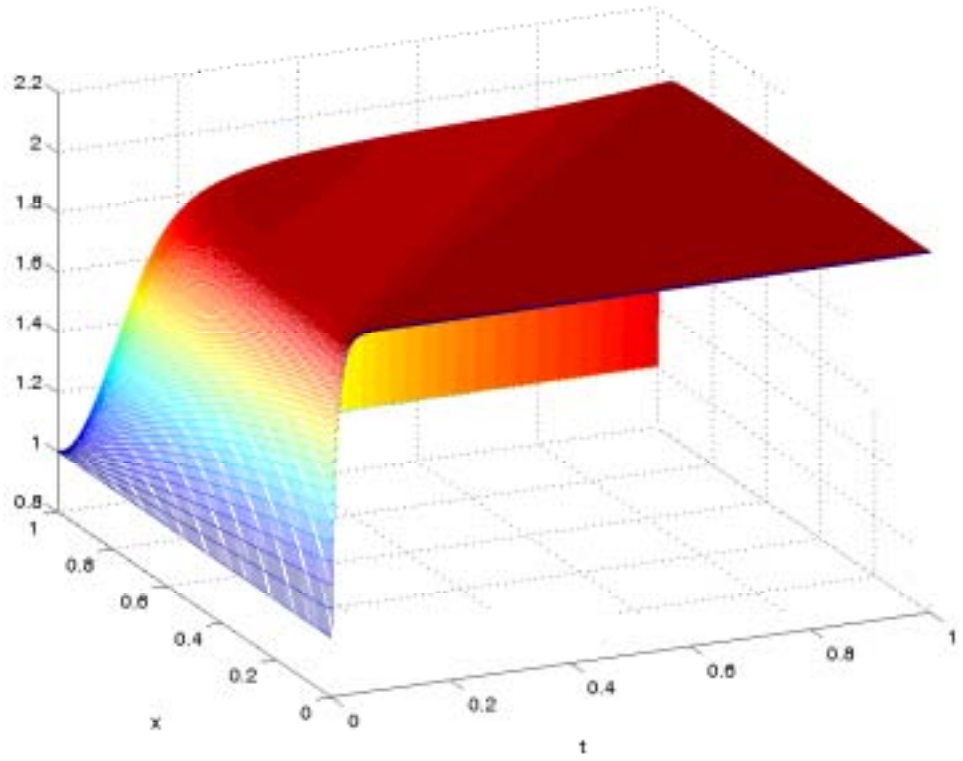


Figure 3: Results for $\varepsilon = 0.01, n = 91, k = 0.001$

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Approximation of fractals generated by Fredholm integral equations

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Abstract

We present the first results of a program recently initiated by us whose main aim is the study of fractals generated by iterated function systems in infinite dimensional spaces (e.g. function spaces). The aim of this paper is to present an example of a fractal, generated by Hutchinson's procedure, embedded in an infinite dimensional Banach space and its finite dimensional approximations. Up to now, the initial finite system of contractions has been given in an Euclidian (finite dimensional) space. Our approach is quite different. Namely, we work in the Banach space of real valued continuous functions on a compact interval and we describe the attractor generated by an iterated function system given by Fredholm integral equations.

Keywords: fractals; iterated function systems; approximation; Fredholm integral equations; function spaces

2000 Mathematics Subject Classification codes: 28A80, 41A65

1. INTRODUCTION

Since the appearance of Hutchinson's paper [4], many papers containing several types of generalizations of the iterated function systems theory appeared, e.g. [3], [5], [6] and [7].

The aim of this paper is to present an example of a fractal, generated by Hutchinson's procedure, embedded in an infinite dimensional Banach space and its finite dimensional approximations. Up to now, the initial finite system of

contractions has been given in an Euclidian (finite dimensional) space. Our approach is quite different. Namely, we work in the Banach space of real valued continuous functions on a compact interval and we describe the attractor generated by an iterated function system given by Fredholm integral equations.

2. PRELIMINARIES

Let $a < b$ be real numbers. In the sequel X (respective Y) will be the Banach space of all continuous $f : [a, b] \to \mathbb{R}$ (respective $K : [a, b] \times [a, b] \to \mathbb{R}$) equipped with the sup norm.

For $f \in X$, $K \in Y$ and $\lambda \in \mathbb{R}$, such that

$$|\lambda| (b - a) \|K\| < 1,$$

we can define the contraction $T : X \to X$, given via $T(u) = v$, where

$$v(x) = f(x) + \lambda \int_a^b K(x, y)u(y)dy,$$

with contraction constant $|\lambda| (b - a) \|K\|$.

The fixed point φ of T is the solution of the Fredholm integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x, y)\varphi(y)dy.$$

The space Y includes the subspace S of all polynomial functions having the form $P(x, y) = \sum_{i,j} a_{i,j} x^i y^j$.

Lemma. *The space S is dense in Y.*

Proof. For instance, in case that $a = 0$ and $b = 1$, for $K \in Y$, one can consider the sequence of polynomial functions $(P_n(K))_n$, defined via

$$P_n(K)(x, y) = \sum_{k,j=0}^{n} K(\frac{k}{n}, \frac{j}{n}) \binom{n}{k} \binom{n}{j} x^k (1 - x)^{n-k} y^j (1 - y)^{n-j}$$

(the bivariate Bernstein polynomials).

Then $P_n(K) \underset{n}{\to} K$ in Y. □

For T as above, take $\varepsilon > 0$ such that

$$|\lambda| (b - a)(\|K\| + \varepsilon) < 1$$

and form a sequence $(P_n)_n$ in Y such that

$$\|P_n - K\| < \varepsilon$$

(which implies $\|P_n\| < \|K\| + \varepsilon$), for all n and such that

$$P_n \underset{n}{\to} K$$

in Y.

Then $T_n : X \to X$, given via $T_n(u) = v_n$, where

$$v_n(x) = f(x) + \lambda \int_a^b P_n(x, y)u(y)dy$$

are contractions and

$$T_n(u) \underset{n}{\to} T(u),$$

for all u in X.

More precisely, we have, for all u in X,

$$\|T_n(u) - T(u)\| \leq |\lambda| (b - a) \|P_n - K\| \|u\|$$

and the supremum of the contraction constants of all T_n is less than or equal to $|\lambda| (b - a)(\|K\| + \varepsilon) < 1$.

If $P_n(x, y) = \sum_{i,j} a_{i,j} x^i y^j$, the fixed point φ of T_n is the solution of the Fredholm integral equation with degenerate kernel P_n,

$$\varphi(x) = f(x) + \lambda \int_a^b (\sum_{i,j} a_{i,j} x^i y^j)\varphi(y)dy,$$

and has the form

$$\varphi(x) = f(x) + \sum_{i=0}^{m} a_i x^i,$$

where $a_i \in \mathbb{R}$ are the solutions of a Cramer system.

It is possible to make the identification $\varphi \equiv (a_0, a_1, ..., a_m)$.

3. THE APPROXIMATION RESULT FOR FRACTALS GENERATED BY FREDHOLM INTEGRAL EQUATIONS

Let $\mathcal{K}(X)$ be the set of all non empty compact subsets of X, which becomes a complete metric space when equipped with the Hausdorff-Pompeiu metric H, given via

$$H(A, B) = \max(d(A, B), d(B, A)),$$

where

$$d(A, B) = \sup\{dist(u, B) \mid u \in A\}$$

and

$$dist(u, B) = \inf\{\|u - b\| \mid b \in B\}.$$

Let us consider an integer $h \geq 2$, $f^1, f^2, ..., f^h$ in X, $K^1, K^2, ..., K^h$ in Y and $\lambda \in \mathbb{R}$ such that

$$|\lambda| (b - a) \left\|K^i\right\| < 1,$$

for all $i \in \{1, 2, ..., h\}$.

We define the Hutchinson contraction (see [2] and [4]) $F : \mathcal{K}(X) \to \mathcal{K}(X)$, given via

$$F(B) = \underset{i=1}{\overset{h}{\cup}} T^i(B),$$

where $T^i : X \to X$ are given via

$$T^i(u) = v^i, v^i(x) = f^i(x) + \lambda \int_a^b K^i(x,y)u(y)dy.$$

The contraction constant of F is less than or equal to

$$\max_{i\in\{1,2,\ldots,h\}} |\lambda| (b-a) \left\|K^i\right\|.$$

Let $A \in \mathcal{K}(X)$ be the attractor of F (i.e. $A = F(A)$ is the fixed point of F). Generally speaking, A is a *fractal*.

Our aim in the sequel is to approximate A.

3.1. First stage of approximation for A.

Let us approximate every K^i via a sequence $(P_n^i)_n$ in S such that

$$P_n^i \underset{n}{\to} K^i$$

and

$$\left\|P_n^i - K^i\right\| < \varepsilon,$$

where $\varepsilon > 0$ is subject to the condition

$$r = |\lambda| (b-a)(C+\varepsilon) < 1,$$

where $C = \max\limits_{i\in\{1,2,\ldots,h\}} \left\|K^i\right\|$.

We define as above the contractions $T_n^i : X \to X$ given via $T_n^i(u) = v_n^i$,

$$v_n^i(x) = f^i(x) + \lambda \int_a^b P_n^i(x,y)u(y)dy.$$

Then

$$T_n^i(u) \underset{n}{\to} T^i(u),$$

for all u in X and all i and the sequence of contraction constants for $(T_n^i)_n$ has supremum less than or equal to $r < 1$ (for all i).

We can construct the sequence of contractions $(F_n)_n$, $F_n : \mathcal{K}(X) \to \mathcal{K}(X)$ given via

$$F_n(B) = \underset{i=1}{\overset{h}{\cup}} T_n^i(B).$$

The sequence of contraction constants for $(F_n)_n$ has supremum less than or equal to $r < 1$.

Now we can prove the following

Theorem. *If A_n is the attractor of F_n, then*

$$A_n \underset{n}{\to} A$$

in the Hausdorff-Pompeiu metric.

Proof. The proof is divided into three steps, namely a), b) and c).

Notice first that the contraction constants of all F_n and of F (and of all T_n and of T) are smaller than r.

a) Fix i and prove that

$$T_n^i(B) \underset{n}{\to} T^i(B),$$

for all B in $\mathcal{K}(X)$.

Assuming, by absurd, the existence of some B in $\mathcal{K}(X)$ and of some $\varepsilon > 0$ such that

$$H(T_n^i(B), T^i(B)) > \varepsilon$$

for infinitely many n, two cases can occur:

First case:

$$d(T_n^i(B), T^i(B)) > \varepsilon$$

for infinitely many n.

For all such n, there exists b_n in B, such that

$$dist(T_n^i(b_n), T^i(B)) > \varepsilon$$

and this implies

$$\left\|T_n^i(b_n) - T^i(b)\right\| > \varepsilon,$$

for all b in B.

Because B is compact, there exists a sequence $(b_{n_k})_k$ and b' in B, such that

$$b_{n_k} \underset{k}{\to} b'.$$

So, for all k, one has

$$\varepsilon < \left\|T_{n_k}^i(b_{n_k}) - T^i(b')\right\| \leq \left\|T_{n_k}^i(b_{n_k}) - T_{n_k}^i(b')\right\| + \left\|T_{n_k}^i(b') - T^i(b')\right\| \leq$$

$$\leq r\left\|b_{n_k} - b'\right\| + \left\|T_{n_k}^i(b') - T^i(b')\right\| \underset{k}{\to} 0,$$

which is a contradiction.

Second case:

$$d(T^i(B), T_n^i(B)) > \varepsilon$$

for infinitely many n.

For all such n, there exists $b_n \in B$, such that

$$dist(T^i(b_n), T_n^i(B)) > \varepsilon$$

and this implies

$$\left\|T^i(b_n) - T_n^i(b)\right\| > \varepsilon$$

for all b in B.

Again, passing to a suitable subsequence $(b_{n_k})_k$, one finds b' in B such that

$$b_{n_k} \underset{k}{\to} b'.$$

So, for all k, one has

$$\varepsilon < \left\|T^i(b_{n_k}) - T_{n_k}^i(b')\right\| \leq \left\|T^i(b_{n_k}) - T^i(b')\right\| + \left\|T^i(b') - T_{n_k}^i(b')\right\| \leq$$

$$\leq r\left\|b_{n_k} - b'\right\| + \left\|T^i(b') - T_{n_k}^i(b')\right\| \underset{k}{\to} 0,$$

which is again a contradiction.

b) Now, it follows that

$$F_n(B) \underset{n}{\to} F(B),$$

for all B in $\mathcal{K}(X)$.

Indeed, by using a classical property of H, we have

$$H(F_n(B), F(B)) = H\left(\cup_{i=1}^h T_n^i(B), \cup_{i=1}^h T^i(B)\right) \leq \max_{i\in\{1,2,\ldots,h\}} H(T_n^i(B), T^i(B)) \underset{n}{\to} 0.$$

c) Finally, we are able to prove that

$$A_n \underset{n}{\to} A.$$

Indeed

$$H(A_n, A) = H(F_n(A_n), F(A)) \leq H(F_n(A_n), F_n(A)) + H(F_n(A), F(A)) \leq$$

$$\leq rH(A_n, A) + H(F_n(A), F(A))$$

and this implies

$$(1-r)H(A_n, A) \leq H(F_n(A), F(A)) \underset{n}{\to} 0. \ \square$$

These A_n constitute the first approximation of A.

3.2. Second stage of approximation for A.

Taking into account the theorem from the first stage of approximation for A, assume that, for a given $\delta > 0$, we succeeded in finding n such that

$$H(A_n, A) < \frac{\delta}{2}.$$

Now it is possible to find a finite set $B_n \subset X$ such that

$$H(B_n, A_n) < \frac{\delta}{2},$$

hence

$$H(B_n, A) < \delta.$$

The finite set B_n is the second (and final) approximation of A.

4. REMARKS

4.1. In order to find B_n, one can use the algorithm described in [1]. Notice that the approximation procedure described in [1] is "effective" because it is based upon the idea of finding fixed points (or approximate fixed points) of the contractions $(T_n^i)_n$ which give the $(F_n)_n$. These fixed points are solutions of the Fredholm equations with degenerate kernel and are of "finite type" (see the identification above).

4.2. One can try to check the proposed procedure as follows: Take $a = 0$, $b = 1$, $h = 2$, $f^1 : [0,1] \to \mathbb{R}$, $f^2 : [0,1] \to \mathbb{R}$, $K^1 : [0,1] \times [0,1] \to \mathbb{R}$ and $K^2 : [0,1] \times [0,1] \to \mathbb{R}$, given by

$$f^1(x) = 1,$$

$$f^2(x) = x,$$

$$K^1(x,y) = e^{xy},$$

$$K^2(x,y) = \cos xy.$$

Here $C = e$.

One can consider, for all integers $n \geq 1$:

$$P_n^1(x,y) = 1 + \frac{xy}{1!} + \frac{(xy)^2}{2!} + ... + \frac{(xy)^{2n}}{(2n)!}$$

and

$$P_n^2(x,y) = 1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!} ... + (-1)^n \frac{(xy)^{2n}}{(2n)!}.$$

Indeed, for $i \in \{1,2\}$ and for all $n \in \mathbb{N}$ and $(x,y) \in [0,1] \times [0,1]$, one has

$$\left|K^i(x,y) - P_n^i(x,y)\right| \leq \sum_{k=2n+1}^{\infty} \frac{(xy)^k}{k!} \leq$$

$$\leq \sum_{k=2n+1}^{\infty} \frac{1}{k!} \leq \frac{1}{(2n+1)!}\left(1 + \frac{1}{2n+2} + \frac{1}{(2n+2)^2} + ...\right) =$$

$$= \frac{1}{(2n+1)!} \cdot \frac{1}{1 - \frac{1}{2n+2}} = \frac{1}{(2n+1)!} \cdot \frac{2n+2}{2n+1}.$$

This implies that, for $i \in \{1,2\}$,

$$P_n^i \underset{n}{\to} K^i,$$

and we can apply the general procedures described above to this concrete case.

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SOME RESULTS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. The purpose of this paper is to study the strong convergence of a general iterative scheme to find a common element of the set of common fixed points of a finite family of nonexpansive mappings, the set of solutions of variational inequality for a relaxed cocoercive mapping and the set of solutions of a equilibrium problems. Our results improve and extend recent results announced by [S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506-515], [H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal. 61 (2005) 341-350.] [S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 336 (2007) 445-469], [Y. Su, M. Shang, X. Qin, An iterative method of solution for equilibrium and optimization problems, Nonlinear Anal. (2007) doi:10.1016/j.na.2007.08.045] and many others.

Key Words and Phrases: Nonexpansive mapping; Viscosity approximation method; Equilibrium problem; Fixed point

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1. Introduction and Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let C be a nonempty closed convex subset of H and let $A : C \to H$ be a nonlinear map. Let P_C be the projection of H onto the convex subset C. The classical variational inequality which denoted by $VI(C, A)$ is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

For a given $z \in H, u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C,$$

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if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality (1.1) is equivalent to a fixed point problem.

The function $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$, where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recall that the following definitions.

(1) the mapping $B : C \to C$ is said to be relaxed (u, v)-cocoercive if there exist two constants $u, v > 0$ such that

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2, \quad \forall x, y \in C.$$

If $u = 0$, then B is said to be v-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings.

(2) A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(3) A mapping $f : C \to C$ is said to be contractive if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

(4) A operator $A : C \to C$ is said to be strong positive if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in C.$$

(5) A set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is said to be maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

Let B be a monotone mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$$

and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$ (see [17]).

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solution of (1.3) is denoted by $EP(F)$. Give a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (1.3). Some methods have been proposed to solve the equilibrium problem (see, for instance, [8, 11]). Combettes and Hirstoaga [8] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Recently, Takahashi et al. [20] also introduced a new iterative scheme:

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n \rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T y_n, \end{cases}$$

where f is a contraction on H, F is a bifunction, for approximating a common element of the set of fixed points of a non-self nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space. Very recently, Su et al. [18] improved the results of [20] and studied the following iterative algorithm

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n \rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T P_C(I - \lambda_n A) y_n, \end{cases}$$

where f is a contraction on H, F is a bifunction and A is inverse-strongly monotone operator of C into H. They proved the sequence $\{x_n\}$ defined by above iterative algorithm converges strongly to a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the equilibrium problems and the set of solutions of variational inequality problems.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see [9, 14, 23-25] and the

references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle, \tag{1.4}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [24], Xu proved that the sequence $\{x_n\}$ defined by the iterative method:

$$\begin{cases} x_0\in H,\\ x_{n+1}=(I-\alpha_nA)Sx_n+\alpha_nb, \quad \forall n\geq 0, \end{cases}$$

converges strongly to the unique solution of the minimization problem (1.4) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [14] introduced a new iterative scheme by the viscosity approximation method [15]:

$$\begin{cases} x_0\in H,\\ x_{n+1}=(I-\alpha_nA)Sx_n+\alpha_n\gamma f(x_n), \quad \forall n\geq 0. \end{cases}$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality:

$$\langle (A-\gamma f)x^*,x-x^*\rangle\geq 0, \quad \forall x\in C,$$

which is the optimality condition for the minimization problem:

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-h(x),$$

where C is the fixed point set of a nonexpansive mapping S, h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

For finding a common element of the set of fixed points of nonexpansive mappings and the set of solution of variational inequalities for α-cocoerceive mapping, Takahashi and Toyoda [20] introduced the following iterative process:

$$\begin{cases} x_0\in C,\\ x_{n+1}=\alpha_nx_n+(1-\alpha_n)SP_C(x_n-\lambda_nAx_n), \quad \forall n\geq 0, \end{cases} \tag{1.5}$$

where A is α-cocoerceive, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They showed that, if $F(S)\cap VI(C,A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.5) converges weakly to some

$z \in F(S) \cap VI(C, A)$. Recently, Iiduka and Takahashi [12] proposed another iterative scheme as following:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that the sequence $\{x_n\}$ converges strongly to some $z \in F(S) \cap VI(C, A)$. Chen et al.[7] studied the following iterative process:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \quad \forall n \geq 0, \end{cases}$$

where a function $f : C \to C$ is contractive, and also obtained a strong convergence theorem by viscosity approximation method.

In this paper, we will consider a finite family of nonexpansive mapping. Let $T_i : C \to C$, where $i = 1, 2, \cdots, N$, be a finite family of nonexpansive mappings. Let $F(T_i)$ denote the fixed point set of T_i, that is, $F(T_i) := \{x \in C : T_i x = x\}$. Finding an optimal point in the intersection $\cap_{i=1}^{N} F(T_i)$ of the fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [2, 5]). The problem of finding an optimal point that minimizes a given cost function over $\cap_{i=1}^{N} F(T_i)$ is of wide interdisciplinary interest and practical importance (see [3, 6, 10, 26]). A simple algorithmic solution to the problem of minimizing a quadratic function over $\cap_{i=1}^{N} F(T_i)$ is of extreme value in many applications including set theoretic signal estimation (see [13, 26]).

Now, we study the mapping W_n defined by

$$\begin{cases} U_{n0} = I, \\ U_{n1} = \lambda_{n1} T_1 U_{n0} + (1 - \lambda_{n1}) I, \\ U_{n2} = \lambda_{n2} T_2 U_{n1} + (1 - \lambda_{n2}) I, \\ \quad \cdots \\ U_{n,N-1} = \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\ W_n = U_{nN} = \lambda_{nN} T_N U_{n,N-1} + (1 - \lambda_{nN}) I, \end{cases} \tag{1.7}$$

where $\{\lambda_{n1}\}$, $\{\lambda_{n2}\}$, $\cdots$, $\{\lambda_{nN}\}$ are sequences in $(0, 1]$. Such a mapping W_n is called the W-mapping generated by $T_1, T_2, \cdots, T_N$ and

$\{\lambda_{n1}\}, \{\lambda_{n2}\}, \cdots, \{\lambda_{nN}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. Moreover, in Lemma 3.1 of [1], it is shown that $F(W_n) = \cap_{i=1}^{N} F(T_i)$.

In this paper, we introduce a more general iterative process as following:

$$\begin{cases} F(z_n, \eta) + \frac{1}{r_n}\langle \eta - z_n, z_n - x_n\rangle \geq 0, \quad \forall \eta \in C, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) W_n P_C (I - s_n B) z_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where W_n is defined by (1.7), F is a bifunction, A is a linear bounded operator and B is relaxed cocoercive. We prove that the sequence $\{x_n\}$ generated by the iterative scheme (1.8) converges strongly to a common element of the set of common fixed points of a finite nonexpansive mappings, the set of solutions of the variational inequalities for relaxed cocoercive mappings and the set of solutions of the equilibrium problem (1.3), which solves another variation inequality:

$$\langle \gamma f(q) - Aq, p - q\rangle \leq 0, \quad \forall p \in F,$$

where $F = \cap_{i=1}^{N} F(T_i) \cap VI(C, B) \cap EP(F)$ and is also the optimality condition for the minimization problem $\min_{x \in F} \frac{1}{2}\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). The results are obtained in this paper improve and extend the recent ones announced by Su et al. [18], Takahashi and Takahashi [20], Iiduka and Takahashi [12], Marino and Xu [14], Chen et al. [7], Combettes and Hirstoaga [8], Plubtieng and Punpaeng [22], Yao, et al. [27] and some others.

We now recall some well-known concepts and results.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. It is well known that, for all $x, y \in H$ and $\lambda \in [0, 1]$,

$$\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

A space X is said to satisfy Opial's condition [16] if, for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$,

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y),$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 1.1. (Xu [23], [24]) *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad \forall n \geq 1,$$

where γ_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n \to \infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

Lemma 1.2. In a real Hilbert space H,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 1.3. (Marino and Xu [14]) *Assume that B is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 1.4. (Blum and Oettli [4]) *Let C be a nonempty closed convex subset of a Hilbert space H and B be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)$~$(A4)$. Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.5. (Combettes and Hirstoaga [8]) *Assume that $F : C \times C \to \mathbf{R}$ satisfies $(A1)$~$(A4)$. For any $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : \ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Then the following hold:

(1) *T_r is single-valued.*
(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle.$$

(3) *$F(T_r) = EP(F)$.*
(4) *$EP(F)$ is closed and convex.*

Lemma 1.6. (Suzuki [19]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and β_n be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n, \quad \forall n \ge 0$$

and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2. Main Results

Now, we are ready to give our main results in this paper.

Theorem 2.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies $(A1)\sim(A4)$, $T_1, T_2, T_3, \cdots, T_N$ be a finite family of nonexpansive mappings of C into H and B be a μ-Lipschitzian relaxed (u,v)-cocoercive mapping of C into H such that $F = \cap_{i=1}^N F(T_i) \cap EP(F) \cap VI(C,B) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let f be a contractive mapping of H into itself with a coefficient α $(0 < \alpha < 1)$ and $\{x_n\}$, $\{y_n\}$ be sequences generated by (1.8), where $\alpha_n \subset [0,1]$ and $\{r_n\}, \{s_n\} \subset [0,\infty)$ satisfy the following:*

(i) $\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty$,
(ii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,
(iii) *there exists $c, d \in (0,1)$ such that $c < \alpha_n < d$ for all $n \ge 0$,*
(iv) $\{s_n\} \in [a,b]$ *for some a, b with $0 \le a \le b \le \frac{2(v-u\mu^2)}{\mu^2}$,*
(v) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all $i = 1, 2, \cdots, N$.*

Then $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))(q)$, which solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \le 0, \quad \forall p \in F.$$

Proof. We divide the proof into seven steps as follows:

Step (I) We prove that the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.

First, we show that $I - s_n B$ is nonexpansive. Indeed, from the relaxed (u,v)-cocoercive and μ-Lipschitzian definition on B and the condition

(iv), we have

$$\begin{aligned}
&\|(I - s_nB)x - (I - s_nB)y\|^2 \\
&= \|(x - y) - s_n(Bx - By)\|^2 \\
&= \|x - y\|^2 - 2s_n\langle x - y, Bx - By\rangle + s_n^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - 2s_n[-u\|Bx - By\|^2 + v\|x - y\|^2] + s_n^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 + 2s_n\mu^2 u\|x - y\|^2 - 2s_n v\|x - y\|^2 + \mu^2 s_n^2\|x - y\|^2 \\
&= (1 + 2s_n\mu^2 u - 2s_n v + \mu^2 s_n^2)\|x - y\|^2 \\
&\leq \|x - y\|^2,
\end{aligned}$$

which implies the mapping $I - s_nB$ is nonexpansive.

Now, we observe that $\{x_n\}$ is bounded. Indeed, pick $p \in F$. Since $z_n = T_{r_n}x_n$, we have

$$\|z_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|.$$

Putting $\rho_n = P_C(I - s_nB)z_n$, we have

$$\|\rho_n - p\| \leq \|(I - s_nB)z_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|.$$

Since $\beta_n \to 0$ by the condition (i), we may assume, without loss of generality, that $\beta_n < \|A\|^{-1}$ for all $n \geq 1$. From Lemma 1.3, we know that, if $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. Therefore, we obtain

$$\begin{aligned}
&\|y_n - p\| \\
&= \|\beta_n(\gamma f(z_n) - Ap) + (I - \beta_n A)(W_n\rho_n - p)\| \\
&\leq \beta_n\|\gamma f(z_n) - Ap\| + \|I - \beta_n A\|\|W_n\rho_n - p\| \\
&\leq \beta_n[\gamma\|f(z_n) - f(p)\| + \|\gamma f(p) - Ap\|] + (1 - \beta_n\bar{\gamma})\|\rho_n - p\| \\
&\leq [1 - (\bar{\gamma} - \gamma\beta_n)\beta_n]\|x_n - p\| + \beta_n\|\gamma f(p) - Ap\|,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n(x_n - p) + (1 - \alpha_n)(y_n - p)\| \\
&\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)\|y_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1 - \alpha_n)[1 - (\bar{\gamma} - \gamma\beta_n)\beta_n]\|x_n - p\| \\
&\quad + (1 - \alpha_n)\beta_n\|\gamma f(p) - Ap\|.
\end{aligned}$$

This in turn implies that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\}, \quad \forall n \geq 0. \tag{2.1}$$

Therefore, we obtain that $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$.

Step (II) We shall estimate the sequence $\{\|z_{n+1} - z_n\|\}$.

Let M denotes the possible different constant appearing in the following argument. Observing that $z_n = T_{r_n}x_n$ and $z_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(z_n, \eta) + \frac{1}{r_n}\langle \eta - z_n, z_n - x_n\rangle \geq 0, \quad \forall \eta \in C, \tag{2.2}$$

and

$$F(z_{n+1}, \eta) + \frac{1}{r_{n+1}}\langle \eta - z_{n+1}, z_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall \eta \in C. \tag{2.3}$$

Putting $\eta = z_{n+1}$ in (2.2) and $\eta = z_n$ in (2.3), we have

$$F(z_n, z_{n+1}) + \frac{1}{r_n}\langle z_{n+1} - z_n, z_n - x_n\rangle \geq 0$$

and

$$F(z_{n+1}, z_n) + \frac{1}{r_{n+1}}\langle z_n - z_{n+1}, z_{n+1} - x_{n+1}\rangle \geq 0.$$

Thus it follows from (A2) that

$$\langle z_{n+1} - z_n, \frac{z_n - x_n}{r_n} - \frac{z_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0,$$

that is,

$$\langle z_{n+1} - z_n, z_n - z_{n+1} + z_{n+1} - x_n - \frac{r_n}{r_{n+1}}(z_{n+1} - x_{n+1})\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number m such that $r_n > m > 0$ for all $n \geq 1$. It follows that

$$\|z_{n+1} - z_n\|^2 \leq \|z_{n+1} - z_n\|(\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|z_{n+1} - x_{n+1}\|),$$

which implies that

$$\begin{aligned} \|z_{n+1} - z_n\| \leq & \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|z_{n+1} - x_{n+1}\| \\ \leq & \|x_{n+1} - x_n\| + \frac{M}{m}|r_{n+1} - r_n|. \end{aligned} \tag{2.4}$$

Step (III) Next, we shall prove that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Note that

$$\begin{aligned} & \|\rho_{n+1} - \rho_n\| \\ = & \|P_C(I - s_{n+1}B)z_{n+1} - P_C(I - s_nB)z_n\| \\ \leq & \|(I - s_{n+1}B)z_{n+1} - (I - s_nB)z_n\| \\ = & \|(I - s_{n+1}B)z_{n+1} - (I - s_{n+1}B)z_n + (s_n - s_{n+1})Bz_n\| \\ \leq & \|z_{n+1} - z_n\| + |s_n - s_{n+1}|\|Bz_n\|. \end{aligned} \tag{2.5}$$

Substituting (2.4) into (2.5) yields that

$$\|\rho_{n+1}-\rho_n\| \leq \|x_{n+1}-x_n\| + M(|r_{n+1}-r_n| + |s_n - s_{n+1}|). \tag{2.6}$$

Observe that

$$\begin{aligned}
&\|y_n - y_{n+1}\| \\
&= \|(I-\beta_{n+1}A)(W_{n+1}\rho_{n+1} - W_n\rho_n) - (\beta_{n+1}-\beta_n)AW_n\rho_n \\
&\quad + \gamma[\beta_{n+1}(f(z_{n+1}) - f(z_n)) + f(z_n)(\beta_{n+1}-\beta_n)]\| \\
&\leq (1-\beta_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\| + \|W_{n+1}\rho_n - W_n\rho_n\|) \\
&\quad + |\beta_{n+1}-\beta_n|\|AW_n\rho_n\| + \gamma[\beta_{n+1}\alpha\|z_{n+1}-z_n\| \\
&\quad + |\beta_{n+1}-\beta_n|\|f(z_n)\|].
\end{aligned} \tag{2.7}$$

Next, we estimate $\|W_{n+1}\rho_n - W_n\rho_n\|$. It follows from the definition of W_n that

$$\begin{aligned}
&\|W_{n+1}\rho_n - W_n\rho_n\| \\
&= \|\lambda_{n+1,N}T_NU_{n+1,N-1}\rho_n + (1-\lambda_{n+1,N})\rho_n \\
&\quad - \lambda_{n,N}T_NU_{n,N-1}\rho_n - (1-\gamma_{n,N})\rho_n\| \\
&\leq |\lambda_{n+1,N}-\lambda_{n,N}|\|\rho_n\| + \|\lambda_{n+1,N}T_NU_{n+1,N-1}\rho_n - \lambda_{n,N}T_NU_{n,N-1}\rho_n\| \\
&\leq |\lambda_{n+1,N}-\lambda_{n,N}|\|\rho_n\| + \|\lambda_{n+1,N}(T_NU_{n+1,N-1}\rho_n - T_NU_{n,N-1}\rho_n)\| \\
&\quad + |\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}\rho_n\| \\
&\leq 2M|\lambda_{n+1,N}-\lambda_{n,N}| + \lambda_{n+1,N}\|U_{n+1,N-1}\rho_n - U_{n,N-1}\rho_n\|.
\end{aligned} \tag{2.8}$$

Next, we consider

$$\begin{aligned}
&\|U_{n+1,N-1}\rho_n - U_{n,N-1}\rho_n\| \\
&= \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}\rho_n + (1-\lambda_{n+1,N-1})\rho_n \\
&\quad - \lambda_{n,N-1}T_{N-1}U_{n,N-2}\rho_n - (1-\lambda_{n,N-1})\rho_n\| \\
&\leq |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|y_n\| \\
&\quad + \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}\rho_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}\rho_n\| \\
&\leq |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|\rho_n\| \\
&\quad + \lambda_{n+1,N-1}\|T_{N-1}U_{n+1,N-2}\rho_n - T_{N-1}U_{n,N-2}\rho_n\| \\
&\quad + |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\|T_{N-1}U_{n,N-2}\rho_n\| \\
&\leq 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}| + \|U_{n+1,N-2}\rho_n - U_{n,N-2}\rho_n\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|U_{n+1,N-1}\rho_n - U_{n,N-1}\rho_n\| \\
&\le 2M|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + 2M|\lambda_{n+1,N-2} - \lambda_{n,N-2}| \\
&\quad + \|U_{n+1,N-3}\rho_n - U_{n,N-3}\rho_n\| \\
&\le 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| + \|U_{n+1,1}\rho_n - U_{n,1}\rho_n\| \\
&= 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\quad + \|\gamma_{n+1,1}T_1\rho_n + (1-\lambda_{n+1,1})\rho_n - \lambda_{n,1}T_1\rho_n - (1-\lambda_{n,1})\rho_n\| \\
&\le 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8) yields that

$$\begin{aligned}
&\|W_{n+1}\rho_n - W_n\rho_n\| \\
&\le 2M|\lambda_{n+1,N} - \lambda_{n,N}| + 2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\le 2M\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned} \tag{2.10}$$

Substituting (2.6) and (2.10) into (2.7) yields that

$$\begin{aligned}
&\|y_n - y_{n+1}\| \\
&\le [1 - \beta_{n+1}(\bar{\gamma} - \alpha\gamma)]\|x_{n+1} - x_n\| \\
&\quad + M(\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}| + |r_{n+1} - r_n| + |s_n - s_{n+1}| + |\beta_n - \beta_{n+1}|).
\end{aligned} \tag{2.11}$$

It follows from the conditions (i), (ii) and (v) that

$$\lim_{n\to\infty}\{\|y_n - y_{n+1}\| - \|x_{n+1} - x_n\|\} \le 0. \tag{2.12}$$

By virtue of Lemma 1.6, we obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.13}$$

On the other hand, from (1.8), we have

$$\|x_{n+1} - x_n\| = (1-\alpha_n)\|x_n - y_n\|, \tag{2.14}$$

which, combining with (2.13) and the condition (iii), yields that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.15}$$

Step (IV) We show that $\lim_{n\to\infty} \|z_n - W_n z_n\| = 0$.

Observing that $y_n = \beta_n \gamma f(W_n z_n) + (I - \beta_n A) W_n \rho_n$, we have

$$\|y_n - W_n \rho_n\| = \beta_n \|\gamma f(z_n) - A W_n \rho_n\|,$$

which, combining with the condition (i), gives

$$\lim_{n\to\infty} \|y_n - W_n \rho_n\| = 0. \tag{2.16}$$

For $p \in F$, we have

$$\begin{aligned}
&\|z_n - p\|^2 \\
&= \|T_{r_n} x_n - T_{r_n} p\|^2 \le \langle T_{r_n} x_n - T_{r_n} p, x_n - p\rangle = \langle z_n - p, x_n - p\rangle \\
&= \frac{1}{2}(\|z_n - p\|^2 + \|x_n - p\|^2 - \|x_n - z_n\|^2)
\end{aligned}$$

and hence

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2.$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n (x_n - p) + (1 - \alpha_n)(y_n - p)\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|\beta_n(\gamma f(z_n) - Ap) \\
&\quad + (I - \beta_n A)(W_n \rho_n - p)\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)(\beta_n \|\gamma f(z_n) - Ap\| + (1 - \beta_n \bar{\gamma})\|\rho_n - p\|)^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\beta_n \|\gamma f(z_n) - Ap\|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n \bar{\gamma})\|\rho_n - p\|^2 \\
&\quad + 2(1 - \alpha_n)\beta_n \|\gamma f(z_n) - Ap\| \|\rho_n - p\| \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\beta_n \|\gamma f(z_n) - Ap\|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n \bar{\gamma})\|z_n - p\|^2 \\
&\quad + 2(1 - \alpha_n)\beta_n \|\gamma f(z_n) - Ap\| \|\rho_n - p\| \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\beta_n \|\gamma f(z_n) - Ap\|^2 \\
&\quad + (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)(1 - \beta_n \bar{\gamma})\|x_n - z_n\|^2 \\
&\quad + 2(1 - \alpha_n)\beta_n \|\gamma f(z_n) - Ap\| \|\rho_n - p\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - z_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\| \\
&\le (\|x_n - p\| - \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|
\end{aligned}$$

It follows from the conditions (i), (iii) and (2.15) that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{2.17}$$

For $p \in F$, we have

$$\begin{aligned}
&\|\rho_n - p\|^2 \\
&= \|P_C(I - s_nA)z_n - P_C(I - s_nA)p\|^2 \\
&\le \|(z_n - p) - s_n(Az_n - Ap)\|^2 \\
&= \|z_n - p\|^2 - 2s_n\langle z_n - p, Az_n - Ap\rangle + s_n^2\|Az_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - 2s_n[-u\|Az_n - Ap\|^2 + v\|z_n - p\|^2] + s_n^2\|Az_n - Ap\|^2 \\
&\le \|x_n - p\|^2 + 2s_nu\|Az_n - Ap\|^2 - 2s_nv\|z_n - p\|^2 + s_n^2\|Az_n - Ap\|^2 \\
&\le \|x_n - p\|^2 + (2s_nu + s_n^2 - \frac{2s_nv}{\mu^2})\|Az_n - Ap\|^2.
\end{aligned} \tag{2.18}$$

Observe that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n(x_n - p) + (1-\alpha_n)(y_n - p)\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\beta_n(\gamma f(z_n) - Ap) \\
&\quad + (I - \beta_nA)(W_n\rho_n - p)\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\beta_n\|\gamma f(z_n) - Ap\| \\
&\quad + \|I - \beta_nA\|\|W_n\rho_n - p\|)^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\beta_n\|\gamma f(z_n) - Ap\| + (1-\beta_n\bar{\gamma})\|\rho_n - p\|)^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 \\
&\quad + (1-\alpha_n)\|\rho_n - p\|^2 + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|.
\end{aligned} \tag{2.19}$$

Substituting (2.18) into (2.19), we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+\|x_n-p\|^2\\
&\quad+(2s_nu+s_n^2-\frac{2s_nv}{\mu^2})\|Az_n-Ap\|^2\\
&\quad+2(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|.
\end{aligned}$$

It follows from the condition (iv) that

$$\begin{aligned}
&(\frac{2av}{\mu^2}-2bu-b^2)\|Az_n-Ap\|^2\\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad+2(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|\\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\quad+2(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|
\end{aligned}$$

Since the condition (i) and (2.15), we have that

$$\lim_{n\to\infty}\|Az_n-Ap\|=0. \tag{2.20}$$

On the other hand, we have

$$\begin{aligned}
&\|\rho_n-p\|^2\\
&=\|P_C(I-s_nA)z_n-P_C(I-s_nA)p\|^2\\
&\le\langle(I-s_nA)z_n-(I-s_nA)p,\rho_n-p\rangle\\
&=\frac{1}{2}[\|(I-s_nA)z_n-(I-s_nA)p\|^2+\|\rho_n-p\|^2\\
&\quad-\|(I-s_nA)z_n-(I-s_nA)p-(\rho_n-p)\|^2]\\
&\le\frac{1}{2}[\|z_n-p\|^2+\|\rho_n-p\|^2-\|(z_n-\rho_n)-s_n(Az_n-Ap)\|^2]\\
&=\frac{1}{2}[\|x_n-p\|^2+\|\rho_n-p\|^2-\|z_n-\rho_n\|^2-s_n^2\|Az_n-Ap\|^2\\
&\quad+2s_n\langle z_n-\rho_n,Az_n-Ap\rangle],
\end{aligned}$$

which yields that

$$\begin{aligned}
&\|\rho_n-p\|^2\\
&\le\|x_n-p\|^2-\|z_n-\rho_n\|^2+2s_n\|z_n-\rho_n\|\|Az_n-Ap\|.
\end{aligned} \tag{2.21}$$

Substituting (2.21) into (2.19) yields that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+\|x_n-p\|^2\\
&\quad+2s_n(1-\alpha_n)\|z_n-p\|\|Az_n-Ap\|\\
&\quad+2(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|-(1-\alpha_n)\|z_n-\rho_n\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(1-\alpha_n)\|z_n-\rho_n\|^2\\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad+2s_n(1-\alpha_n)\|z_n-p\|\|Az_n-Ap\|\\
&\quad+2(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|\\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\quad+2s_n(1-\alpha_n)\|z_n-p\|\|Az_n-Ap\|\\
&\quad+2(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|
\end{aligned}$$

From the conditions (i) (ii), (2.15) and (2.20), we have

$$\lim_{n\to\infty}\|z_n-\rho_n\|=0. \tag{2.22}$$

Observe that

$$\begin{aligned}
&\|z_n-W_nz_n\|\\
&\le \|W_nz_n-W_n\rho_n\|+\|W_n\rho_n-y_n\|+\|y_n-x_n\|+\|x_n-z_n\|\\
&\le \|z_n-\rho_n\|+\|W_n\rho_n-y_n\|+\|y_n-x_n\|+\|x_n-z_n\|.
\end{aligned}$$

From (2.13), (2.16), (2.17) and (2.22), we have

$$\lim_{n\to\infty}\|z_n-W_nz_n\|=0. \tag{2.23}$$

Step (V) We prove that the mapping $P_F(\gamma f+(I-A))$ has a unique fixed point.

Since $\beta_n\to 0$ by the condition (i), we may assume, without loss of generality, that $\beta_n<\|A\|^{-1}$ for all $n\ge 1$. From Lemma 1.3, we know that, if $0<\rho\le\|A\|^{-1}$, then $\|I-\rho A\|\le 1-\rho\bar{\gamma}$. In this part, we will assume that $\|I-A\|\le 1-\bar{\gamma}$. Observe that $P_F(\gamma f+(I-A))$ is a contractive mapping.

Indeed, for all $x, y \in H$, we have

$$\begin{aligned}
&\|P_F(\gamma f + (I - A))(x) - P_F(\gamma f + (I - A))(y)\| \\
&\leq \|(\gamma f + (I - A))(x) - (\gamma f + (I - A))(y)\| \\
&\leq \gamma\|f(x) - f(y)\| + \|I - A\|\|x - y\| \\
&\leq \gamma\alpha\|x - y\| + (1 - \bar{\gamma})\|x - y\| \\
&< \|x - y\|.
\end{aligned}$$

Thus Banach's Contraction Principle guarantees that $P_F(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$, that is,

$$q = P_F(\gamma f + (I - A))(q).$$

Step (VI) Next, we show that $\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle \leq 0$.

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Aq, x_{n_i} - q\rangle.$$

Correspondingly, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$. Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $z_{n_i} \rightharpoonup w$.

Next, we show $w \in F$. First, we prove $w \in EP(F)$. Since $z_n = T_{r_n}x_n$, we have

$$F(z_n, \eta) + \frac{1}{r_n}\langle \eta - z_n, z_n - x_n\rangle \geq 0, \quad \forall \eta \in C.$$

It follows from the condition (A2) that

$$\langle \eta - z_n, \frac{z_n - x_n}{r_n}\rangle \geq F(\eta, z_n),$$

$$\langle \eta - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(\eta, z_{n_i}).$$

Since $\frac{z_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, $z_{n_i} \rightharpoonup w$ and (A4), we have $F(\eta, w) \leq 0$ for all $\eta \in C$. For t with $0 < t \leq 1$ and $\eta \in C$, let $\eta_t = t\eta + (1 - t)w$. Since $\eta \in C$ and $w \in C$, we have $\eta_t \in C$ and hence $F(\eta_t, w) \leq 0$. So, from (A1) and (A4), we have

$$0 = F(\eta_t, \eta_t) \leq tF(\eta_t, \eta) + (1 - t)F(\eta_t, w) \leq tF(\eta_t, \eta).$$

That is, $F(\eta_t, \eta) \geq 0$. It follows from (A3) that $F(w, \eta) \geq 0$ for all $\eta \in C$ and hence $w \in EP(F)$. Since every Hilbert space has Opial's

condition, it follows from (2.23) that

$$\begin{aligned}\liminf_{i\to\infty}\|z_{n_i}-w\| &< \liminf_{i\to\infty}\|z_{n_i}-W_nw\|\\ &= \liminf_{i\to\infty}\|z_{n_i}-W_nz_{n_i}+W_nz_{n_i}-W_nw\|\\ &\le \liminf_{i\to\infty}\|W_nz_{n_i}-W_nw\|\\ &\le \liminf_{i\to\infty}\|z_{n_i}-w\|,\end{aligned}$$

which derives a contradiction. Thus we have $w \in F(W_n)$. It follows from $F(W_n) = \cap_{i=1}^N F(T_i)$ that $w \in \cap_{i=1}^N F(T_i)$.

Next, let us first show that $w \in VI(C, A)$. Put

$$Tw_1 = \begin{cases} Bw_1 + N_Cw_1, & w_1 \in C,\\ \emptyset, & w_1 \notin C.\end{cases}$$

Since B is relaxed (u, v)-cocoercve, from the condition (iv), we have

$$\begin{aligned}\langle Bx - By, x - y\rangle &\ge (-u)\|Bx - By\|^2 + v\|x - y\|^2\\ &\ge (v - u\mu^2)\|x - y\|^2 \ge 0,\end{aligned}$$

which yields that B is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_Cw_1$ and $\rho_n \in C$, we have

$$\langle w_1 - \rho_n, w_2 - Bw_1\rangle \ge 0.$$

On the other hand, from $\rho_n = P_C(I - s_nB)z_n$, we have

$$\langle w_1 - \rho_n, \rho_n - (I - s_nB)z_n\rangle \ge 0$$

and hence

$$\langle w_1 - \rho_n, \frac{\rho_n - z_n}{s_n} + Bz_n\rangle \ge 0.$$

Thus it follows that

$$\begin{aligned}\langle w_1 - \rho_{n_i}, w_2\rangle &\ge \langle w_1 - \rho_{n_i}, Bw_1\rangle \ge \langle w_1 - \rho_{n_i}, Bw_1\rangle\\ &\quad - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}} + Bz_{n_i}\rangle\\ &= \langle w_1 - \rho_{n_i}, Bw_1 - \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}} - Bz_{n_i}\rangle\\ &= \langle w_1 - \rho_{n_i}, Bw_1 - B\rho_{n_i}\rangle + \langle w_1 - \rho_{n_i}, B\rho_{n_i} - Bz_{n_i}\rangle\\ &\quad - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}}\rangle\\ &\ge \langle w_1 - \rho_{n_i}, B\rho_{n_i} - Bz_{n_i}\rangle - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}}\rangle,\end{aligned}$$

which implies that $\langle w_1 - w, w_2\rangle \geq 0$. We have $w \in T^{-1}0$ and hence $w \in VI(C, A)$, that is, $w \in F$. Since $q = P_F(\gamma f + (I - A))(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle &= \lim_{n\to\infty}\langle \gamma f(q) - Aq, x_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Aq, w - q\rangle \leq 0.\end{aligned} \tag{2.24}$$

Step (VII) Finally, we prove $x_n \to q$ strongly.

It follows from Lemma 1.2 that

$$\begin{aligned}&\|y_n - q\|^2 \\ &= \|(I - \beta_n A)(W_n\rho_n - q) + \beta_n(\gamma f(z_n) - Aq)\|^2 \\ &\leq \|(I - \beta_n A)(W_n\rho_n - q)\|^2 + 2\beta_n\langle \gamma f(z_n) - Aq, y_n - q\rangle \\ &\leq (1 - \beta_n\bar{\gamma})^2\|\rho_n - q\|^2 + 2\beta_n\langle \gamma f(z_n) - Aq, y_n - q\rangle \\ &\leq (1 - \beta_n\bar{\gamma})^2\|z_n - q\|^2 + 2\beta_n\gamma\langle f(z_n) - f(q), y_n - q\rangle \\ &\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle \\ &\leq (1 - \beta_n\bar{\gamma})^2\|x_n - q\|^2 + 2\beta_n\gamma\alpha\|z_n - q\|\|y_n - q\| \\ &\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle \\ &\leq (1 - \beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\alpha(\|z_n - q\|^2 + \|y_n - q\|^2) \\ &\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle \\ &\leq (1 - \beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\alpha(\|x_n - q\|^2 + \|y_n - q\|^2) \\ &\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}&\|y_n - q\|^2 \\ &\leq \frac{(1 - \beta_n\bar{\gamma})^2 + \beta_n\gamma\alpha}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\beta_n}{1 - \beta_n\gamma\alpha}\langle \gamma f(q) - Aq, y_n - q\rangle \\ &= \frac{(1 - 2\beta_n\bar{\gamma} + \beta_n\alpha\gamma)}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{\beta_n^2\bar{\gamma}^2}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 \\ &\quad + \frac{2\beta_n}{1 - \beta_n\gamma\alpha}\langle \gamma f(q) - Aq, y_n - q\rangle \\ &\leq [1 - \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\alpha}]\|x_n - q\|^2 \\ &\quad + \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\alpha}[\frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Aq, y_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M].\end{aligned} \tag{2.25}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2.\end{aligned} \tag{2.26}$$

Substituting (2.25) into (2.26) yields that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\le [1-(1-\alpha_n)\frac{2\beta_n(\bar\gamma-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n-q\|^2\\ &+(1-\alpha_n)\frac{2\beta_n(\bar\gamma-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar\gamma-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle\\ &\quad+\frac{\beta_n\bar\gamma^2}{2(\bar\gamma-\alpha\gamma)}M].\end{aligned} \tag{2.27}$$

In (2.27), put $l_n=(1-\alpha_n)\frac{2\beta_n(\bar\gamma-\alpha_n\gamma)}{1-\beta_n\alpha\gamma}$ and

$$t_n=\frac{1}{\bar\gamma-\alpha\gamma}\langle\gamma f(q)-Aq,x_{n+1}-q\rangle+\frac{\beta_n\bar\gamma^2}{2(\bar\gamma-\alpha\gamma)}M.$$

Then we have

$$\|x_{n+1}-q\|^2\le(1-l_n)\|x_n-q\|^2+l_nt_n. \tag{2.28}$$

It follows from the condition (i) and (2.24) that

$$\lim_{n\to\infty}l_n=0,\sum_{n=1}^{\infty}l_n=\infty \text{ and } \limsup_{n\to\infty}t_n\le 0.$$

If we apply Lemma 2.1 to (2.28), it follows that $x_n\to q$ strongly.

3. Applications

By using Theorem 2.1, we have the following:

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a nonexpansive mappings of C into H and B be a μ-Lipschitzian relaxed (u,v)-cocoercive mapping of C into H such that $F=F(T)\cap EP(F)\cap VI(C,B)\neq\emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\bar\gamma>0$. Assume that $0<\gamma<\frac{\bar\gamma}{\alpha}$. Let f be a contractive mapping of H into itself with coefficient α $(0<\alpha<1)$ and $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases}x_1\in H,\\ y_n=\beta_n\gamma f(z_n)+(I-\beta_nA)TP_C(I-s_nB)z_n,\\ x_{n+1}=\alpha_nx_n+(1-\alpha_n)y_n,\end{cases}$$

where $z_n = P_C x_n$ for all $n \geq 0$ and P_C is the metric projection form H onto C, $\alpha_n \subset [0,1]$ and $\{r_n\}, \{s_n\} \subset [0,\infty)$ satisfy the following:

(i) $\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty$,

(ii) $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,

(iii) *there exists $c, d \in (0,1)$ such that $c < \alpha_n < d$ for all $n \geq 0$,*

(iv) *$\{s_n\} \in [a,b]$ for some a, b with $0 \leq a \leq b \leq \frac{2(v-u\mu^2)}{\mu^2}$.*

Then $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - A))(q)$, which solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F.$$

Proof. Taking $F(x,y) = 0$ for all $x, y \in C$ and $\{r_n\} = 1$ for all $n \geq 1$ in Theorem 2.1, we have

$$\langle \eta - z_n, z_n - x_n \rangle \geq 0, \quad \forall \eta \in C,$$

which implies that $z_n = P_C x_n$. Putting $N = 1$ and $\lambda_{n1} = 1$ in (1.8), we can obtain the desired conclusion from Theorem 2.1 immediately.

Furthermore, If taking $\{s_n\} = 0$, $\gamma = 1$ and $A = I$ in Theorem 3.1, we can also obtain the following theorem easily:

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a nonexpansive mappings of C into H such that $F = F(T) \cap EP(F) \neq \emptyset$. Let f be a contractive mapping of H into itself with coefficient α $(0 < \alpha < 1)$ and $\{x_n\}$, $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 \in H, \\ y_n = \beta_n f(z_n) + (I - \beta_n A) T z_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \end{cases}$$

where $z_n = P_C x_n$ for all $n \geq 0$ and P_C is the metric projection form H onto C, $\alpha_n \subset [0,1]$ and $\{r_n\}, \{s_n\} \subset [0,\infty)$ satisfy the following:

(i) $\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty$,

(ii) *there exists $c, d \in (0,1)$ such that $c < \alpha_n < d$ for all $n \geq 0$.*

Then $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F f(q)$, which solves the following variational inequality:

$$\langle f(q) - q, p - q \rangle \leq 0, \quad \forall p \in F.$$

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Weighted composition operators between weighted Bergman spaces and weighted Bloch type spaces

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Abstract

Let $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C}$ be analytic maps. They induce a weighted composition operator $C_{\phi,\psi}$ acting between weighted Bergman spaces and weighted Bloch type spaces. Under some assumptions on the weights we give a necessary as well as a sufficient condition when such an operator is continuous resp. compact.

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I. Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ and ψ be an analytic map defined on $\mathbb{D}$. These maps induce a weighted composition operator $C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \to \psi(f \circ \phi)$, where $H(\mathbb{D})$ denotes the set of all holomorphic functions on $\mathbb{D}$. Furthermore let v and w be strictly positive continuous and bounded functions (*weights*) on $\mathbb{D}$. We consider weighted Bloch type spaces B_v of functions $f \in H(\mathbb{D})$ satisfying $\|f\|_{B_v} := \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty$. Provided we identify functions that differ by a constant, $\|.\|_{B_v}$ becomes a norm and B_v a Banach space. Moreover let $A_{w,p}$ denote the weighted Bergman space

$$A_{w,p} := \{f \in H(\mathbb{D}); \|f\|_{w,p} = \left(\int_{\mathbb{D}} |f(z)|^p w(z)\, dA(z)\right)^{\frac{1}{p}} < \infty\},$$

where A is the area measure, normalized so that area of $\mathbb{D}$ is one. In case $w \equiv 1$ we write A_p. Composition operators and weighted composition operators acting between various spaces of analytic functions have been investigated by several authors, see e.g. [12], [8], [10], [2], [4], [3], [9], [11]. In this article we want to give necessary and sufficient conditions for a weighted composition operator acting between weighted Bergman spaces and weighted Bloch type spaces to be continuous resp. compact. These conditions are given in terms of the weights as well as the involved analytic functions ϕ and ψ.

II. Preliminaries

For notation on composition operators we refer the reader to the monographs [5] and [13]. We want to consider some special weights, the so-called *standard weights*, $v_p(z) = (1 - |z|^2)^p$, $p > 0$.

We need some auxiliary results.

Lemma 1 (Hedenmalm-Korenblum-Zhu, Lemma 3.2) *Let $f \in A_{v_\alpha,p}$, then for every $z \in \mathbb{D}$ we have*

$$|f(z)| \leq \frac{\|f\|_{v_\alpha,p}}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}.$$

Lemma 2 (Wolf, [14], Lemma 1) *Let w be a weight of the form $w = |u|$, where u is a holomorphic function without any zeros on $\mathbb{D}$. Then*

$$|f(z)| \leq \frac{1}{(1 - |z|^2)^{\frac{2}{p}} w^{\frac{1}{p}}(z)} \|f\|_{w,p}$$

for all $z \in \mathbb{D}$, $f \in A_{w,p}$.

For the study of the compactness of the operator $C_{\phi,\psi}$ we need the following result.

Proposition 3 (Cowen-MacCluer, [5] Proposition 3.11) *Let X and Y be $A_{w,p}$ or B_v. Then $C_{\phi,\psi} : X \to Y$ is compact if and only if for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $C_{\phi,\psi} f_n \to 0$ in Y.*

III. Main Result

Proposition 4 *Let w be a weight of the form $w = |u|$, where u is a holomorphic function without any zeros on $\mathbb{D}$. If*

(a) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|}{w^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{2}{p}}} < \infty$,

(b) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)||\phi'(z)||u'(\phi(z))|}{(1-|\phi(z)|^2)^{\frac{2}{p}} w(\phi(z))} < \infty$,

(c) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)||\phi'(z)|}{w^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{(2+p)}{p}}} < \infty$,

then the weighted composition operator $C_{\phi,\psi} : A_{w,p} \to B_v$ is continuous.

Proof. We have $f \in A_{w,p}$ if and only if $fu^{\frac{1}{p}} \in A_p$. By a theorem of Hardy-Littlewood and Flett (see [6]) this yields that $(fu^{\frac{1}{p}})' \in A_{v_p}$.
Next, we fix $f \in A_{w,p}$ and obtain using Lemma 1

$$|f'(z)u^{\frac{1}{p}}(z) - \frac{1}{p}f(z)u^{\frac{1}{p}-1}(z)u'(z)| \le c_p \frac{\|fu^{\frac{1}{p}}\|_p}{(1-|z|^2)^{\frac{2+p}{p}}} = c_p \frac{\|f\|_{w,p}}{(1-|z|^2)^{\frac{2+p}{p}}}$$

for every $z \in \mathbb{D}$. Thus we get applying Lemma 2

$$\begin{aligned} |f'(z)| &\le c_p \frac{\|f\|_{w,p}}{w^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2+p}{p}}} + \frac{1}{p}|f(z)|w^{\frac{1}{p}-1}(z)|u'(z)| \\ &\le c_p \frac{\|f\|_{w,p}}{w^{\frac{1}{p}}(z)(1-|z|^2)^{\frac{2+p}{p}}} + \frac{1}{p}\frac{|u'(z)|\|f\|_{w,p}}{(1-|z|^2)^{\frac{2}{p}}w(z)} \end{aligned} \tag{0.1}$$

for every $z \in \mathbb{D}$. Now, for every $z \in \mathbb{D}$ we get using Lemma 2 once again

$$\begin{aligned} \|C_{\phi,\psi}f\|_{B_v} &= \sup_{z\in\mathbb{D}} v(z)|(C_{\phi,\psi}f)'(z)| \\ &\le \sup_{z\in\mathbb{D}} v(z)|\psi'(z)||f(\phi(z))| + \sup_{z\in\mathbb{D}} v(z)|\psi(z)||f'(\phi(z))||\phi'(z)| \\ &\le \sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|\|f\|_{w,p}}{(1-|\phi(z)|^2)^{\frac{2}{p}}w^{\frac{1}{p}}(\phi(z))} + \sup_{z\in\mathbb{D}} \frac{c_p v(z)|\psi(z)||\phi'(z)|\|f\|_{w,p}}{w^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{2+p}{p}}} \\ &+ \frac{1}{p}\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)||\phi'(z)||u'(\phi(z)))|\|f\|_{w,p}}{(1-|\phi(z)|^2)^{\frac{2}{p}}w(\phi(z))}. \end{aligned}$$

Thus, the claim follows. □

Proposition 5 *Let w be a weight of the form $w = |u|$ where u is a holomorphic function without any zeros on $\mathbb{D}$. If the weighted composition operator $C_{\phi,\psi} : A_{w^{2p},p} \to B_v$ is continuous, then*

(a) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi'(z)|}{w(\phi(z))} < \infty$.

(b) $\sup_{z\in\mathbb{D}} \frac{v(z)|\psi(z)||\phi'(z)||u'(\phi(z))|}{w(\phi(z))} < \infty$.

Proof. Fix $a \in \mathbb{D}$ and put $f_a(z) = \frac{2}{u(z)} - \frac{u(\phi(a))}{u^2(z)}$. Then obviously $f_a \in A_{w^{2p},p}$ since

$$\|f_a\|^p_{w^{2p},p} = \int_{\mathbb{D}} |f_a(z)|^p w^{2p}(z)\, dA(z) \quad = \quad \int_{\mathbb{D}} |2u(z) - u(\phi(a))|^p\, dA(z) \leq 3M^p,$$

where $M = \sup_{z\in\mathbb{D}} w(z)$. Moreover we have

$$f_a'(z) = -2\frac{u'(z)}{u^2(z)} + 2\frac{u(\phi(a))u'(z)}{u^3(z)}$$

and thus $f_a(\phi(a)) = \frac{1}{u(\phi(a))}$ and $f_a'(\phi(a)) = 0$. Finally

$$v(a)|(C_{\phi,\psi}f_a)'(a)| = v(a)\frac{|\psi'(a)|}{w(\phi(a))} \leq \|C_{\phi,\psi}\|\|f_a\|_{w^{2p},p} \leq 3^{\frac{1}{p}} M\|C_{\phi,\psi}\|.$$

Thus, we have proved that condition (a) holds.
Next, we consider $g_a(z) = \frac{u(\phi(a))}{u^2(z)} - \frac{1}{u(z)}$. Then, $g_a \in A_{w^{2p},p}$, since

$$\|g_a\|^p_{w,p} = \int_{\mathbb{D}} |g_a(z)|^p w^{2p}(z)\, dA(z) = \int_{\mathbb{D}} |u(\phi(a)) - u(z)|^p\, dA(z) \leq 2M^p.$$

Moreover we have $g_a'(z) = -2\frac{u(\phi(a))u'(z)}{u^3(z)} + \frac{u'(z)}{u^2(z)}$. Then $g_a(\phi(a)) = 0$ and $g_a'(\phi(a)) = -\frac{u'(\phi(a))}{u^2(\phi(a))}$ and

$$\frac{|\psi(a)||\phi'(a)||u'(\phi(a))|v(a)}{w^2(\phi(a)} = v(a)|(C_{\phi,\psi}f)'(a)| \leq 2^{\frac{1}{p}} M\|C_{\phi,\psi}\|,$$

and condition (b) follows. □

Proposition 6 *Let w be a weight of the form $w = |u|$ where u is a holomorphic function without any zeros on $\mathbb{D}$. If*

(a) $\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{v(z)|\psi'(z)|}{w^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{2}{p}}} = 0,$

(b) $\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{v(z)|\psi(z)||\phi'(z)||u'(\phi(z))|}{(1-|\phi(z)|^2)^{\frac{1}{p}} w(\phi(z))} = 0,$

(c) $\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{v(z)|\psi(z)||\phi'(z)|}{w^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{2+p}{p}}} = 0,$

(d) $\sup_{z\in\mathbb{D}} \frac{v(z)}{w(\phi(z))}|\psi(z)||\phi'(z)| < \infty,$

then the weighted composition operator $C_{\phi,\psi} : A_{w,p} \to B_v$ is compact.

Proof. Let $(f_n)_n$ be a sequence in $A_{w,p}$ with $\|f_n\|_{w,p} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. By assumption, for any $\varepsilon > 0$, there is a constant δ, $0 < \delta < 1$ such that $\delta < |\phi(z)| < 1$ implies

$$\begin{aligned}
\frac{v(z)|\psi(z)||\phi'(z)||u'(\phi(z))|}{w(\phi(z))(1-|\phi(z)|^2)^{\frac{1}{p}}} &< \frac{\varepsilon}{3} \text{ and} \\
\frac{v(z)|\psi(z)||\phi'(z)|}{w^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{2+p}{p}}} &< \frac{\varepsilon}{3} \text{ and} \\
\frac{v(z)|\psi'(z)|}{w^{\frac{1}{p}}(\phi(z))(1-|\phi(z)|^2)^{\frac{2}{p}}} &< \frac{\varepsilon}{3}.
\end{aligned}$$

Let K denote the compact set $K = \{w \in \mathbb{D};\ |w| \leq \delta\}$. Then we obtain

$$\begin{aligned}
\|C_{\phi,\psi} f_n\|_{B_v} &= \sup_{z\in\mathbb{D}} v(z)|(C_{\phi,\psi} f_n)'(z)| \\
&\leq \sup_{z\in\mathbb{D}} v(z)|\psi'(z)||f_n(\phi(z))| + \sup_{z\in D} v(z)|\psi(z)||f_n'(\phi(z))||\phi'(z)| \\
&\leq \sup_{\{z\in\mathbb{D};\ \phi(z)\in K\}} v(z)|\psi'(z)||f_n(\phi(z))| + \sup_{\{z\in\mathbb{D};\ \phi(z)\in K\}} v(z)|\psi(z)||f_n'(\phi(z))||\phi'(z)| \\
&+ \varepsilon \\
&\leq \|\psi\|_{B_v} \sup_{s\in K} |f_n(s)| + M \sup_{s\in K} w(s)|f_n'(s)|,
\end{aligned}$$

where $M = \sup_{z\in\mathbb{D}} \frac{v(z)}{w(\phi(z)}|\psi(z)||\phi'(z)|$. By condition (a) $\psi \in B_v$. Hence $\|C_{\phi,\psi} f_n\|_{B_v} \to 0$ if $n \to \infty$. Thus, $C_{\phi,\psi} : A_{w,p} \to B_v$ is a compact operator. □

Proposition 7 *Let w be a weight of the form $w = |u|$ where u is a holomorphic function without any zeros on $\mathbb{D}$. If the weighted composition operator $C_{\phi,\psi} : A_{w^{2p},p} \to B_v$ is compact, then the following conditions hold:*

(a) $\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{|\psi'(z)|v(z)}{w^{\frac{1}{2}}(\phi(z))} = 0.$

(b) $\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{|\psi(z)||\phi'(z)||u'(z)|v(z)}{w^{\frac{3}{2}}(\phi(z))} = 0.$

Proof. Let $(z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ be a sequence such that $|\phi(z_n)| \to 1$ if $n \to \infty$. Next, consider

$$f_n(z) = u^{\frac{1}{2}}(\phi(z_n)) \left(\frac{2}{u(z)} - \frac{u(\phi(z_n))}{u^2(z)} \right), z \in \mathbb{D}.$$

Then $f_n \in A_{w^{2p},p}$, $\|f_n\|_{w,p} \leq 3^{\frac{1}{p}} M^2$ for every $n \in \mathbb{N}$ and $(f_n)_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Moreover $f_n'(z) = u^{\frac{1}{2}}(\phi(z_n)) \left(-2\frac{u'(z)}{u^2(z)} + 2\frac{u(\phi(z_n))u'(z)}{u^3(z)} \right)$ and thus $f_n(\phi(z_n)) = \frac{1}{u^{\frac{1}{2}}(\phi(z_n))}$ and $f_n'(\phi(z_n)) = 0$. Since $C_{\phi,\psi}$ is compact, we have $\|C_{\phi,\psi} f_n\|_{B_v} \to 0$ if $n \to \infty$. Furthermore

$$\begin{aligned}
\|C_{\phi,\psi} f_n\|_{B_v} &= \sup_{z\in\mathbb{D}} v(z)|(C_{\phi,\psi} f_n)'(z)| \\
&\geq v(z_n)|(C_{\phi,\psi} f_n)'(z_n)| = |\psi'(z_n)| \frac{v(z_n)}{w^{\frac{1}{2}}(\phi(z_n))}.
\end{aligned}$$

Finally, (a) follows. In order to show (b) we consider the functions

$$g_n(z) = u^{\frac{1}{2}}(\phi(z_n)) \left(\frac{u(\phi(z_n))}{u^2(z)} - \frac{1}{u(z)} \right).$$

Then $g_n \in A_{w^{2p},p}$, $\|g_n\|_{B_v} \leq 2^{\frac{1}{p}} M^2$ for every $n \in \mathbb{N}$ and $(g_n)_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Moreover $g_n'(z) = u^{\frac{1}{2}}(\phi(z_n)) \left(-2\frac{u(\phi(z_n))u'(z)}{u^3(z)} + \frac{u'(z)}{u^2(z)} \right)$. Thus, $g_n(\phi(z_n)) = 0$ and $g_n'(\phi(z_n)) = -\frac{u'(\phi(z_n))}{u^{\frac{3}{2}}(\phi(z_n))}$ and $\|C_{\phi,\psi} f_n\|_{B_v} \geq v(z_n) \frac{|\phi'(z_n)||u'(z_n)|}{w^{\frac{3}{2}}(\phi(z_n))}$. □

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The TTF Method for the Inverse Problem of Finding Unknown Source in a Heat Equation

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Abstract: An inverse problem of finding unknown source function in a heat equation is considered. A method based on the Trace-Type Functional (TTF) formulation and implicit finite difference scheme is applied for the numerical solution. The convergence and the stability of the method are investigated for some test example. The performance of the method in comparision with other methods are presented.

Keywords: parabolic equation, inverse problem, unknown source, finite-difference method, TTF method

1 Introduction

The approach of the TTF method is to use the overspecified condition to eliminate the unknown function from the partial differential equation. The resulting problem then has the form of a standart boundary value or initial boundary value problem but having coefficients which are functionals of the unknown solution. If the overspesified data is measured on the boundary then these functionals will depend on restriction to the boundary of the solution (i.e. they will be "trace-type functionals"). After solving this TTF (Trace-type functional) problem, the unknown function can be determined using the overspecified conditions. Trace-type functionals can be used to establish existence of a solution in an inverse problems[1-4]. This approach can also be implemented numerically [5,6]. The usual numerical methods such as finite diffrence and finite element can be used for numerical solution of TTF problem.

A classical approach to solve the considered problem referred to in the literature as the method of output least squares is to assume that the unknown function is a specific functional form depending on some parameters and then seek to determine optimal parameter values so as to minimize an error functional based on the overspecified data. However, this approach has some drawbacks. For example, it is usually not evident that the solution to the optimization problem solves the original inverse problem and the error functional may be based on data which do not uniquely determine the unknown function. Another methods to solve this problem are residual update methods such as Newton, Homotopy, FPP(Fixed Point Projection) methods[7] or montonicity methods [8,9]. The

main difficulty with these methods is the form of the nonlinearity. In this work we apply TTF method for the solution of considered problem and show effectiveness of this method by comparison with other methods.

This paper is organized as follows. In section 2 we give the formulation of the direct and inverse problems and the properties of solution of the direct problem. In section 3 we describe the trace-type functional formulation of the problem. The finite difference scheme for solving of TTF formulation of the problem is also described in this section 3 and the results of numerical calculations are presented in section 4.

2 Formulation of the Problem

We consider the problem of structural identification of an unknown source term in a heat equation subject to the specification of the solution at the boundary. This problem is described by the following inverse problem:

Find $u = u(x,t)$ and $F = F(u)$ which satisfy

$$u_t(x,t) = u_{xx}(x,t) + F(u(x,t)), \quad (x,t) \in Q_T = (0,1) \times (0,T), \tag{1}$$

$$u(x,0) = 0, \quad x \in (0,1), \tag{2}$$

$$u_x(0,t) = g(t), \quad t \in (0,T), \tag{3}$$

$$u_x(1,t) = 0, \quad t \in (0,T), \tag{4}$$

subject to the overspecification

$$u(0,t) = f(t), \quad t \in (0,T) \tag{5}$$

where $f(t)$ and $g(t)$ are known functions.

In the context of heat conduction and diffusion when u represents temperature and concentration the unknown function $F(u)$ is interpreted as a heat and material source, respectively, while in a chemical or biochemical application F may be interpreted as a reaction term. Although the results in this paper apply to each of these interpretations, the unknown function $F(u)$ will be referred to here as a source term.

The problem above in such formulation have been studied, for instance, in [10-12].

If F is continuous and piecewise differentiable on R and $g \in C(0,\infty)$ with $g(0) = 0$ functions, then initial boundary value problem (1)-(4) has a unique classical solution in $Q_{T'}$ for sufficiently small $T^{'}$(local existence of a solution)[13]. It is known that if the local solution is known to satisfy an a priori estimate then the local solution may be extended to a global solution. In particular, if it is known a priori that the solution of (1)-(4) satisfies

$$|u(x,t)| \leq C_1 \quad for \quad 0 \leq x \leq 1 \quad and \quad 0 \leq t \leq T \tag{6}$$

for some $T > 0$, then it can be concluded that $T^{'} = T$. One refers to (1)-(4) as a direct problem.

Let $u = u(x,t;F,g)$ denote the solution of (1)-(4) for boundary input $g = g(t)$ and source term $F = F(u)$. Assume that this solution is known to satisfy (6) for a fixed $T > 0$ so that $u(x,t)$ is then a solution in Q_T. Then $u(x,t)$ will be said to be a solution of the direct problem. The function $f(t) = u(0,t;F,g)$ will be viewed as an output corresponding to the input $g(t)$ in the presence of

the source term F. In [9] the following properties were deduced for the direct problem solution:

suppose that $g(t) < 0$ for $0 < t < T$. Then

a) for each $t,\ 0 < t < T,\ f(t) = u(0,t) > u(x,t) > u(1,t)$ for $0 < x < 1$. If, in addition, $F(0) \geq 0$ with $F(u) \geq 0$ for $0 \leq u \leq U_1$ for some $U_1 > 0$, then exists $T_1 > 0$ such that $f^{'}(t) > 0$ for $0 < t < T_1$ and

$b) f(t) = u(0,t) > u(x,t) > u(1,t) \geq 0$ for $0 < x < 1,\ 0 < t < T_1$.

We shall use these properties to construct numerical procedure for the solution of considered problem.

Now the inverse problem can be defined as follows: suppose $g \in C[0,T],\ g(0) = 0,\ g(t) < 0$ and $f \in C^{'}[0,T],\ f(0) = 0,\ f^{'}(t) > 0$ for $0 < t < T$. Then the problem of determining $F(u)$ on an interval $[0, f(T)]$ from the data $f(t)$ and $g(t)$ whose values are known on the interval $[0,T]$ will be said the inverse problem. The uniqueness of formulated and similar inverse problem has been established in [10, 15].

3 The TTF Method

3.1 Trace-Type Functional Formulation of the Problem

If the function pair $\{u, F\}$ solves the inverse problem $(1) - (5)$ then it follows that,

$$F(u(0,t)) = f'(t) - u_{xx}(0,t) \tag{7}$$

Let $s = u(0,t) = f(t)$ then

$$F(s) = f'(f^{-1}(s)) - u_{xx}(0, f^{-1}(s))$$

If we eliminate F from (1) using the last formula we obtain the Trace Type Functional (or TTF) equation

$$u_t(x,t) - u_{xx}(x,t) = f'(f^{-1}(u(x,t))) - u_{xx}(0, f^{-1}(u(x,t)) \tag{8}$$

The equation (8) along with initial condition (2) and boundary condition (3), (4) is called as the Trace-Type Functional Formulation of the inverse problem $(1) - (5)$.

Then from the solution $u(x,t)$ of the TTF formulation of the problem

$$u_t(x,t) - u_{xx}(x,t) = f^{'}(f^{-1}(u(x,t)) - u_{xx}(0, f^{-1}(u(x,t))) \tag{9}$$

$$u(x,0) = 0, \qquad x \in (0,1) \tag{10}$$

$$u_x(0,t) = g(t), \qquad t \in (0,T) \tag{11}$$

$$u_x(1,t) = 0, \qquad t \in (0,T) \tag{12}$$

we can find $F(u)$ by using (7). Numerical solution of $(9) - (12)$ is realized by the implicit finite-difference scheme.

3.2 Finite-Difference Approximation of TTF problem.

Let $\tau = \Delta t > 0$ and $h = \Delta x > 0$ be step length on time and space coordinate, $\{0 = t_0 < t_1 < ... < t_M = T\}$ and $\{0 = x_0 < x_1 < ... < x_N = 1\}$ denotes a partitions of the $[0,T]$ and $[0,1]$ respectively.

The implicit finite-difference approximation of this system can be written in the form

$$\frac{U_{i,n}-U_{i,n-1}}{\tau} = \frac{U_{i-1,n}-2U_{i,n}+U_{i-1,n}}{h^2} + f^{'}(f^{-1}(U_{i,n})) - U_{xx}(0,f^{-1}(U_{i,n})) \quad 1 \leq i \leq N-1, \quad 1 \leq n \leq M \tag{13}$$

$$U_{i,0} = u_0, \qquad 0 \leq i \leq N \tag{14}$$

$$\frac{U_{1,n}-U_{0,n}}{h} = g(t_n), \qquad 1 \leq n \leq M \tag{15}$$

$$\frac{U_{N,n}-U_{N-1,n}}{h} = 0, \qquad 1 \leq n \leq M \tag{16}$$

where $U_{i,n}$ is the approximate value of $u_{i,n} = u(x_i, t_n)$,

$$f'(f^{-1}(U_{i,n})) = \frac{f(f^{-1}(U_{i,n})) - f(f^{-1}(U_{i,n-1}))}{\tau},$$

$$U_{xx}(0, f^{-1}(U_{i,n}) = \frac{U_{2,n^*} - 2U_{1,n^*} + U_{0,n^*}}{h^2},$$

and n^* is such that $t_{n^*} < f^{-1}(U_{i,n}) < t_{n^*+1}$.

The difference scheme $(13)-(16)$ has a second order approximation on x at the interior nodal points and first order approximation on t. A solution of this finite–difference scheme can be realized by the standard solver. From the numerically obtained solution U the unknown F can be calculated through the formula (7) via numerical differentiation.

4 Numerical Results

In this section we report some results of our numerical calculations using the numerical algorithm proposed in the previous section.

Example 1. The data function $g(t)$, the source $F(u), u_0$ and T were given by

$$g(t) = -3t,\ F(u) = 5 - 4/(u+1),\ T = 1,\ u_0 = 0.$$

By solving the direct problem with these data the solution values of $f(t)$ were recorded. Then the inverse problem was solved with this overspesification to determine the unknown source $F(u)$.

At first, we investigate the convergence of the numerical solution with respect to the number of nodal points.

Results of determination of $F(u)$ by the presented numerical method, are illustrated in figures 1-3, corresponds to results with grids $N \times M = 50 \times 5, 50 \times 10, 50 \times 30$, respectively, where the symbols correspond to approximate results and the ones without symbols correspond to exact $F(u)$. It is seen that approximation of $F(u)$ is improved by increasing the number of nodes and that for sufficiently large number of nodes the agreement between numerical and exact solution becomes uniformly good. It should be noted that for $M = 5$ the numerical solution is already an accurate approximation to the exact solution. Although not presented here it is reported that other test examples have been investigated and it was found that in general the TTF method produces an accurate numerical solution even for small numbers of nodal points.

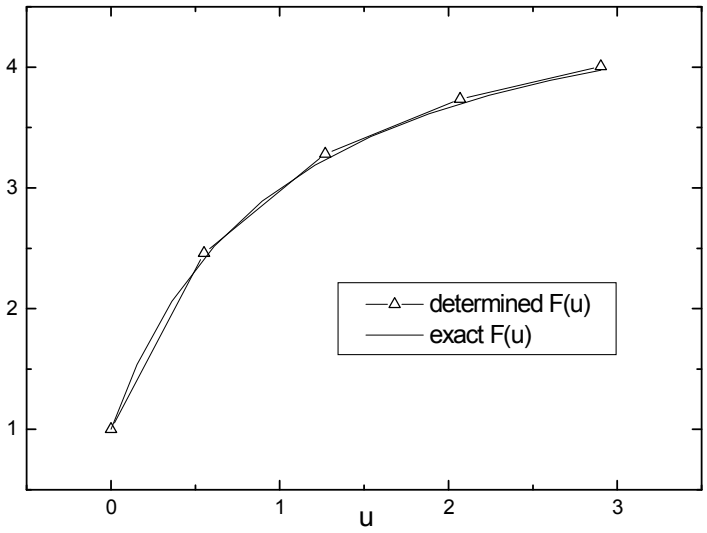


Figure 1. The exact and determined values of F(u), for N=50,M=5.

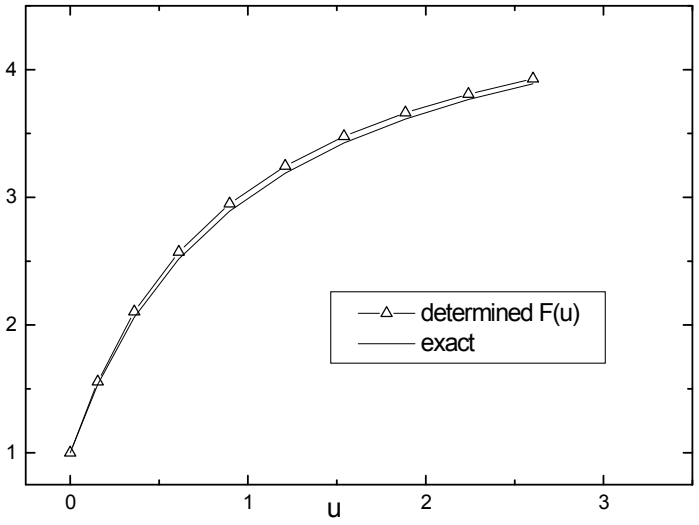


Figure 2. The exact and determined values of F(u) for N=50,M=10.

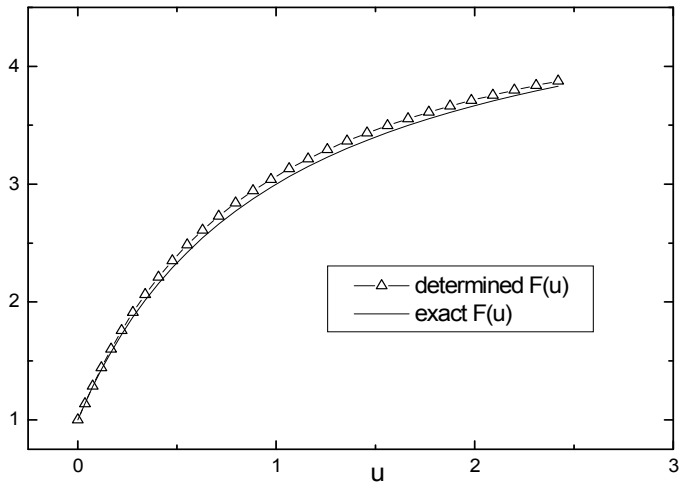


Figure 3. The exact and determined values of F(u) for N=50,M=30.

Example 2.In the next example input data were used from the previous example. The stability of the TTF method proposed is investigated by perturbing the overspecification data function $f(t)$ as

$$f^*(t_n) = f(t_n)(1 + \delta(t_n, d)), \qquad n = 1, 2, ..., M.$$

Here $\delta(t, d)$ is random function of t uniformly distributed on $(-d, d)$ (function of random errors represents the level of relative error in the corresponding piece of data).

Calculation results with grid $N \times M = 50 \times 20$ with the random errors $\delta(t, 0.02)$ and $\delta(t, 0.005)$ are presented in figure 4. As seen from the figure that in the case of random errors results are worsening and there are approximations in some integral norm. It is also seen that there is a good approximation for the small values of artificial errors, as it was expected.

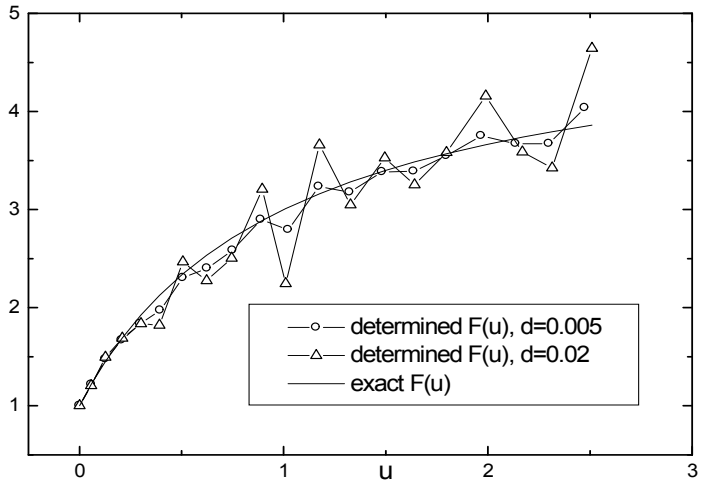


Figure 4. The exact and determined values of F(u)
for N=50, M=20 and various level of random errors.

Example 3.Since u is a monotone increasing function of the unknown coefficient f it is seems that monotonicity type successive determinig (SD) method [9, 16] is readly available for considered problem[7] . The drawbacks of other method for considered problem have been mentioned and more thorough survey of the methods can be also found in [7]**.** Next, we investigate the performance of the TTF method in comparision with the SD method. Results of determination of $F(u)$ are illustrated in Figs. 5-7.

As seen from the figures that the results for the TTF method approximates better than the SD method. It is also seen that TTF method is less sensitive to the random errors. Various other numerical examples have been investigated and similar results have been obtained.

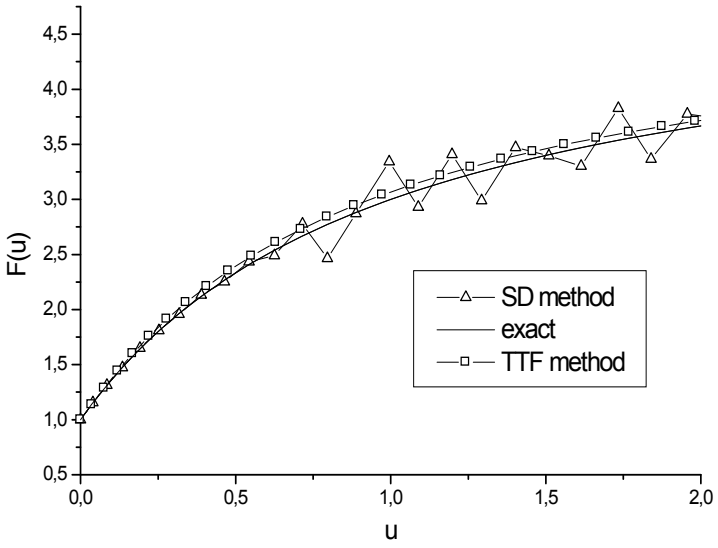


Figure 5. The exact and determined values of F(u) for N=50,M=30 and d=0.

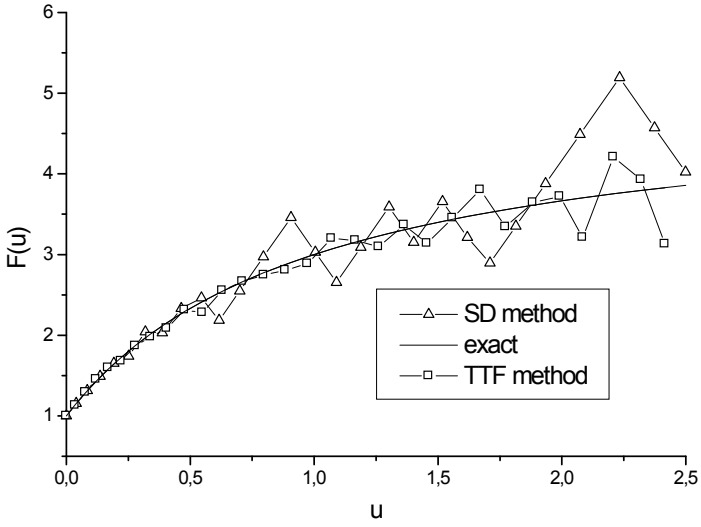


Figure 6. The exact and determined values of F(u) for N=50,M=30, d=0.02

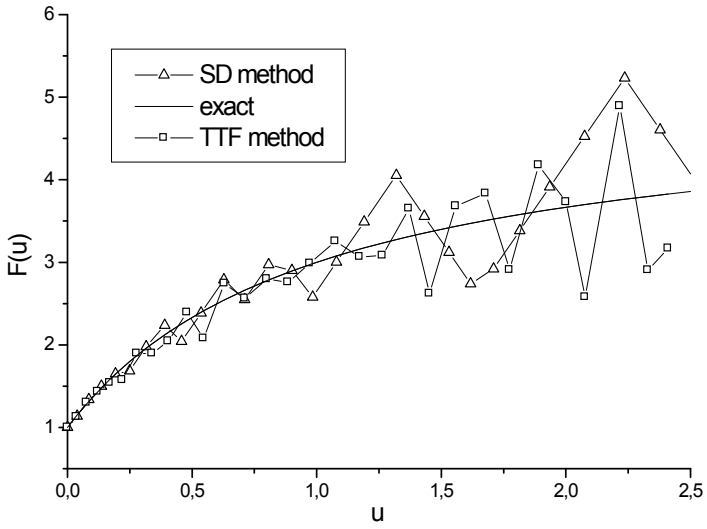


Figure 7. The exact and determined values of F(u) for N=50, M=30 and d=0.04.

5 Conclusions

In this paper the TTF method is applied for solving the inverse problem of finding unknown source function in a parabolic equation. The numerical results show effectiveness of this method by comparison with other methods and suggest that the TTF method is an accurate and reliable numerical technique. The method is very efficient from a computational point of view since it was found that it produces accurate results even if small numbers nodal points are used. Moreover the method is easy to adapt to other type of inverse coefficient problems for parabolic equations.

Acknowledgements

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RANDOM FIXED POINT AND RANDOM BEST APPROXIMATION

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ABSTRACT

In the present paper, random fixed point results for generalized nonexpansive mapping in the setup of compact and weakly compact subset of q-normed space have been established. Random best approximation results have also been derived as its application. Our results give stochastic version generalization of Dotson and Singh.

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Key Words and Phrases: Banach space; Random best approximation; Random fixed point; Random operator; q-normed space.

1. INTRODUCTION

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The theory of random fixed point theorems was initiated by the Prague school of probabilistic in the 1950s.The interest in this subject enhanced after publication of the survey paper by Bharucha Reid [5]. Random fixed point theory has received much attention in recent years(see, e.g. [2, 12, 13, 14, 17]).

Interesting and valuable results applying various random fixed point theorems appeared in the literature of approximation theory. In this direction, some of the authors are Beg and Shahzad [3, 4], Lin [10], Tan and Yuan [17] and Papageorgion [13, 14]. In the subject of best approximation, we often wish to know whether there are some useful property of the function being approximation which can be inherited by the approximating function.

In fact, Meinardus [11] was the first who observed the gereral principle and employed a fixed point theorem to established the existence of an invariant approximation. Later on, number of results were developed in this direction under different conditions following the line made by Meinardus [11].

The aim of this paper is to establish existence of random fixed point as random best approximation in the setting of q-normed space. While doing so, however, we need to prove such result for compact and weakly compact subset by using result of O'Regan, Shazad and Agrawal [12]. In this way, we give random best approximation generalization of best approximation theorem

obtained by Singh [16] and random fixed point generalization of fixed point theorem of Dotson [6]

2. PRELIMINARIES

In the material to be produced here, the following definitions have been used:

Let $\mathcal{X}$ be a linear space. A q-norm on $\mathcal{X}$ is a real-valued function $\|.\|_q$ on $\mathcal{X}$ with $0 < q \leq 1$, satisfying the following conditions :

(a) $\|x\|_q \geq 0$ and $\|x\|_q = 0$ iff $x = 0$,

(b) $\|\lambda x\|_q = |\lambda|^q \|x\|_q$,

(c) $\|x + y\|_q \leq \|x\|_q + \|y\|_q$,

for all $x, y \in \mathcal{X}$ and all scalars λ. The pair $(\mathcal{X}, \|.\|_q)$ is called a q-normed space. It is a metric space with $d_q(x, y) = \|x-y\|_q$ for all $x, y \in \mathcal{X}$, defining a translation invariant metric d_q on $\mathcal{X}$. If $q = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some q-norm, $0 < q \leq 1$. The spaces l_q and $\mathcal{L}_q[0, 1]$, $0 < q \leq 1$ are q-normed space. A q-normed space is not necessarily a locally convex space. Recall that, if $\mathcal{X}$ is a topological linear space, then its continuous dual space $\mathcal{X}^*$ is said to separate the points of $\mathcal{X}$, if for each $x \neq 0$ in $\mathcal{X}$, there exists an $g \in \mathcal{X}^*$ such that $gx \neq 0$. In this case the weak topology on $\mathcal{X}$ is well-defined. We mention that, if $\mathcal{X}$ is not locally convex, then $\mathcal{X}^*$ need not separates the points of $\mathcal{X}$. For example, if $\mathcal{X} = \mathcal{L}_q[0, 1]$, $0 < q < 1$, then $\mathcal{X}^* = \{0\}$ [15, page 36-37]. However, there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual separates the points [8].

Definition 2.1. [12]. Let $(\Omega, \mathcal{A})$ be a measurable space and $\mathcal{X}$ be a metric space. Let $2^{\mathcal{X}}$ be the family of all nonempty subsets of $\mathcal{X}$ and $\mathcal{C}(\mathcal{X})$ denote the family of all nonempty compact subsets of $\mathcal{X}$. Now, we call a mapping $\mathcal{F} : \Omega \to 2^{\mathcal{X}}$ measurable (respectively, weakly measurable) if, for any closed (respectively, open) subset $\mathcal{B}$ of $\mathcal{X}$, $\mathcal{F}^{-1}(\mathcal{B}) = \{\omega \in \Omega : \mathcal{F}(\omega) \cap \mathcal{B} \neq \phi\} \in \mathcal{A}$. Note that, if $\mathcal{F}(\omega) \in \mathcal{C}(\mathcal{X})$ for every $\omega \in \Omega$, then $\mathcal{F}$ is weakly measurable if and only if measurable.

A mapping $\xi : \Omega \to \mathcal{X}$ is called a measurable selector of a measurable mapping $\mathcal{F} : \Omega \to 2^{\mathcal{X}}$, if ξ is measurable and, for any $\omega \in \Omega, \xi(\omega) \in \mathcal{F}(\omega)$. A mapping $f : \Omega \times \mathcal{X} \to \mathcal{X}$ is called a random operator if for any $x \in \mathcal{X}$, $f(., x)$ is measurable. A measurable mapping $\xi : \Omega \to \mathcal{X}$ is called a random fixed point of a random operator $f : \Omega \times \mathcal{X} \to \mathcal{X}$ if for every $\omega \in \Omega$, $\xi(\omega) = f(\omega, \xi(\omega))$. A random operator $f : \Omega \times \mathcal{X} \to \mathcal{X}$ is continuous if, for each $\omega \in \Omega$, $f(\omega, .)$ is continuous.

Definition 2.2. Let $\mathcal{M}$ be a nonempty subset of a Banach space $\mathcal{X}$. For $x_0 \in \mathcal{X}$, define

$$d(x_0, \mathcal{M}) = \inf_{y \in \mathcal{M}} \|x_0 - y\|$$

and

$$\mathcal{P}_{\mathcal{M}}(x_0) = \{y \in \mathcal{M} : \|x_0 - y\| = d(x_0, \mathcal{M})\}.$$

Then an element $y \in \mathcal{P}_{\mathcal{M}}(x_0)$ is called a *best approximant* of x_0 of $\mathcal{M}$. The set $\mathcal{P}_{\mathcal{M}}(x_0)$ is the set of all best approximants of x_0 of $\mathcal{M}$.

We also use the following result of O'Regan, Shahzad and Agarwal [12]:

Theorem 2.3. [12]. *Let $(\mathcal{X}, d)$ be a Polish space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ be a continuous random operator. Suppose, there is some $h \in (0, 1)$ such that for $x, y \in \mathcal{X}$ and $\omega \in \Omega$, we have*

$$d(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \leq h \max\{d(x, y), d(x, \mathcal{T}(\omega, x)), d(y, \mathcal{T}(\omega, y)),$$

$$\tfrac{1}{2}[d(x, \mathcal{T}(\omega, y)) + d(y, \mathcal{T}(\omega, x))]\}.$$

Then $\mathcal{T}$ have a random fixed point.

3. MAIN RESULT

We first prove our main result for compact subset of a q-normed space.

Theorem 3.1. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{M}$ be a subset of $\mathcal{X}$, and $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be a continuous random operator. Suppose $\mathcal{M}$ is nonempty, compact and starshaped to a point $p \in \mathcal{M}$. If $\mathcal{T}$ satisfies*

(3.1)

$$\|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q \leq \max\{\|x - y\|_q, dist(x, [\mathcal{T}(\omega, x), p]), dist(y, [\mathcal{T}(\omega, y), p]),$$

$$\tfrac{1}{2}[dist(x, [\mathcal{T}(\omega, y), p]) + dist(y, [\mathcal{T}(\omega, x), p])]\}$$

for $x, y \in \mathcal{M}$, $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Proof. Choose a sequence $k_n \subset (0, 1)$ such that $\{k_n\} \to 1$ as $n \to \infty$. Then for each n, define a random operator $\mathcal{T}_n : \Omega \times \mathcal{M} \to \mathcal{M}$ as

$$\mathcal{T}_n(\omega, x) = k_n \mathcal{T}(\omega, x) + (1 - k_n)p \ \ for \ \ each \ \ x \in \mathcal{M}. \tag{3.2}$$

Then each $\mathcal{T}_n$ is a self map from $\mathcal{M}$ into $\mathcal{M}$ and $\omega \in \Omega$. Also (3.1) and (3.2) imply that

$$\|\mathcal{T}_n(\omega, x) - \mathcal{T}_n(\omega, y)\|_q = (k_n)^q \|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q$$

$$\leq (k_n)^q \max\{\|x - y\|_q, dist(x, [\mathcal{T}(\omega, x), p]), dist(y, [\mathcal{T}(\omega, y), p]),$$

$$\tfrac{1}{2}[dist(x, [\mathcal{T}(\omega, y), p]) + dist(y, [\mathcal{T}(\omega, x), p])]\}$$

$$\leq (k_n)^q \max\{\|x - y\|_q, \|x - \mathcal{T}_n(\omega, x)\|_q, \|y - \mathcal{T}_n(\omega, y)\|_q,$$

$$\tfrac{1}{2}[\|x - \mathcal{T}_n(\omega, y)\|_q + \|y - \mathcal{T}_n(\omega, x)\|_q]\}$$

i.e.,

$$\|\mathcal{T}_n(\omega, x) - \mathcal{T}_n(\omega, y)\|_q \leq (k_n)^q \max\{\|x - y\|_q, \|x - \mathcal{T}_n(\omega, x)\|_q, \|y - \mathcal{T}_n(\omega, y)\|_q,$$

$$\tfrac{1}{2}[\|x - \mathcal{T}_n(\omega, y)\|_q + \|y - \mathcal{T}_n(\omega, x)\|_q]\}$$

for all $x, y \in \mathcal{M}, \omega \in \Omega$. By the continuity of $\mathcal{T}_n(., x)$ $(x \in \mathcal{M})$, the inverse image of any open subset $\mathcal{K}$ of $\mathcal{M}$ is open in $\omega = [0, 1]$ and hence Lebsegue measurable. Thus each $\mathcal{T}_n(., x)$ is a random operator. By Theorem 2.3, $\mathcal{T}_n$ has a random fixed point ξ_n of $\mathcal{T}_n$ such that $\xi_n(\omega) = \mathcal{T}_n(\omega, \xi_n)$ for all $n \in \mathbb{N}$.
For each n, define $\mathcal{G}_n : \Omega \to \mathcal{C}(\mathcal{M})$ by $\mathcal{G}_n = cl\{\xi_i(\omega) : i \geq n\}$ where $\mathcal{C}(\mathcal{M})$ is the set of all nonempty compact subset of $\mathcal{M}$.
Let $\mathcal{G} : \Omega \to \mathcal{C}$ $(\mathcal{M})$ be a mapping defined as $\mathcal{G}(\omega) = \cap_{n=1}^{\infty} \mathcal{G}_n(\omega)$. Then, by a result of Himmelberg [7, Theorem 4.1] we see that $\mathcal{G}$ is measurable. The Kuratowski and Ryll-Nardzewski selection Theorem [9] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{M}$. We now show that ξ is the random fixed point of $\mathcal{T}$. We first fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges to $\xi(\omega)$; that is $\xi_m(\omega) \to \xi(\omega)$. Since $\mathcal{T}_m(\omega, \xi_m(\omega)) = \xi_m(\omega)$, we have $\mathcal{T}_m(\omega, \xi_m(\omega)) \to \xi(\omega)$.
Proceeds to the limit as $m \to \infty$, $k_m \to 1$, we have $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$. This completes the proof.

□

An immediately consequences of the Theorem 3.1 are as follow:

Corollary 3.2. *Let $\mathcal{M}$ be a subset of a q-normed space $\mathcal{X}$ and $\mathcal{X}$ and $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be continuous random operator. Suppose $\mathcal{M}$ is nonempty, compact and starshaped to a point $p \in \mathcal{M}$. If $\mathcal{T}$ satisfies*
(3.3)

$$\|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q \leq \max\{\|x - y\|_q, dist(x, [\mathcal{T}(\omega, x), p]), dist(y, [\mathcal{T}(\omega, y), p]),$$
$$\tfrac{1}{2} dist(x, [\mathcal{T}(\omega, y), p]), \tfrac{1}{2} dist(y, [\mathcal{T}(\omega, x), p])\}$$

for $x, y \in \mathcal{M}$, $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Corollary 3.3. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{M}$ be a subset of $\mathcal{X}$, and $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be continuous random operator. Suppose $\mathcal{M}$ is nonempty, compact and starshaped to a point $p \in \mathcal{M}$. If $\mathcal{T}$ is nonexpansive for $x, y \in \mathcal{M}$, $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.*

As application of Theorem 3.1, following is a random fixed point theorem for random best approximation:

Theorem 3.4. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ be continuous random operator. Let $\mathcal{M} \subset \mathcal{X}$ such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty compact and starshaped to a point $p \in \mathcal{D}$. If $\mathcal{T}$ satisfies for $x \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$*
(3.4)

$$\|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q \leq \begin{cases} \|x - x_0\|_q, \ \ if \ \ y = x_0, \\ \max\{\|x - y\|_q, dist(x, [\mathcal{T}(\omega, x), p]), dist(y, [\mathcal{T}(\omega, y), p]), \\ \tfrac{1}{2}[dist(x, [\mathcal{T}(\omega, y), p]) + dist(y, [\mathcal{T}(\omega, x), p])]\}, \ \ if \ y \in \mathcal{D}, \end{cases}$$

then there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Proof. Let $y \in \mathcal{D}$. Also, if $y \in \partial\mathcal{M}$ then $\mathcal{T}(\omega, y) \in \mathcal{M}$, because $\mathcal{T}(\omega, \partial\mathcal{M}) \subseteq \mathcal{M}$ for each $\omega \in \Omega$. Now since $x_0 = \mathcal{T}(\omega, x_0)$, we have

$$\|\mathcal{T}(\omega, y) - x_0\|_q = \|\mathcal{T}(\omega, y) - \mathcal{T}(\omega, x_0)\|_q \leq \|x - x_0\|_q,$$

yielding thereby $\mathcal{T}(\omega, y) \in \mathcal{D}$; consequently $\mathcal{D}$ is $\mathcal{T}(\omega, .)$-invariant, that is, $\mathcal{T}(\omega, .) \subseteq \mathcal{D}$. Now, Theorem 3.1 guarantees that there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

□

Next, an immediate consequences of the Theorem 3.4 are as follow:

Corollary 3.5. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ be continuous random operator. Let $\mathcal{M} \subset \mathcal{X}$ such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty compact and starshaped to a point $p \in \mathcal{D}$. If $\mathcal{T}$ satisfies for $x \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$*

(3.5)

$$\|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, y)\|_q \leq \begin{cases} \|x - x_0\|_q, \quad \text{if} \quad y = x_0, \\ \max\{\|x - y\|_q, dist(x, [\mathcal{T}(\omega, x), p]), dist(y, [\mathcal{T}(\omega, y), p]), \\ \frac{1}{2} dist(x, [\mathcal{T}(\omega, y), p]), \frac{1}{2} dist(y, [\mathcal{T}(\omega, x), p])\}, \ \text{if } y \in \mathcal{D}, \end{cases}$$

then there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Corollary 3.6. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ be continuous random operator. Let $\mathcal{M} \subset \mathcal{X}$ such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty compact and starshaped to a point $p \in \mathcal{D}$. If $\mathcal{T}$ is nonexpansive for $x \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.*

An analogue of the Theorem 3.1 for weakly compact subset is as follows:

Theorem 3.7. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{M}$ be a subset of $\mathcal{X}$. Let $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be weakly continuous random operator. Suppose $\mathcal{M}$ is nonempty separable weakly compact and starshaped to a point $p \in M$. If T satisfies (3.1) for $x, y \in \mathcal{M}$, $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$, provided $\mathcal{I} - \mathcal{T}(\omega, .)$ is demiclosed at zero for each $\omega \in \Omega$, where $\mathcal{I}$ is a identity mapping.*

Proof. For each $n \in \mathbb{N}$, define $\{k_n\}, \{\mathcal{T}_n\}$ as in the proof of the Theorem 3.1. Also, we have

$$\|\mathcal{T}_n(\omega, x) - \mathcal{T}_n(\omega, y)\|_q \leq (k_n)^q \max\{\|x - y\|_q, \|x - \mathcal{T}_n(\omega, x)\|_q, \|y - \mathcal{T}_n(\omega, y)\|_q,$$

$$\frac{1}{2}[\|x - \mathcal{T}_n(\omega, y)\|_q + \|y - \mathcal{T}_n(\omega, x)\|_q]\}$$

for all $x, y \in \mathcal{M}, \omega \in \Omega$. Since weak topology is Hausdorff and $\mathcal{M}$ is weakly compact, it follows that $\mathcal{M}$ is strongly closed and is a completely metric space. Thus,

weak continuity of $\mathcal{T}$, joint weakly continuous family Δ and Theorem 2.3 guarantee that there exists a random fixed point ξ of $\mathcal{T}_n$ such that $\xi_n = \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$.
For each n, define $\mathcal{G}_n : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}_n = w - cl\{\xi_i(\omega) : i \geq n\}$, where $\mathcal{WC}(\mathcal{M})$ is the set of all nonempty weakly compact subset of $\mathcal{M}$ and $w - cl$ denotes the weak closure. Define a mapping $\mathcal{G} : \Omega \to \mathcal{WC}(\mathcal{M})$ by $\mathcal{G}(\omega) = \cap_{n=1}^{\infty} \mathcal{G}_n(\omega)$. Because $\mathcal{M}$ is weakly compact and separable, the weak topology on $\mathcal{M}$ is a metric topology. Then by result of Himmelberg [7, Theorem 4.1] implies that $\mathcal{G}$ is $w-$measurable. The Kuratowski and Ryll-Nardzewski selection Theorem [9] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{M}$. We now show that ξ is the random fixed point of $\mathcal{T}$. We first fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, therefore there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges weakly to $\xi(\omega)$; that is $\xi_m(\omega) \to^w \xi(\omega)$.
Now,

$$\xi_m(\omega) - \mathcal{T}(\omega, \xi_m(\omega)) = \xi_m(\omega) - \tfrac{1}{k_m}(\mathcal{T}_m(\omega, \xi_m(\omega)) - (1 - k_m)p)$$

$$= (1 - \tfrac{1}{k_m})(\xi_m(\omega) - p).$$

Since $\mathcal{M}$ is bounded and $k_m \to 1$, it follows that $\xi_m(\omega) - \mathcal{T}(\omega, \xi_m(\omega)) \to 0$. Now, $y_m = \xi_m(\omega) - \mathcal{T}(\omega, \xi_m(\omega)) = (\mathcal{I} - \mathcal{T})(\omega, \xi_m(\omega))$ and $y_m \to 0$. Since $(\mathcal{I} - \mathcal{T})(\omega, .)$ is demiclosed at 0, so $0 \in (\mathcal{I} - \mathcal{T})(\omega, \xi(\omega))$. This implies that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$. This completes the proof.

□

An immediate consequences of the Theorem 3.7 are as follow:

Corollary 3.8. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{M}$ be a subset of $\mathcal{X}$. Let $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be weakly continuous random operator. Suppose $\mathcal{M}$ is nonempty separable weakly compact and starshaped to a point $p \in \mathcal{M}$. If T satisfies (3.3) for $x, y \in \mathcal{M}$, $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$, provided $\mathcal{I} - \mathcal{T}(\omega, .)$ is demiclosed at zero for each $\omega \in \Omega$, where $\mathcal{I}$ is a identity mapping.*

Corollary 3.9. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{M}$ be a subset of $\mathcal{X}$. Let $\mathcal{T} : \Omega \times \mathcal{M} \to \mathcal{M}$ be weakly continuous random operator. Suppose $\mathcal{M}$ is nonempty separable weakly compact and starshaped to a point $p \in \mathcal{M}$. If $\mathcal{T}$ is nonexpansive for $x, y \in \mathcal{M}$, $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$, provided $\mathcal{I} - \mathcal{T}(\omega, .)$ is demiclosed at zero for each $\omega \in \Omega$, where $\mathcal{I}$ is a identity mapping.*

As application of Theorem 3.7, we prove following random fixed point theorem for random best approximation:

Theorem 3.10. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ be weakly continuous random operator. Let $\mathcal{M} \subset \mathcal{X}$ such that $T(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty separable weakly compact and starshaped to a point $p \in \mathcal{D}$. Further, Suppose $\mathcal{T}$ satisfies the condition (3.4) for $x \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$ and $t \in (0, 1)$. Then there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$, provided $\mathcal{I} - \mathcal{T}(\omega, .)$ is demiclosed at zero for each $\omega \in \Omega$, where $\mathcal{I}$ is a identity mapping.*

Proof. It follows from the proof of the Theorem 3.4 and Theorem 3.7. □

Next, an immediate consequences of the Theorem 3.10 are as follow:

Corollary 3.11. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ be weakly continuous random operator. Let $\mathcal{M} \subset \mathcal{X}$ such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty separable weakly compact and starshaped to a point $p \in \mathcal{D}$. Further, Suppose $\mathcal{T}$ satisfies the condition (3.5) for $x \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$. Then there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$, provided $\mathcal{I} - \mathcal{T}(\omega, .)$ is demiclosed at zero for each $\omega \in \Omega$, where $\mathcal{I}$ is a identity mapping.*

Corollary 3.12. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ be weakly continuous random operator. Let $\mathcal{M} \subset \mathcal{X}$ such that $\mathcal{T}(\omega, .) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0)$. Suppose $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty separable weakly compact and starshaped to a point $p \in \mathcal{D}$. Further, Suppose T is nonexpansive for $x \in \mathcal{D} \cup \{x_0\}$, $\omega \in \Omega$. Then there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ for each $\omega \in \Omega$, provided $\mathcal{I} - \mathcal{T}(\omega, .)$ is demiclosed at zero for each $\omega \in \Omega$, where $\mathcal{I}$ is a identity mapping.*

Remark 3.13. Theorem 3.1, Corollary 3.2, Corollary 3.3, Theorem 3.7, Corollary 3.8 and Corollary 3.9 are stochastic version generalization of Dotson [6] to q-normed space.

Remark 3.14. Theorem 3.4, Corollary 3.5, Corollary 3.6, Theorem 3.10, Corollary 3.11 and Corollary 3.12 are stochastic version generalization of Singh [16] to q-normed space.

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On Behaviour of Solutions for Third Order Nonlinear Ordinary Differential Equations with Damping Terms.

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Abstract

In this work, we consider $(r(t)y'')' + q(t)k(y') + p(t)h(y) = f(t)$,where $r(t) > 0$, $f(t) \geq 0$ are real -valued continuous functions of $[0, \infty)$, $k(y')$ and $h(y)$ is a continuous function of $y' \in (-\infty, \infty)$ such that $k(y')y' > 0$ for $y' \neq 0$ and $h(y)y > 0$ for $y \neq 0$. We obtain sufficient conditions so that solutions of considered equations are nonoscillatory.

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1 Introduction

In this paper, we consider

$$(r(t)y'')' + q(t)k(y') + p(t)h(y) = f(t) \tag{1}$$

and various particular cases of (1), where p, q, r and f are real -valued continuous functions of $[0, \infty)$ such that $r(t) > 0$, $f(t) \geq 0$ and k is a continuous function of $y' \in (-\infty, \infty)$ satisfying $k(y')y' > 0$ for $y' \neq 0$ and h is a continuous function of $y \in (-\infty, \infty)$ satisfying

$h(y)y > 0$ for $y \neq 0$. We restrict our considerations to those real solutions of (1) which exist on the half line $[T, \infty)$, where $T \geq 0$ depends on the particular solution and are non-trivial in any neighborhood of infinity. We give sufficient conditions under which solutions of (1) are nonoscillatory and oscillatory.

We classify solutions of (1) as follows :

(i) A solution y(t), $t \in [T, \infty)$, is said to be nonoscillatory if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geq t_1$; (ii) y(t) is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$, and (iii) it is said to be a z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive. y(t) is said to be weakly oscillatory if it is oscillatory or z-type.

Equation (1) with $r(t) \equiv 1$, $f(t) \equiv 0$, $k(y') = y'$ has been consider by Heidel [5], Nelson[6], Waltman[4] and for the case $\alpha = 1$, $k(y') = y'$, $r(t) \equiv 1$ and $f(t) \equiv 0$ we mention the papers of Lazer [3], Barrett[7], and the book of Swanson[12]. Erbe [8] has considered equation (1)with r(t) once continuously differentiable, $f(t) \equiv 0$, $k(y') = y'$. They have given sufficient conditions for the existence of oscillatory and nonoscillatory solutions and have studied their asymptotic behaviour. Parhi [9] has considered equation (1) with $p(t) \leq 0$, $q(t) \leq 0$,in addition to Parhi [10] has considered equation (1) with (i)$p(t) \geq 0$, $q(t) \leq 0$ and (ii)$p(t) \leq 0$, $q(t) \geq 0$, $h(y) = y^{\alpha}$ and $k(y') = y'^{\beta}$ for $\alpha > 0$ and $\beta > 0$ are ratio of odd integers. He has given sufficient conditions under which solutions of (1) are nonoscillatory and has studied the the qualitative behaviour of these solutions. Temtek [11] has considered equation (1) with $p(t) \geq 0$, $q(t) \leq 0$ and $r(t) \equiv 1$, $k(y') = y'^{\alpha}$, $\alpha > 0$ is ratio of odd integers. She obtained sufficient conditions so that solutions of (1) are nonoscillatory.

2. The Case of $\mathbf{p(t) \leq 0, q(t) \leq 0}$ in Equation (1).

In this section, we obtain the sufficient conditions for the nonoscillatory solutions of Equation(1).

Theorem 2.1. If $p(t) \equiv 0$, then all solution of (1) are nonoscillatory.

Proof. Let y(t) be a solution of (1) on $[T, \infty)$,$T \geq 0$. If possible let y(t) be non-

negative z-type. Let a and b ($T \leq a < b$) be consecutive double zeros of y(t). So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Multiplying (1) through by $y'(t)$, we get

$$[r(t)y''(t)y'(t)]' = r(t)y''^2(t) - q(t)k(y'(t))y'(t) + f(t)y'(t). \tag{2}$$

Integration of (2) from a to c yields

$$0 = \int_a^c [r(t)y''^2(t) - q(t)k(y'(t))y'(t) + f(t)y'(t)]dt > 0,$$

a contradiction.

Similarly, if integration of (2) from c to b , it can be shown that y(t) cannot be a nonpositive z-type solution.

Suppose that y(t) is oscillatory. Let b', a and b ($T \leq b' < a < b$) be any three consecutive zeros of y(t) such that $y'(b') \leq 0$, $y'(a) \geq 0$, $y'(b) \leq 0$, $y(t) < 0$ for $t \in (b', a)$ and $y(t) > 0$ for $t \in (a, b)$. So there exist points $c' \in (b', a)$ and $c \in (a, b)$ such that $y'(c') = 0 = y'(c)$ and $y'(t) > 0$ for $t \in (c', c)$. We consider two cases viz. (i) $y''(a) \geq 0$ and (ii) $y''(a) < 0$. Suppose that $y''(a) \geq 0$. Integrating (2) from a to c we obtain

$$0 \geq -r(a)y'(a)y''(a) = \int_a^c [r(t)y''^2(t) - q(t)k(y'(t))y'(t) + f(t)y'(t)]dt > 0,$$

a contradiction. Let $y''(a) < 0$. Integrating (2) from c' to a, we get

$$0 \geq r(a)y'(a)y''(a) = \int_{c'}^a [r(t)y''^2(t) - q(t)k(y'(t))y'(t) + f(t)y'(t)]dt > 0,$$

a contradiction.

Hence the theorem.

Theorem 2.2. Let $h(y) = y^\alpha$ in (1) where $\alpha > 0$ is the ratio of odd integers. If $p'(t) \geq 0$, $f'(t) \geq 0$ and $(\alpha + 1)f(t) + p(t) \geq 0$ then all solutions y(t) of (1) for which $|y(t)| \leq 1$ ultimately are nonoscillatory.

Proof. Let y(t) be a solution of (1) on $[T, \infty)$, $T \geq 0$ such that $|y(t)| \leq 1$ for $t \geq T_1 > T$. If possible let y(t) be non-negative z-type. Let a and b ($T_1 \leq a < b$) be

consecutive double zeros of y(t). So there exists a $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a,c)$. Multiplying (1) through by $y'(t)$, we get

$$[r(t)y''(t)y'(t)]' = r(t)y''^2(t) - q(t)k(y'(t))y'(t) - p(t)y^\alpha(t)y'(t) + f(t)y'(t). \tag{3}$$

Integration of (3) from a to c, we obtain

$$0 = \int_a^c [r(t)y''^2(t) - q(t)k(y'(t))y'(t) - p(t)y^\alpha(t)y'(t) + f(t)y'(t)]dt > 0$$

a contradiction.

Suppose that y(t) is of non-positive z-type. Let a and b ($T_1 \le a < b$) be consecutive double zeros of y(t). So there exists a $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$. Integration of (3) from c to b, we get

$$0 = \int_c^b [r(t)y''^2(t) - q(t)k(y'(t))y'(t) - p(t)y^\alpha(t)y'(t) + f(t)y'(t)]dt.$$

But

$$\begin{aligned}\int_c^b p(t)y^\alpha(t)y'(t)dt &= \frac{1}{\alpha+1}[p(t)y^{\alpha+1}(t)]_c^b - \frac{1}{\alpha+1}\int_c^b p'(t)y^{\alpha+1}(t)dt \\ &\le -\frac{1}{\alpha+1}p(c)y^{\alpha+1}(c)\end{aligned}$$

and

$$\int_c^b f(t)y'(t)dt = [f(t)y(t)]_c^b - \int_c^b f'(t)y(t) \ge -f(c)y(c).$$

So

$$\begin{aligned}\int_c^b f(t)y'(t)dt - \int_c^b p(t)y^\alpha(t)y'(t) &\ge -f(c)y(c) + \frac{1}{\alpha+1}p(c)y^{\alpha+1}(c) \\ &\ge \frac{1}{\alpha+1}p(c)[y(c) + y^{\alpha+1}(c)] \\ &> 0\end{aligned}$$

since $|y(t)| \le 1$ for $t \ge T_1$, a contradiction.

If possible let y(t) be oscillatory. Let b', a and b ($T \le b' < a < b$) be any three consecutive zeros of y(t) such that $y'(b') \le 0$, $y'(a) \ge 0$, $y'(b) \le 0$, $y(t) < 0$ for $t \in (b',a)$ and $y(t) > 0$

for $t \in (a,b)$. So there exist points $c' \in (b',a)$ and $c \in (a,b)$ such that $y'(c') = 0 = y'(c)$ and $y'(t) > 0$ for $t \in (c',c)$. We consider two cases viz. (i) $y''(a) \geq 0$ and (ii) $y''(a) < 0$. Integrating (3) from a to c, we get

$$0 \geq -r(a)y'(a)y''(a) = \int_a^c [r(t)y''^2(t) - q(t)k(y'(t))y'(t) - p(t)y^\alpha(t)y'(t) + f(t)y'(t)]dt > 0,$$

a contradiction.

Now let $y''(a) < 0$. Integrating (3) from c' to a

$$0 \geq r(a)y'(a)y''(a) = \int_{c'}^a [r(t)y''^2(t) - q(t)k(y'(t))y'(t) - p(t)y^\alpha(t)y'(t) + f(t)y'(t)]dt. \quad (4)$$

But

$$\int_{c'}^a p(t)y^\alpha(t)y'(t)dt = \frac{1}{\alpha+1}[p(t)y^{\alpha+1}(t)]_{c'}^a - \frac{1}{\alpha+1}\int_{c'}^a p'(t)y^{\alpha+1} \leq \frac{1}{\alpha+1}p(c')y^{\alpha+1}(c')$$

and

$$\int_{c'}^a f(t)y'(t)dt = [f(t)y(t)]_{c'}^a - \int_{c'}^a f'(t)y(t)dt \geq -f(c')y(c').$$

So

$$\begin{aligned}\int_{c'}^a [f(t)y'(t) - p(t)y^\alpha(t)y'(t)]dt &\geq -f(c')y(c') + \frac{1}{\alpha+1}p(c')y^{\alpha+1}(c') \\ &\geq \frac{1}{\alpha+1}[y(c') + y^{\alpha+1}(c')]p(c') \\ &\geq 0,\end{aligned}$$

since $|y(t)| \leq 1$ for $t \geq T_1$. Hence (4) yields

$$0 \geq r(a)y'(a)y''(a) > 0$$

a contradiction. So y(t) is nonoscillatory. This completes the proof of the theorem.

Example. Consider the third order differential equation

$$((e^{6t} + \frac{1}{4}e^{2t})y'')' - e^{-3t}(y'^2 + y') - e^{-t}y^3 = \frac{1}{4}e^t - e^{-5t}, t > 0.$$

The equations satisfies conditions of Theorem 2.2. Clearly $y(t) = e^{-t}$ is a nonoscillatory solution of equations.

3. The case of $\mathbf{p(t) \leq 0}$ and $\mathbf{q(t) \equiv 0}$ in Equation (1).

Theorem 3.1. Consider (1) with $f(t) \equiv 0$. Let r(t) be once continuously differentiable such that $r'(t) \leq 0$. If $\int_0^\infty p(t)dt = -\infty$ and $h(y)$ is increasing, then all bounded solution of (1) are oscillatory.

Proof. Let y(t) be a a bounded solution of (1) on $[T, \infty)$, $T \geq 0$. If y(t) is of z-type with consecutive double zeros at a and b ($T \leq a < b$), then we integrate

$$[r(t)y(t)y''(t)]' = r(t)y'(t)y''(t) - p(t)y(t)h(y(t)) \tag{5}$$

from a to b to get the necessary contradiction.

If possible , let y(t) be nonoscillatory. Without any loss of generality , we can assume $y(t) > 0$ for $t \geq t_0 \geq T$. From (1) it is clear that $y'(t)$ cannot have more than two zeros. So $y'(t)$ is ultimately positive or negative, let $y'(t) > 0$ for $t \geq t_1 \geq t_0$. Integrating (1) from t_1 to t,

$$\int_{t_1}^{t} [r(s)y''(s)]'ds = -\int_{t_1}^{t} p(s)h(y(s))ds$$

$$r(t)y''(t) \geq r(t_1)y''(t_1) - h(y(t_1)) \int_{t_1}^{t} p(s)ds$$

we get $\lim_{t\to\infty} r(t)y''(t) = +\infty$ which in turn implies that $\lim_{t\to\infty} y(t) = +\infty$, a contradiction.

Let $y'(t) < 0$ for $t \geq t_1 \geq t_0$. Clearly $r(t)y''(t)$ is nondecreasing. We claim that $y''(t) \leq 0$ for $t \geq t_1$. If not, there exists $t_2 > t_1$ such that $y''(t_2) > 0$. Now $t \geq t_2$ implies that $y''(t) \geq y''(t_2)$ and hence $y'(t) > 0$ for large t, a contradiction. So our claim holds. Hence $t \geq t_1$ implies that $y'(t) \leq y'(t_1)$. Integrating we get $y(t) < 0$ for large t. This contradiction completes the proof of the theorem.

Theorem 3.2. Let $p(t) \leq -1$ and r(t) satisfy the conditions of Theorem 3.1. If $f(t) \equiv 0$ in (1) , then all solutions y(t) of (1) which satisfy the inequality

$$r(t)z'z'' + zh(z) > 0 \tag{6}$$

in any interval where $y(t) > 0$ and $y'(t) > 0$ are nonoscillatory.

Proof. From (5) it is clear that y(t) cannot be of z-type. If possible, let y(t) be oscillatory with consecutive zeros at a and b ($a < b$) such that $y(t) > 0$ for $t \in (a, b)$. So there exist a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. If $y''(t) \geq 0$, then integrating (5) from c to b, we get

$$\begin{aligned} 0 \geq -r(c)y(c)y''(c) = & - \int_c^b p(t)y(t)h(y(t))dt + \frac{1}{2}r(b)(y'(b))^2 \\ & - \frac{1}{2}\int_c^b r'(t)(y'(t))^2 dt > 0, \end{aligned}$$

a contradiction. Hence $y''(t) < 0$. Again integrating (5) from a to c, we get

$$\begin{aligned} 0 > r(c)y(c)y''(c) &= \int_a^c [r(t)y'(t)y''(t) - p(t)y(t)h(y(t))]dt \\ &\geq \int_a^c [r(t)y'(t)y''(t) + y(t)h(y(t))]dt \\ &> 0, \end{aligned}$$

a contradiction. Hence the theorem.

Example. Consider $y''' - 8y = 0$. Clearly $y(t) = e^{2t}$ satisfies (6) and is a nonoscillatory.

4. The case of $\mathbf{p(t) \geq 0, f(t) \equiv 0}$ and $\mathbf{q(t) \equiv 0}$ in Equation (1).

In this section we consider

$$(r(t)y'')' + p(t)h(y) = 0. \tag{7}$$

Theorem 4.1. Let $p(t) \geq 1$, $f(t) \equiv 0$ and r(t) be once continuously differentiable such that $r'(t) \geq 0$. Then all solutions y(t) of (7) which satisfy the inequality $r(t)z'z'' - zh(z) \leq 0$ in any interval where $y(t) < 0$ and $y'(t) > 0$ are nonoscillatory.

Proof. Let y(t) be a solution of (7) on $[T, \infty)$, $T \geq 0$ and satisfy the hypothesis of theorem. If possible, let y(t) be of z-type with consecutive double zeros at a and b ($T \leq a < b$). Integrating

$$[r(t)y''(t)y(t)]' = r(t)y'(t)y''(t) + p(t)y(t)h(y(t)) \tag{8}$$

from a to b, we get

$$0 = -\frac{1}{2}\int_a^b r'(t)(y'(t))^2dt - \int_a^b p(t)y(t)h(y(t))dt < 0$$

a contradiction.

Suppose that y(t) is oscillatory. Let a and b ($T \leq a < b$) be consecutive zeros of y(t) such that $y(t) < 0$ for $t \in (a, b)$ and $y'(a) \leq 0, y'(b) \geq 0$. So there exists a $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$. We consider two cases, viz., $y''(c) \leq 0$ and $y''(c) > 0$. Let $y''(c) \leq 0$. Integrating (8) from a to c, we get

$$r(c)y(c)y''(c) = -\frac{1}{2}r(a)(y'(a))^2 - \frac{1}{2}\int_a^c r'(t)(y'(t))^2dt - \int_a^c p(t)y(t)h(y(t))dt < 0$$

a contradiction. Let $y''(c) > 0$. Integrating (8) from c to b, we obtain

$$\begin{aligned} -r(c)y(c)y''(c) &= \int_c^b r(t)y'(t)y''(t)dt - \int_c^b p(t)y(t)h(y(t))dt \\ &\leq \int_c^b [r(t)y'(t)y''(t) - y(t)h(y(t))]dt \\ &< 0, \end{aligned}$$

a contradiction. Hence the theorem.

Following example illustrates Theorem 4.1.

Example. Consider $y''' + 8y = 0, t > 0$. Clearly $y(t) = -e^{-2t}$ satisfies the condition of the theorem and is a nonoscillatory solution of the equation.

Theorem 4.2. Consider equation (7). Let r(t) be once continuous differentiable such that $r'(t) \geq 0$. Let $\lim_{t\to\infty} \frac{1}{r(t)} \int_0^t p(s)ds = +\infty$ and $h(y)$ is decreasing. If y(t) is a solution of (7) such that $\lim_{t\to\infty} y(t) \neq 0$ when the limit exists , then y(t) is oscillatory.

Proof. Clearly y(t) cannot be of z-type. If possible, let y(t) be nonoscillatory. Without any loss of generality we can assume that y(t) is ultimately positive. Let $y(t) > 0$ for $t \geq t_0 > 0$. From (8) it follows that $y'(t)$ cannot be of z-type or oscillatory. If possible, let $y'(t) > 0$ for $t \geq t_1 \geq t_0$. Integrating (7) from t_1 to t , we get

$$y''(t) \leq \frac{r(t_1)y''(t_1)}{r(t)} - \frac{h(y(t_1))}{r(t)}\int_{t_1}^t p(s)ds.$$

If $y''(t_1) \leq 0$, then

$$y''(t) \leq -\frac{1}{r(t)} h(y(t_1)) \int_{t_1}^{t} p(s)ds.$$

If $y''(t_1) > 0$, then

$$y''(t) \leq y''(t_1) - \frac{h(y(t_1))}{r(t)} \int_{t_1}^{t} p(s)ds.$$

Hence in each case $\lim_{t\to\infty} y''(t) = -\infty$. This in turn implies that $y'(t) < 0$ for large t, a contradiction. Suppose that $y'(t) < 0$ for $t \geq t_1 \geq t_0$. So $\lim_{t\to\infty} y(t)$ exists . From the given hypothesis it follows that $\lim_{t\to\infty} y(t) \neq 0$. Now integrating (7) from t_1 to t , we get $\lim_{t\to\infty} y''(t) = -\infty$. From this it follows that $y(t) < 0$ for large t, a contradiction. Hence the theorem.

COROLLARY 4.3. Consider (7) with $r(t) \equiv 1$ and $p(t) \geq 1$. Let y(t) be a solution of (7) which satisfies the inequality $z'z'' - zh(z) \leq 0$ in any interval where $y(t) < 0$ and $y'(t) > 0$. If $\lim_{t\to\infty} y(t)$ exists, then $\lim_{t\to\infty} y(t) = 0$.

This follows from Theorems 4.1 and 4.2.

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EXTENDED CESÀRO OPERATORS FROM H^∞ TO ZYGMUND TYPE SPACES IN THE UNIT BALL

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Abstract: Let $H(B)$ denote the space of all holomorphic functions on the unit ball B of $\mathbb{C}^n$ and $\Re h(z) = \sum_{j=1}^n z_j \frac{\partial h}{\partial z_j}(z)$ the radial derivative of h. In this paper we investigate the boundedness and compactness of the extended Cesàro operator

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \quad f \in H(B), \quad z \in B,$$

from H^∞ to the Zygmund type space.

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1 Introduction

Let $H(B)$ be the space of all holomorphic functions in the unit ball B of $\mathbb{C}^n$. Denote by $H^\infty(B)$ the bounded holomorphic function space in B. For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|\alpha|\geq 0} a_\alpha z^\alpha$, where α is a multi-index, let $\Re f(z) = \sum_{|\alpha|\geq 0} |\alpha| a_\alpha z^\alpha$ stand for the radial derivative of $f \in H(B)$. It is well known that(see [20])

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

We write $\Re^m f = \Re(\Re^{m-1} f)$.

A positive continuous function μ on $[0,1)$ is called normal, if there exist positive numbers s and t, $0 < s < t$, such that(see, for example, [4])

$$\frac{\mu(r)}{(1-r)^s} \downarrow 0, \qquad \frac{\mu(r)}{(1-r)^t} \uparrow \infty \ \text{ as } \ r \to 1.$$

Recall that the Bloch space $\mathcal{B}(B)$, is the space of all $f \in H(B)$ such that([20])

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in B}(1-|z|^2)\,|\Re f(z)| < \infty.$$

The little Bloch space $\mathcal{B}_0(B)$ is the space of all $f \in H(B)$ such that

$$\lim_{|z|\to 1} (1-|z|^2)\,|\Re f(z)| = 0.$$

Let $\Lambda = \Lambda(B)$ denote the class of all $f \in H(B)$ for which

$$\sup_{z\in B}(1-|z|^2)|\Re^2 f(z)| < \infty. \tag{1}$$

From [20] we see that $f \in \Lambda$ if and only if $f \in A(B)$ and there exists a constant $C > 0$ such that

$$|f(\zeta + h) + f(\zeta - h) - 2f(\zeta)| < Ch,$$

for all $\zeta \in \partial B$ and $\zeta \pm h \in \partial B$, where $A(B)$ is the ball algebra on B. The quantity in (1) is only a semi norm. From [20] we see that class Λ with the following norm

$$\|f\|_\Lambda = |f(0)| + \sup_{z\in B}(1-|z|^2)|\Re^2 f(z)| \tag{2}$$

is a Banach space, which is called the Zygmund space. Let Λ_0 denote the closure in Λ of the set of all polynomials. From Theorem 7.12 of [20] we see that

$$f \in \Lambda_0 \quad \Leftrightarrow \quad \lim_{|z|\to 1}(1-|z|^2)|\Re^2 f(z)| = 0.$$

It is natural to generalize the Zygmund space to a more general form. Let μ be a normal function on $[0,1)$. Define the space $\Lambda_\mu = \Lambda_\mu(B)$, called the Zygmund type space, which consisting of all $f \in H(B)$ such that

$$\sup_{z\in B} \mu(|z|)|\Re^2 f(z)| < \infty. \tag{3}$$

Similarly, under the norm

$$\|f\|_{\Lambda_\mu} = |f(0)| + \sup_{z\in B} \mu(|z|)|\Re^2 f(z)|, \tag{4}$$

Λ_μ becomes a Banach space. Let $\Lambda_{\mu,0}$ denote the closure in Λ_μ of the set of all polynomials. Similar to Theorem 7.12 of [20] we see that

$$f \in \Lambda_{\mu,0} \quad \Leftrightarrow \quad \lim_{|z|\to 1} \mu(|z|)|\Re^2 f(z)| = 0. \tag{5}$$

Suppose that $g : B \to \mathbb{C}^1$ is a holomorphic map, define

$$T_g f(z) = \int_0^1 f(tz)\frac{dg(tz)}{dt} = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \qquad f \in H(B), \quad z \in B. \tag{6}$$

This operator is called the extended Cesàro operator (or the Riemann-Stieltjes operator), which was introduced in [4], and studied in [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19].

In this paper, we study the boundedness and compactness of the extended Cesàro operator T_g from H^∞ to the Zygmund type space.

Throughout the paper, constants are denoted by C, they are positive and may not be the same in every occurrence.

2 Main results and proofs

In this section, we give our main results and proofs. First, we give several auxiliary results which will be used in the proofs of our main results.

Lemma 1. *Assume that $g \in H(B)$ and μ is a normal function on $[0,1)$. Then $T_g : H^\infty \to \Lambda_\mu$ is compact if and only if $T_g : H^\infty \to \Lambda_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in H^∞ which converges to zero uniformly on compact subsets of B as $k \to \infty$, we have $\|T_g f_k\|_{\Lambda_\mu} \to 0$ as $k \to \infty$.*

Proof. The proof follows by standard arguments similar to those outlined in Proposition 3.11 of [2]. We omit the details. □

Lemma 2. [6] *For every $f, g \in H(B)$ it holds*

$$\Re[T_g(f)](z) = f(z)\Re g(z).$$

Lemma 3. *Let $f \in H^\infty(B)$. Then $f \in \mathcal{B}$. Moreover,*

$$\|f\|_{\mathcal{B}} \le C\|f\|_\infty.$$

Lemma 4. *A closed set K in $\Lambda_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(|z|)|\Re^2 f(z)| = 0.$$

Proof. The proof is similar to the proof of Lemma 1 in [15]. We omit the details. □

Theorem 1. *Assume that $g \in H(B)$ and μ is a normal function on $[0,1)$. Then $T_g : H^\infty \to \Lambda_\mu$ is bounded if and only if $g \in \Lambda_\mu$ and*

$$\sup_{z\in B} \frac{\mu(|z|)}{(1-|z|^2)}|\Re g(z)| < \infty. \tag{7}$$

Proof. Assume that $T_g : H^\infty \to \Lambda_\mu$ is bounded. Taking the function $f(z) = 1$, we see that $g \in \Lambda_\mu$. Let $f(z) = z$. For $a \in B$, set

$$f_a(z) = \frac{1-|a|^2}{1-\langle z,a\rangle}. \tag{8}$$

It is easy to see that $f_a \in H^\infty$. By Lemma 2 we have

$$\begin{aligned}
\infty > \|T_g f_a\|_{\Lambda_\mu} &\ge \sup_{z\in B} \mu(|z|)|\Re^2(T_g f_a)(z)| = \sup_{z\in B} \mu(|z|)|\Re(f_a \cdot \Re g)(z)| \\
&\ge \sup_{z\in B} \mu(|z|)|\Re f_a(z)\Re g(z) + f_a(z)\Re^2 g(z)| \\
&\ge \mu(|a|)|\Re f_a(a)\Re g(a) + f_a(a)\Re^2 g(a)| \\
&\ge \mu(|a|)|\Re f_a(a)\Re g(a)| - \mu(|a|)|\Re^2 g(a)| \\
&= \frac{\mu(|a|)|a|^2}{1-|a|^2}|\Re g(a)| - \mu(|a|)|\Re^2 g(a)|.
\end{aligned} \tag{9}$$

Since $g \in \Lambda_\mu$, from (9) we see that

$$\begin{aligned}\sup_{|a|>\frac{1}{2}} \frac{\mu(|a|)}{1-|a|^2}|\Re g(a)| &< 4 \sup_{|a|>\frac{1}{2}} \frac{\mu(|a|)|a|^2}{1-|a|^2}|\Re g(a)| \\ &\leq 4\|T_g f_a\|_{\Lambda_\mu} + 4 \sup_{a\in B} \mu(|a|)|\Re^2 g(a)| < \infty. \quad (10)\end{aligned}$$

Since(see [18])

$$\sup_{0<r\leq 1/2} \mu(r) \leq C\mu(1/2), \quad (11)$$

we get

$$\begin{aligned}\sup_{|a|\leq\frac{1}{2}} \frac{\mu(|a|)}{1-|a|^2}|\Re g(a)| &< \frac{4}{3}\mu(1/2) \sup_{|a|\leq\frac{1}{2}} |\Re g(a)| \\ &< C\mu(1/2) \sup_{|a|\leq(1+\frac{1}{2})/2} |g(a)| < \infty. \quad (12)\end{aligned}$$

Combining (10) with (12) we get (7).

Conversely, assume that $g \in \Lambda_\mu$ and (7) holds. For an $f \in H^\infty$, using Lemmas 2 and 3 we have

$$\begin{aligned}\mu(|z|)|\Re^2(T_g f)(z)| &= \mu(|z|)|\Re(f \cdot \Re g)(z)| \\ &= \mu(|z|)|\Re f(z)\Re g(z) + f(z)\Re^2 g(z)| \\ &\leq C\|f\|_{\mathcal{B}} \frac{\mu(|z|)}{1-|z|^2}|\Re g(z)| + C\|f\|_\infty \mu(|z|)|\Re^2 g(z)| \\ &\leq C\|f\|_\infty \frac{\mu(|z|)}{1-|z|^2}|\Re g(z)| + C\|f\|_\infty \mu(|z|)|\Re^2 g(z)|. \quad (13)\end{aligned}$$

On the other hand, we have $T_g(f)(0) = 0$. From these, by taking the supremum in (13) over B, and using $g \in \Lambda_\mu$ and (7) the boundedness of the operator $T_g : H^\infty \to \Lambda_\mu$ follows. □

Theorem 2. *Assume that $g \in H(B)$ and μ is a normal function on $[0, 1)$. Then $T_g : H^\infty \to \Lambda_\mu$ is compact if and only if $T_g : H^\infty \to \Lambda_\mu$ is bounded and $g \in \Lambda_{\mu,0}$ and*

$$\lim_{|z|\to 1} \frac{\mu(|z|)}{1-|z|^2}|\Re g(z)| = 0. \quad (14)$$

Proof. Assume that $T_g : H^\infty \to \Lambda_\mu$ is compact. Then it is clear that $T_g : H^\infty \to \Lambda_\mu$ is bounded.

Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $\lim_{k\to\infty} |z_k| = 0$. Set

$$f_k(z) = \frac{(1-|z_k|^2)^2}{(1-\langle z, z_k\rangle)^2} - \frac{1-|z_k|^2}{1-\langle z, z_k\rangle}. \quad (15)$$

It is easy to see that $f_k \in H^\infty$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By Lemma 1, it holds

$$\lim_{k\to\infty} \|T_g f_k\|_{\Lambda_\mu} = 0. \tag{16}$$

By Lemma 2 we have

$$\begin{aligned} \|T_g f_k\|_{\Lambda_\mu} &\geq \sup_{z\in B} \mu(|z|)|\Re f_k(z)\Re g(z) + f_k(z)\Re^2 g(z)| \\ &\geq \mu(|z_k|)|\Re f_k(z_k)\Re g(z_k) + f_k(z_k)\Re^2 g(z_k)| \\ &= \mu(|z_k|)|\Re g(z_k)|\frac{|z_k|^2}{1-|z_k|^2}. \end{aligned} \tag{17}$$

From (16) and (17) we see that

$$\lim_{k\to\infty} \frac{\mu(|z_k|)}{1-|z_k|^2}|\Re g(z_k)| = \lim_{k\to\infty} \mu(|z_k|)|\Re g(z_k)|\frac{|z_k|^2}{1-|z_k|^2} = 0, \tag{18}$$

which means that (14) holds.

Now set

$$h_k(z) = \frac{1-|z_k|^2}{1-\langle z, z_k\rangle}. \tag{19}$$

Then $h_k \in H^\infty$ and $h_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By Lemma 1, it holds

$$\lim_{k\to\infty} \|T_g h_k\|_{\Lambda_\mu} = 0. \tag{20}$$

By Lemma 2 we have

$$\begin{aligned} \|T_g h_k\|_{\Lambda_\mu} &\geq \sup_{z\in B} \mu(|z|)|\Re h_k(z)\Re g(z) + h_k(z)\Re^2 g(z)| \\ &\geq \mu(|z_k|)|\Re h_k(z_k)\Re g(z_k) + h_k(z_k)\Re^2 g(z_k)| \\ &\geq \mu(|z_k|)|\Re^2 g(z_k)| - \mu(|z_k|)|\Re g(z_k)|\frac{|z_k|^2}{1-|z_k|^2}. \end{aligned} \tag{21}$$

From (17), (20) and (21) we get

$$\lim_{k\to\infty} \mu(|z_k|)|\Re^2 g(z_k)| = 0, \tag{22}$$

which implies that $g \in \Lambda_{\mu,0}$.

Now assume that $T_g : H^\infty \to \Lambda_\mu$ is bounded and $g \in \Lambda_{\mu,0}$ and (14) holds. Since $T_g : H^\infty \to \Lambda_\mu$ is bounded, from the proof of Theorem 1 we see that

$$M_1 = \sup_{z\in B} \mu(|z|)|\Re^2 g(z)| < \infty \quad \text{and} \quad M_2 = \sup_{z\in B} \frac{\mu(|z|)}{1-|z|^2}|\Re g(z)| < \infty. \tag{23}$$

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in H^∞ such that $\sup_{k\in\mathbb{N}}\|f_k\|_\infty \le L$ and that $f_k \to 0$ uniformly on compact subsets of B as $k\to\infty$. Now note that for every $\varepsilon>0$, there is a $\delta\in(0,1)$, such that

$$\mu(|z|)|\Re^2 g(z)|<\varepsilon \quad \text{and} \quad \frac{\mu(|z|)}{1-|z|^2}|\Re g(z)|<\varepsilon, \tag{24}$$

whenever $\delta<|z|<1$. Let $K=\{z\in B:|z|\le\delta\}$. Note that K is a compact subset of B. In view of (23-24) and $T_g f_k(0)=0$, we have

$$\begin{aligned}
&\|T_g f_k\|_{\Lambda_\mu} = |f_k(0)||\Re g(0)| + \sup_{z\in B}\mu(|z|)|\Re^2(T_g f)(z)| \\
= \;& |f_k(0)||\Re g(0)| + \sup_{z\in B}\mu(|z|)|\Re f_k(z)\Re g(z) + f_k(z)\Re^2 g(z)| \\
\le \;& |f_k(0)||\Re g(0)| + \sup_{z\in K}\mu(|z|)|\Re f_k(z)\Re g(z)| + \sup_{z\in B\setminus K}\mu(|z|)|\Re f_k(z)\Re g(z)| \\
& + \sup_{z\in K}\mu(|z|)|f_k(z)\Re^2 g(z)| + \sup_{z\in B\setminus K}\mu(|z|)|f_k(z)\Re^2 g(z)| \\
\le \;& |f_k(0)||\Re g(0)| + M_2\sup_{z\in K}|\Re f_k(z)| + 2\varepsilon\|f_k\|_\infty + M_1\sup_{z\in K}|f_k(z)|.
\end{aligned} \tag{25}$$

Since $f_k\to 0$ uniformly on compact subsets of B, it follows from Cauchy's estimate that $\Re f_k\to 0$ uniformly on compact subsets of B, in particular on K. Using this and that ε is an arbitrary positive number, by letting $k\to\infty$ in (25), it shows that $\lim_{k\to\infty}\|T_g f_k\|_{\Lambda_\mu}=0$. According to Lemma 1, the compactness of the operator $T_g: H^\infty\to\Lambda_\mu$ follows. □

Theorem 3. *Assume that $g\in H(B)$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent.*

(a) $T_g: H^\infty\to\Lambda_{\mu,0}$ is bounded;

(b) $T_g: H^\infty\to\Lambda_{\mu,0}$ is compact;

(c) $g\in\Lambda_{\mu,0}$ and

$$\lim_{|z|\to 1}\frac{\mu(|z|)|\Re g(z)|}{1-|z|^2}=0. \tag{26}$$

Proof. $(b)\Rightarrow(a)$ is obvious.

$(a)\Rightarrow(c)$ Assume that $T_g: H^\infty\to\Lambda_{\mu,0}$ is bounded. Taking $f(z)=1$, we get $g\in\Lambda_{\mu,0}$.

Now we prove that (26) holds. Assume to the contrary that there is a sequence $(z^{(k)})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}|z^{(k)}|=1$ and

$$\frac{\mu(|z^{(k)}|)|\Re g(z^{(k)})|}{1-|z^{(k)}|^2}\ge\varepsilon_0>0.$$

Without loss of generality we assume that $(z^{(k)})\to(1,0,\dots,0)$ as $k\to\infty$ and that $(1-|z^{(k)}|^2)\ge\frac{1}{2}(1-|z_1^{(k)}|^2)$.

We may also assume that the sequence $(z_1^{(k)})_{k\in\mathbb{N}}$ is an interpolating sequence on the unit disk $\mathbb{D}$, that is, there exists a $\delta > 0$ such that

$$\inf_{k\in\mathbb{N}} \prod_{m\neq k} \frac{|z_1^{(k)} - z_1^{(m)}|}{|1 - \overline{z_1^{(m)}} z_1^{(k)}|} > \delta > 0.$$

It is well known that the Blaschke product(see [3])

$$b(z) = \prod_{m=1}^{\infty} \frac{z - z_1^{(m)}}{1 - \overline{z_1^{(m)}} z}, \quad z \in \mathbb{D},$$

is a holomorphic function on $\mathbb{D}$ and converges uniformly on compact subsets. Moreover(see [16])

$$(1 - |z_1^{(k)}|^2)|b'(z_1^{(k)})| = \prod_{m\neq k} \frac{|z_1^{(k)} - z_1^{(m)}|}{|1 - \overline{z_1^{(m)}} z_1^{(k)}|} \geq \delta > 0.$$

Let $f(z) = b(z_1)$. Then $f \in H^\infty$. Hence for sufficiently large k, we have

$$\begin{aligned}
\mu(|z^{(k)}|)|\Re^2(T_g f)(z^{(k)})| &= \mu(|z^{(k)}|)|\Re f(z^{(k)})\Re g(z^{(k)}) + f(z^{(k)})\Re^2 g(z^{(k)})| \\
&= \mu(|z^{(k)}|)|\Re f(z^{(k)})\Re g(z^{(k)})| \\
&= \frac{\mu(|z^{(k)}|)}{1 - |z^{(k)}|^2}|\Re g(z^{(k)})|(1 - |z_1^{(k)}|^2)|z_1^{(k)} b'(z_1^{(k)})| \\
&\geq \delta\varepsilon_0 > 0.
\end{aligned}$$

Since $T_g(f) \in \Lambda_{\mu,0}$, it follows that

$$\lim_{k\to\infty} \mu(|z^{(k)}|)|\Re^2(T_g f)(z^{(k)})| = 0,$$

which is a contradiction. It follows that (26) holds.

$(c) \Rightarrow (b)$ Assume that (c) holds. Then, for any $f \in H^\infty$, from (13) we have

$$\begin{aligned}
\mu(|z|)|\Re^2(T_g f)(z)| &= \mu(|z|)|\Re f(z)\Re g(z) + f(z)\Re^2 g(z)| \\
&\leq C\|f\|_\infty \frac{\mu(|z|)|\Re g(z)|}{1 - |z|^2} + \|f\|_\infty \mu(|z|)|\Re^2 g(z)|.
\end{aligned}$$

Employing Lemma 4 and the condition (c), the compactness of the operator $T_g : H^\infty \to \Lambda_{\mu,0}$ follows. The proof of the theorem is completed. □

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An optimal order error estimate of a linear finite element method for smooth solutions of 2D systems of conservation laws

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Abstract. In this paper we consider approximating smooth solutions of systems of nonlinear conservation laws by a linear finite element method with uniform mesh in two spatial dimensions, where the time discretization is carried out by a second order explicit Runge-Kutta method. An optimal error estimate $O(h^2)$ in L^2-norm for continuous linear finite elements is obtained under the CFL condition $\Delta t \leq Ch^{\frac{4}{3}}$, where h and Δt are the spatial meshsize and the time step, respectively, and the positive constant C is independent of h and Δt.

Keywords. finite element method, hyperbolic conservation laws, error estimates.

1 Introduction

The numerical solution of nonlinear hyperbolic conservation laws is an important but difficult problem. The main difficulty is due to the nonlinearity. In recent years, numerical methods for nonlinear multi-dimensional hyperbolic conservation laws on unstructured grids, and some related analysis of their convergence behavior, have attracted extensive attention. The numerical methods include: finite volume schemes [2, 7, 8], shock capturing streamline diffusion finite element method [9, 10], and Rung-Kutta discontinuous Galerkin finite element (RKDG) methods, [3, 4, 5], etc.

Runge-Kutta discontinuous Galerkin (RKDG) method was developed by Cockburn et al. [3, 4, 5] for solving nonlinear hyperbolic conservation laws. It has important advantages in its stability and highly parallelizable structure. When the scheme was extended to the multi-dimensional case, the construction of the generalized 'slope limiter' represented a serious challenge. The purpose of the slope limiter is to enforce the nonlinear stability of the scheme. In the multi-dimensional case, the constraints imposed by nonlinear stability on the accuracy of a scheme are even greater than that in the one-dimensional case. In [14] and [15] respectively, error estimates are obtained for RKDG methods to sufficiently smooth solutions in scalar conservation laws and symmetrizable systems of conservation laws. In [15], time discretization is the second order explicit TVD (total variation diminishing) Runge-Kutta method, and the $\bar{\mathbb{P}}^k$ (discontinuous piecewise polynomials of degree $\leq k$) finite element is implemented. Error estimate in the $\bar{\mathbb{P}}^1$ finite element space is obtained under CFL condition $\tau \leq \beta h$ for nonlinear systems in one dimension and for linear systems in the multi-dimensional case, where h and τ are the maximum element lengths and time

steps respectively; β is a positive constant independent of h and τ. Error estimates for $\bar{\mathbb{P}}^k$ $(k > 1)$ finite element spaces are obtained under further CFL condition.

Both finite element and finite volume methods are suitable for unstructured meshes; finite volume methods are dominant in upwinding effects. These methods can therefore resolve strong discontinuities without exhibiting oscillations. High order finite volume schemes can be extended to multi-dimensional systems of conservation laws. At present, error estimates of the form $\|u(,t) - u_h(,t)\|_{L^1(R^2)} \leq ch^\alpha$, where α is sufficiently large, for the numerical solution to high order finite volume schemes have been established. In [8], LeVeque pointed out that the error estimate is at most $O(h^{\frac{1}{2}})$.

A shock-capturing streamline diffusion finite element method in [9, 10] is a general finite element method for hyperbolic problems which may be regarded as a combination of a standard Galerkin finite element method and a least squares method, giving added stability through the weighted least squares control of the residual. It combines $O(h^{k+\frac{1}{2}})$ accuracy for smooth solutions approximated by polynomials of degree k, with good stability obtained through the least squares control of the residual and the shocking-capturing artificial viscosity. This method is very efficient in solving scalar hyperbolic conservation laws, but it has not been extended to systems.

At present, general statements are not available for error estimates of smooth solutions to systems of conservation laws for both finite volume and a shock-capturing streamline diffusion finite element methods.

In this paper, we consider error estimates for a general finite element method based upon that of Ying et al. [13]. The method implements second order Runge-Kutta scheme for the temporal discretization and continuous piecewise linear finite elements for the spatial discretization. We employ an artificial viscosity term which is at most first order accurate in the neighborhood of shock waves, but is at least second order accurate away from shock waves. The numerical flux functions can be reduced to one-dimensional calculations along the edges of elements by introducing integrating factors ([11]). Below we denote by C a positive constant independent of h and Δt, not necessarily the same at each occurrence. We obtain an optimal second order error estimate in L^2-norm under the restrictive CFL condition $\Delta t \leq Ch^{\frac{4}{3}}$. The main techniques used in this paper, such as Taylor expansions, energy analysis, superapproximation estimate, hyperbolic and parabolic properties of systems and the *a priori* assumption, play important roles in error estimates.

The scheme is efficient for solving convection dominated problems and discontinuous solutions of multi-dimensional hyperbolic conservation laws, where the threshold parameter, C_0, used for measuring the magnitude of gradients of numerical solutions, can be taken as a kind of limiter. The definition of C_0 is further discussed in Step 3 of §3, and unchanged between one-dimensional and multi-dimensional cases. Numerical computations with this method [13] demonstrate the resolution of sharp shock transitions with no oscillations and accurate approximate solutions in smooth areas. The error analysis in [12], which proved second order accuracy for smooth solutions, can be extended straightforwardly to symmetric systems with the same results. However the analysis cannot be extended to symmetrizable systems (such as 2D fully compressible Euler equations). The analysis shown here is more difficult than that in symmetric case, where a norm condition (condition (f) in Theorem 3.1) is added under which we present that second order accuracy is maintained for systems.

The rest of the paper is organized as follows. In §2, we discuss the general finite element scheme. In §3, we examine the truncation error and prove our main result, Theorem 3.1.

2 The finite element scheme

We consider systems of conservation laws of the form

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{N} \frac{\partial \mathbf{f}_i(\mathbf{u})}{\partial x_i} = 0, \qquad (\mathbf{x}, t) \in \mathbb{R}^N \times (0, T^*], \tag{1}$$

where T^* is an arbitrary positive constant. Here, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) : \mathbb{R}^N \times (0, T^*] \to \mathbb{R}^m$ is the vector of dependent solution variables, and $\mathbf{f}_i(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^m$, $i = 1, 2, \ldots, N$, is the vector-valued flux function. Let $\mathbf{f}(\mathbf{u}) = (\mathbf{f}_1(\mathbf{u}), \mathbf{f}_2(\mathbf{u}), \ldots, \mathbf{f}_N(\mathbf{u}))$ denote the matrix composed of flux-vector columns. Without loss of generality, we assume that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, otherwise $\mathbf{f}(\mathbf{u})$ can be replaced by $\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{0})$. We assume (1) is strictly hyperbolic. Let us consider the Cauchy problem with the initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0. \tag{2}$$

By adding artificial viscosity, the system becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{N} \frac{\partial \mathbf{f}_i(\mathbf{u})}{\partial x_i} = \nabla \cdot (\varepsilon \nabla \mathbf{u}). \tag{3}$$

In a numerical implementation, ε may vary from grid to grid. Based on [6], the weak formulation of the initial value problem (2), (3) of the system is:

Find $\mathbf{u} = (u_1(\mathbf{x}, t), \ldots, u_m(\mathbf{x}, t))^T \in L^2(0, T^*; H^1(\mathbb{R}^N; \mathbb{R}^m))$ with $\mathbf{u}' \in L^2(0, T^*; H^{-1}(\mathbb{R}^N; \mathbb{R}^m))$, such that

$$\frac{d}{dt} \int_{\mathbb{R}^N} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\mathbb{R}^N} \sum_{j=1}^{N} \left(\varepsilon \frac{\partial \mathbf{u}}{\partial x_j} - \mathbf{f}_j(\mathbf{u}) \right) \cdot \frac{\partial \mathbf{v}}{\partial x_j} \, d\mathbf{x} = 0, \tag{4}$$

$$\int_{\mathbb{R}^N} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} \Big|_{t=0} = \int_{\mathbb{R}^N} \mathbf{u}_0 \cdot \mathbf{v} \, d\mathbf{x}, \tag{5}$$

for all $\mathbf{v} = (v_1, \ldots, v_m)^T \in H^1(\mathbb{R}^N; \mathbb{R}^m)$.
Defining the flux matrix $\mathbf{J}(\mathbf{u}; \varepsilon) = \varepsilon \nabla \mathbf{u} - \mathbf{f}(\mathbf{u})$, (i.e. $J_{kj} = \varepsilon \partial u_k / \partial x_j - f_{kj}$), (4) can be written

$$\frac{d}{dt} \int_{\mathbb{R}^N} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\mathbb{R}^N} \sum_{k=1}^{m} \sum_{j=1}^{N} J_{kj} \frac{\partial v_k}{\partial x_j} \, d\mathbf{x} = 0. \tag{6}$$

Let $\mathbb{R}^N$ be partitioned into simplex finite elements by regular triangulation, a typical element being denoted by T. Assume $h_T / \rho_T \leq C$ for any element T, where ρ_T is the supremum of the diameters of all balls contained in T, and the diameter of T is denoted by h_T. Let $h = \max_T h_T$, $\theta_E^T \leq \frac{\pi}{2}$, where θ_E^T is the angle between the faces $\mathbf{F_i}$ and $\mathbf{F_j}$, where $\mathbf{F_i}$, $\mathbf{F_j}$ are $N-1$ dimensional simplices opposite to the vertices $\mathbf{X_i}$ and $\mathbf{X_j}$. We also introduce the following notations: let the nodes be $\mathbf{X_j}$, and $\mathbf{E_{ij}}$ be an edge connecting nodes $\mathbf{X_i}$ and $\mathbf{X_j}$; h_{ij} is the length of $\mathbf{E_{ij}}$ and $\tau_{\mathbf{ij}}$ is the unit directional vector pointing from $\mathbf{X_i}$ to $\mathbf{X_j}$. We will also carry over the use of $\mathbf{E_{ij}}$, h_{ij} and $\tau_{\mathbf{ij}}$ locally to T. We denote the shape functions on each T by $\varphi_i(\mathbf{x}) : \mathbb{R}^{\mathbf{N}} \to \mathbb{R}$, $i = 1, 2, ...N+1$, which are the linear interpolation functions with respect to $N+1$ vertices on each T, satisfying

that $\varphi_i(\mathbf{X_j}) = \delta_{ij}$, $i, j = 1, 2, 3, \ldots, N+1$. The shape functions are linear and continuous on each T.

For a given time step Δt, we denote the approximate solution, which is linear on each T and continuous, by $\mathbf{u_h}$. Let $\mathbf{u}_h^n = \mathbf{u}_h(\mathbf{x}, n\Delta t)$ and $\mathbf{u}_i^n = \mathbf{u}_h(\mathbf{X_i}, n\Delta t)$. We develop an explicit scheme to obtain $\mathbf{u}_i^{n+1} = \mathbf{u}_h(\mathbf{X_i}, (n+1)\Delta t)$.

In [12], to guarantee the maximum principle holds and to ensure a second order scheme, elements are divided into two categories, where we expand the technique into systems case, labeled $\mathcal{T}_{i,1}$ and $\mathcal{T}_{i,2}$. For a given node $\mathbf{X_i}$, we let $\mathcal{T}_i$ denote the set of elements neighboring $\mathbf{X_i}$. These elements define the region $\Omega_i = \underset{T\in\mathcal{T}_i}{\cup} T$. For any $T \in \mathcal{T}_i$, let I_T denote the index of the nodes of T excluding $\mathbf{X_i}$. At any fixed time value, the categories $\mathcal{T}_{i,1}$ and $\mathcal{T}_{i,2}$ are defined relative to a fixed constant threshold $C_0 > 0$ as follows:

$$\begin{array}{rcl} \mathcal{T}_{i,1} & = & \{T \mid T \in \mathcal{T}_i, |\nabla \mathbf{u}_h| > C_0\}, \\ \mathcal{T}_{i,2} & = & \{T \mid T \in \mathcal{T}_i, |\nabla \mathbf{u}_h| \le C_0\}. \end{array} \tag{7}$$

Elements in $\mathcal{T}_{i,1}$ are located near shock waves; let $\varepsilon = \varepsilon_1$ be as same order as h in $\mathcal{T}_{i,1}$. Elements in $\mathcal{T}_{i,2}$ are located in regions where the solution is smooth; let $\varepsilon = \varepsilon_2 \le Ch^2$ in $\mathcal{T}_{i,2}$.

We apply the weak solution formulation (6) to Ω_i, requiring that the formulation hold for each of the basis functions $\mathbf{v}_l(\mathbf{x}) = \mathbf{e}_l\varphi(\mathbf{x})$, $l = 1, \ldots, m$, where $\mathbf{e}_l = (0, \ldots, 0, 1, 0 \ldots, 0)$ is the l-th unit vector in $\mathbb{R}^m$. The resultant system of m equations, with the flux integrals split into contributions from $\mathcal{T}_{i,1}$ and $\mathcal{T}_{i,2}$, is

$$\frac{d}{dt}\int_{\mathbb{R}^N} \mathbf{u}\varphi_i d\mathbf{x} + \sum_{T\in\mathcal{T}_{i,1}} \int_T \mathbf{J}(\mathbf{u};\varepsilon_1)\cdot\nabla\varphi_i d\mathbf{x} + \sum_{T\in\mathcal{T}_{i,2}} \int_T \mathbf{J}(\mathbf{u};\varepsilon_2)\cdot\nabla\varphi_i d\mathbf{x} = 0. \tag{8}$$

We quote two elementary formulae from [11] which will simplify handling the integration terms on $\mathcal{T}_{i,1}$ and some of the terms on $\mathcal{T}_{i,2}$. Let $\mathbb{P}^1(T)$ be the space of linear polynomials on the element T. We have the following results.

(a) If $u, v \in \mathbb{P}^1(T)$, then

$$\int_T \nabla u \cdot \nabla v d\mathbf{x} = \sum_{i<j} a_{ij}^T (u_i - u_j)(v_j - v_i), \tag{9}$$

where u_i, v_i, u_j, v_j are the values of u, v at node $\mathbf{X_i}, \mathbf{X_j}$ respectively, and

$$a_{ij}^T = \int_T \nabla\varphi_i \cdot \nabla\varphi_j d\mathbf{x}, i, j = 1, 2, 3, ..., N+1. \tag{10}$$

(b) If $v \in \mathbb{P}^1(T)$ and $\mathbf{c}$ is a constant vector, then

$$\int_T \mathbf{c}\cdot\nabla v\, d\mathbf{x} = \sum_{i<j} a_{ij}^T\ \mathbf{c}\cdot\tau_{\mathbf{ij}}\ h_{ij}(v_i - v_j). \tag{11}$$

Requiring that approximate solution $\mathbf{u}_h$ satisfies (8), we can use (9) to handle the $\varepsilon\nabla\mathbf{u}_h$ integrand terms on $\mathcal{T}_{i,2}$. Further, assuming that $\mathbf{J}(\mathbf{u_h};\varepsilon_1)$ is approximately constant matrix on

simplices in $\mathcal{T}_{i,1}$, we use (11) to approximate the integration terms on $\mathcal{T}_{i,1}$. As a result, we can write the spatially semi-discretized scheme

$$\int_{\Omega_i} \frac{d\mathbf{u}_h^n}{dt}\varphi_i d\mathbf{x} + \sum_{T\in\mathcal{T}_{i,1}}\sum_{j\in I_T} a_{ij}^T \mathbf{J}(\mathbf{u}_h^n;\varepsilon_1)\cdot\tau_{\mathbf{ij}} h_{ij} + \sum_{T\in\mathcal{T}_{i,2}}\left\{\sum_{j\in I_T}\varepsilon_2 a_{ij}^T(\mathbf{u}_j^n-\mathbf{u}_i^n) - \int_T \mathbf{f}(\mathbf{u}_h^n)\cdot\nabla\varphi_i d\mathbf{x}\right\} = 0\,. \quad (12)$$

To handle the first term on the left hand side of (12), we use standard "mass lumping" quadrature and obtain

$$\int_{\Omega_i} \frac{d\mathbf{u}_h^n}{dt}\varphi_i d\mathbf{x} \approx A_i\left(\frac{d\mathbf{u}_h}{dt}\right)_i^n, \quad (13)$$

where $A_i = \int_{\Omega_i}\varphi_i(\mathbf{x})\mathbf{dx}$. For future reference in §3 we note that $A_i = O(h^2)$.

Before discussing the time discretization, we discuss the computation of the "edge projection" terms $\mathbf{J}\cdot\tau_{\mathbf{ij}}$.

We further assume $\mathbf{f}(\mathbf{u})\cdot\tau_{\mathbf{ij}} = \mathbf{A}(\mathbf{u})\cdot\mathbf{u}$, where $A(\mathbf{u})$ is an $m\times m$ matrix. This holds true for many systems, e.g. 2D fully compressible Euler equations of gas dynamics, where

$$\mathbf{u} = \begin{pmatrix}\rho\\ \rho u\\ \rho v\\ e\end{pmatrix}, f(\mathbf{u})\cdot\tau_{\mathbf{ij}} = \begin{pmatrix} (u,v)\cdot\tau_{ij} & 0 & 0 & 0\\ (\frac{p}{\rho},0)\cdot\tau_{\mathbf{ij}} & (u,v)\cdot\tau_{\mathbf{ij}} & 0 & 0\\ (0,\frac{p}{\rho})\cdot\tau_{\mathbf{ij}} & 0 & (u,v)\cdot\tau_{\mathbf{ij}} & 0\\ 0 & (\frac{p}{\rho},0)\cdot\tau_{\mathbf{ij}} & (0,\frac{p}{\rho})\cdot\tau_{\mathbf{ij}} & (u,v)\cdot\tau_{\mathbf{ij}}\end{pmatrix}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ e\end{pmatrix} = \mathbf{A}(\mathbf{u})\cdot\mathbf{u}.$$

By definition,

$$\mathbf{J}\cdot\tau_{\mathbf{ij}} = \varepsilon_1\frac{\partial\mathbf{u}}{\partial\tau_{\mathbf{ij}}} - \mathbf{A}(\mathbf{u})\cdot\mathbf{u}\,. \quad (14)$$

Considering (14) as an ODE system along $\mathbf{E_{ij}}$, we introduce the integrating factor (matrix) $\mathbf{Y}$ for the right hand side of (14):

$$\begin{cases}\dfrac{d\mathbf{Y}}{d\tau_{\mathbf{ij}}} + \mathbf{Y}\cdot\dfrac{\mathbf{A}(\mathbf{u})}{\varepsilon_1} = 0,\\ \\ \mathbf{Y}(\mathbf{X_{ij}}) = \mathbf{I},\end{cases} \quad (15)$$

where $\mathbf{I}$ is the $m\times m$ identity matrix, and $\mathbf{X_{ij}}$ denotes the midpoint of $\mathbf{E_{ij}}$. $\mathbf{Y}$ is related to $\mathbf{J}$ along $\mathbf{E_{ij}}$ via

$$\frac{\partial(\mathbf{Y}\cdot\mathbf{u})}{\partial\tau_{\mathbf{ij}}} = \mathbf{Y}\cdot\frac{\partial\mathbf{u}}{\partial\tau_{\mathbf{ij}}} + \frac{\partial\mathbf{Y}}{\partial\tau_{\mathbf{ij}}}\cdot\mathbf{u} = \mathbf{Y}\cdot\left(\frac{\partial\mathbf{u}}{\partial\tau_{\mathbf{ij}}} - \frac{\mathbf{A}(\mathbf{u})\cdot\mathbf{u}}{\varepsilon_1}\right) = \frac{\mathbf{Y}\cdot\mathbf{J}\cdot\tau_{\mathbf{ij}}}{\varepsilon_1}\,. \quad (16)$$

Integrating along $\mathbf{E_{ij}}$ gives

$$(\mathbf{Y}\cdot\mathbf{u})_j - (\mathbf{Y}\cdot\mathbf{u})_i = \frac{1}{\varepsilon_1}\int_{\mathbf{X_i}}^{\mathbf{X_j}}\mathbf{Y}\cdot\mathbf{J}\cdot\tau_{\mathbf{ij}}\,ds. \quad (17)$$

Assuming $\mathbf{J}\cdot\tau_{\mathbf{ij}}$ is constant on $\mathbf{E_{ij}}$, we have

$$\mathbf{J}\cdot\tau_{\mathbf{ij}} \approx \varepsilon_1\left(\int_{\mathbf{X_i}}^{\mathbf{X_j}}\mathbf{Y}ds\right)^{-1}\cdot((\mathbf{Y}\cdot\mathbf{u})_j - (\mathbf{Y}\cdot\mathbf{u})_i)\,. \quad (18)$$

In the numerical scheme, $\mathbf{u}$ varies linearly, so (15) is a variable coefficient system. To simply its solution, we rewrite (15) in the equivalent form

$$\frac{d\mathbf{Y}}{d\tau_{\mathbf{ij}}} + \mathbf{Y}\cdot\frac{\mathbf{A}(\mathbf{u}_{ij})}{\varepsilon_1} = \mathbf{Y}\cdot\left(\frac{\mathbf{A}(\mathbf{u}_{ij})}{\varepsilon_1} - \frac{\mathbf{A}(\mathbf{u})}{\varepsilon_1}\right), \tag{19}$$

where $\mathbf{u}_{ij} = \mathbf{u}(\mathbf{X_{ij}})$. If we replace $\mathbf{Y}$ in the right hand of (19) with the solution of the constant coefficient ODE

$$\frac{d\mathbf{Y}}{d\tau_{\mathbf{ij}}} + \frac{\mathbf{Y}\cdot\mathbf{A}(\mathbf{u}_{ij})}{\varepsilon_1} = 0, \tag{20}$$

we have the approximate representation for $\mathbf{Y}$ as follows: let $x = (\mathbf{X}-\mathbf{X_i})\cdot\tau_{\mathbf{ij}}$, $x_{ij} = (\mathbf{X_{ij}}-\mathbf{X_i})\cdot\tau_{\mathbf{ij}}$, then we notice the point $\mathbf{X}$ is varying along $\mathbf{E_{ij}}$ and

$$\mathbf{Y} = e^{-(x-x_{ij})\frac{\mathbf{A}(\mathbf{u}_{ij})}{\varepsilon_1}}\cdot\left\{I + \int_{x_{ij}}^{x} e^{-(s-x_{ij})\frac{\mathbf{A}(\mathbf{u}_{ij})}{\varepsilon_1}}\cdot\left(\frac{\mathbf{A}(\mathbf{u}_{ij})}{\varepsilon_1} - \frac{\mathbf{A}(\mathbf{u})}{\varepsilon_1}\right)\cdot e^{(s-x_{ij})\frac{\mathbf{A}(\mathbf{u}_{ij})}{\varepsilon_1}}\,ds\right\}.$$

Completing the discussion of the discretization, we use second order Runge-Kutta for the time discretization, arriving at the fully discrete scheme

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n + \frac{1}{2}(\mathbf{K}(\mathbf{u}_i^n) + \mathbf{L}(\mathbf{w}_i^n)). \tag{21}$$

In (21),

$$\begin{aligned}\mathbf{K}_i^n = \mathbf{K}(\mathbf{u}_i^n) &= -\tfrac{\Delta t}{A_i}\sum_{T\in\mathcal{T}_{i,1}}\sum_{j\in I_T} a_{ij}^T\mathbf{J}(\mathbf{u}_i^n,\mathbf{u}_j^n;\varepsilon_1)\cdot\tau_{\mathbf{ij}}h_{ij}\\ &\quad -\tfrac{\Delta t}{A_i}\sum_{T\in\mathcal{T}_{i,2}}\left\{\sum_{j\in I_T}\varepsilon_2 a_{ij}^T(\mathbf{u}_j^n-\mathbf{u}_i^n) - \int_T \mathbf{f}(\mathbf{u}^n)\cdot\nabla\varphi_i d\mathbf{x}\right\},\end{aligned} \tag{22}$$

and

$$\mathbf{w}_j^n = \mathbf{u}_j^n + \mathbf{K}_j^n, \quad \forall j. \tag{23}$$

Let $\mathbf{w}_h^n$ denote the linear function interpolating $\mathbf{w}_j^n$ in T. Then with

$$\begin{aligned}\widehat{\mathcal{T}}_{i,1} &= \{T \mid T\in\mathcal{T}_i, |\nabla\mathbf{w}_h^n| > C_0\},\\ \widehat{\mathcal{T}}_{i,2} &= \{T \mid T\in\mathcal{T}_i, |\nabla\mathbf{w}_h^n| \le C_0\},\end{aligned} \tag{24}$$

we have

$$\begin{aligned}\mathbf{L}_i^n = \mathbf{L}(\mathbf{w}_i^n) &= -\tfrac{\Delta t}{A_i}\sum_{T\in\widehat{\mathcal{T}}_{i,1}}\sum_{j\in I_T} a_{ij}^T\mathbf{J}(\mathbf{w}_i^n,\mathbf{w}_j^n;\varepsilon_1)\cdot\tau_{\mathbf{ij}}h_{ij}\\ &\quad -\tfrac{\Delta t}{A_i}\sum_{T\in\widehat{\mathcal{T}}_{i,2}}\left\{\sum_{j\in I_T}\varepsilon_2 a_{ij}^T(\mathbf{w}_j^n-\mathbf{w}_i^n) - \int_T \mathbf{f}(\mathbf{w}_h^n)\cdot\nabla\varphi_i d\mathbf{x}\right\},\end{aligned} \tag{25}$$

In (25),

$$\mathbf{J}(\mathbf{w}_i^n,\mathbf{w}_j^n;\varepsilon_1)\cdot\tau_{\mathbf{ij}} = \varepsilon_1\left(\int_{\mathbf{X_i}}^{\mathbf{X_j}}\mathbf{Y}ds\right)^{-1}\cdot\left((\mathbf{Y}\cdot\mathbf{w}_h^n)_j - (\mathbf{Y}\cdot\mathbf{w}_h^n)_i\right). \tag{26}$$

In the calculation of $\mathbf{Y}$ in (26), we replace $\mathbf{u}_h^n$ with $\mathbf{w}_h^n$.

The initial condition for (21) is

$$\mathbf{u}_i^0 = \frac{1}{A_i}\int_{\Omega_i}\mathbf{u}_0\varphi_i\,d\mathbf{x}. \tag{27}$$

3 Error analysis

In this section we prove our main result, Theorem 3.1, for smooth solutions to systems of hyperbolic conservation laws in $\mathbb{R}^2$ obeying reasonably general assumptions. We show a second order error estimate for the explicit finite element solution. This result is more difficult to prove than that in the scalar case in [12].

As in §2, we denote the exact solution by $\mathbf{u}(\mathbf{x},t)$, and the numerical solution by $\mathbf{u}_h$. The numerical solution is denoted $\mathbf{u}_h^n$ on $[n\Delta t,(n+\frac{1}{2})\Delta t)$ and $\mathbf{w}_h^n$ on $[(n+\frac{1}{2})\Delta t,(n+1)\Delta t)$, $\forall n$. Let $\mathbf{u}_I(\mathbf{x},t)$ denote the spatial linear interpolation to the exact solution $\mathbf{u}(\mathbf{X}_i,t)$ at nodes $\mathbf{X}_i$. In contrast to §2, we now let $\mathbf{u}_i^n$ denote values of the exact solution $\mathbf{u}$ at nodes, and employ $\mathbf{u}_{hi}^n$ and $\mathbf{w}_{hi}^n$ to denote nodal values of the numerical solution.

We continue with standard vector and matrix notation (e.g. [6]). For $N=2$,

$$\mathbf{u}=\begin{pmatrix}u_1\\ \dots\\ u_m\end{pmatrix},\quad \mathbf{f}_1(\mathbf{u})=\begin{pmatrix}f_{11}(\mathbf{u})\\ \dots\\ f_{m1}(\mathbf{u})\end{pmatrix},\quad \mathbf{f}_2(\mathbf{u})=\begin{pmatrix}f_{12}(\mathbf{u})\\ \dots\\ f_{m2}(\mathbf{u})\end{pmatrix},\quad \mathbf{f}(\mathbf{u})=(\mathbf{f}_1(\mathbf{u}),\mathbf{f}_2(\mathbf{u}))\,.$$

We also employ Jacobean matrix notation,

$$D\mathbf{f}_1(\mathbf{u})=\begin{pmatrix}\frac{\partial f_{11}(\mathbf{u})}{\partial u_1} & \dots & \frac{\partial f_{11}(\mathbf{u})}{\partial u_m}\\ \dots & \dots & \\ \frac{\partial f_{m1}(\mathbf{u})}{\partial u_1} & \dots & \frac{\partial f_{m1}(\mathbf{u})}{\partial u_m}\end{pmatrix},\quad D\mathbf{f}_2(\mathbf{u})=\begin{pmatrix}\frac{\partial f_{12}(\mathbf{u})}{\partial u_1} & \dots & \frac{\partial f_{12}(\mathbf{u})}{\partial u_m}\\ \dots & \dots & \\ \frac{\partial f_{m2}(\mathbf{u})}{\partial u_1} & \dots & \frac{\partial f_{m2}(\mathbf{u})}{\partial u_m}\end{pmatrix},$$

$$\mathbf{f}'(\mathbf{u}):=D\mathbf{f}(\mathbf{u})=(D\mathbf{f}_1(\mathbf{u}),D\mathbf{f}_2(\mathbf{u}))\,.$$

Common notations for norms and semi-norms in Sobolev spaces will be employed.

Throughout this section, as we mentioned in introduction, C will denote a positive constant independent of h and Δt. The value of C is not necessarily the same at each occurrence of use.

For $N=2$, let $\mathcal{T}_h$ be a uniform triangulated partition of $\mathbb{R}^2$; that is $\mathbb{R}^2$ is divided into squares $\{(x_1,x_2)|\ i_1h\le x_1\le(i_1+1)h,\ i_2h\le x_2\le(i_2+1)h,\ i_1,i_2=0,\pm1,\pm2,\cdots\}$, and each square is further divided into two triangles along the diagonal $x_2=x_1+(i_2-i_1)h$.

Theorem 3.1. *Let Ω be an arbitrary compact subdomain in $\mathbb{R}^2$. Assume*

(a) the system (1)–(2) is strictly hyperbolic;

(b) the initial data $\mathbf{u}_0\in L^\infty(\mathbb{R}^2)$ such that $M_0=\max\|\mathbf{u}_0\|$;

(c) the solution $\mathbf{u}$ and the flux functions $\mathbf{f}_i(\mathbf{u})$, $i=1,2$, are sufficiently smooth with bounded derivatives;

(d) $\varepsilon_2\le Ch^2$ and $\varepsilon_1=0$;

(e) $\Delta t\le Ch^{\frac{4}{3}}$;

(f) for all $0<\eta<h$, there exists a $\mathbf{u}^*$, such that $\|\mathbf{u}-\mathbf{u}^*\|<\eta$ and

$$\sum_{i=1}^{2}\|D\mathbf{f}_i(\mathbf{u}^*)-D\mathbf{f}_i^T(\mathbf{u}^*)\|<\eta.$$

Then

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \le C(T^*, C_0)h^2, \qquad t \in [\,0, T^*\,],$$

provided that C_0 is large enough and h is small enough. Here $||\cdot||$ is an ordinary matrix norm, and C_0 is the threshold constant appearing in (7) and (24).

Because of assumption (c), we will write $\varepsilon_2 = \varepsilon$. The uniformity of the grid implies $A_i = A = O(h^2)$ for all regions Ω_i associated with an interior node $\mathbf{X}_i$. In proving Theorem 3.1, we first derive the truncation error in time.

Lemma 3.2. *Assume conditions (b), (c) and (d) of Theorem 3.1 are satisfied. Let*

$$\mathbf{w}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) + \frac{\Delta t}{A}\int_{\Omega_0} \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x},t))\cdot\nabla\varphi_0(\mathbf{z})\,\mathbf{dz}, \tag{28}$$

where φ_0 is the shape function at the origin, $\mathcal{T}_0$ denotes the set of elements neighboring the origin, and $\Omega_0 = \underset{T\in\mathcal{T}_0}{\cup} T$. Then

$$\mathbf{w}_i^n = \mathbf{u}_i^n - \frac{\Delta t}{A}\left\{\varepsilon\int_{\Omega_i}\nabla\mathbf{u}_I^n\cdot\nabla\varphi_i\,d\mathbf{x} - \int_{\Omega_i} f(\mathbf{u}^n)\cdot\nabla\varphi_i\,d\mathbf{x} + O(h^4)\right\}, \tag{29}$$

$$\begin{aligned}\mathbf{u}_i^{n+1} = {}& \frac{1}{2}\mathbf{u}_i^n + \frac{1}{2}\mathbf{w}_i^n \\ & - \frac{\Delta t}{2A}\left\{\varepsilon\int_{\Omega_i}\nabla\mathbf{w}_I^n\cdot\nabla\varphi_i\,d\mathbf{x} - \int_{\Omega_i}\mathbf{f}(\mathbf{w}^n)\cdot\nabla\varphi_i\,d\mathbf{x} + O(h^2\Delta t^2 + h^4)\right\}.\end{aligned} \tag{30}$$

Proof of Lemma 3.2: Applying (28) to the node X_i gives

$$\mathbf{w}(\mathbf{X}_i,t) = \mathbf{u}(\mathbf{X}_i,t) + \frac{\Delta t}{A}\int_{\Omega_0}\mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{X}_i,t))\cdot\nabla\varphi_0(\mathbf{z})\,d\mathbf{z}\,. \tag{31}$$

By the uniformity of the mesh $\mathcal{T}_h$, as $\mathbf{z}$ varies over Ω_0, $\mathbf{X}_i+\mathbf{z}$ varies over Ω_i and property of the shape functions, we have

$$\mathbf{w}(\mathbf{X}_i,t) = \mathbf{u}(\mathbf{X}_i,t) + \frac{\Delta t}{A}\int_{\Omega_i}\mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{X}_i,t))\cdot\nabla\varphi_i(\mathbf{z})\,d\mathbf{z}\,, \tag{32}$$

where, in (32), $\mathbf{z}$ is a coordinate variable local to φ_i (i.e. $\mathbf{z}=0$ at $\mathbf{X}_i$). Adding and subtracting common terms, and using the notation $\mathbf{u}_i^n = \mathbf{u}(\mathbf{X}_i, n\Delta t)$, (32) can be rewritten:

$$\begin{aligned}\mathbf{w}_i^n = \mathbf{u}_i^n - {}& \frac{\Delta t}{A}\left\{\varepsilon\int_{\Omega_i}\nabla\mathbf{u}_I^n\cdot\nabla\varphi_i\,d\mathbf{x} - \int_{\Omega_i}\mathbf{f}(\mathbf{u}^n)\cdot\nabla\varphi_i\,\mathbf{dx} - \varepsilon\int_{\Omega_i}\nabla\mathbf{u}^n\cdot\nabla\varphi_i\,d\mathbf{x}\right\} \\ & - \frac{\varepsilon\Delta t}{A}\int_{\Omega_i}\nabla(\mathbf{u}^n - \mathbf{u}_I^n)\cdot\nabla\varphi_i\,d\mathbf{x}.\end{aligned} \tag{33}$$

The third and fourth terms on the right hand side of (33) can be bounded as

$$\left\|\varepsilon\int_{\Omega_i}\nabla\mathbf{u}^n\cdot\nabla\varphi_i\,d\mathbf{x}\right\| = \left\|-\varepsilon\int_{\Omega_i}\triangle\mathbf{u}^n\cdot\varphi_i\,d\mathbf{x}\right\| \le Ch^4, \tag{34}$$

and

$$\left\| \varepsilon \int_{\Omega_i} \nabla(\mathbf{u}^n - \mathbf{u}_I^n) \cdot \nabla \varphi_i \, d\mathbf{x} \right\| \leq Ch^4, \tag{35}$$

where we have relied on: the smoothness of $\mathbf{u}^n$ and Taylor expansion, $\| \bigtriangledown (\mathbf{u}^n - \mathbf{u}_I) \| \leq Ch$; and $\| \bigtriangledown \varphi_i \| \leq \frac{C}{h}$. Equation (29) follows directly.

In order to show (30), we begin with the Taylor's expansions,

$$\mathbf{u}(\mathbf{x}, t+\Delta t) - \mathbf{u}(\mathbf{x}, t) - \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\Delta t - \frac{\Delta t^2}{2}\frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} = O(\Delta t^3), \tag{36}$$

$$\frac{\Delta t}{2}\frac{\partial \mathbf{u}(\mathbf{x}, t+\Delta t)}{\partial t} - \frac{\Delta t}{2}\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} - \frac{\Delta t^2}{2}\frac{\partial^2 \mathbf{u}(\mathbf{x}, t)}{\partial t^2} = O(\Delta t^3). \tag{37}$$

Subtraction of (36) and (37) yields

$$\mathbf{u}(\mathbf{x}, t+\Delta t) - \mathbf{u}(\mathbf{x}, t) - \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\frac{\Delta t}{2} - \frac{\partial \mathbf{u}(\mathbf{x}, t+\Delta t)}{\partial t}\frac{\Delta t}{2} = O(\Delta t^3). \tag{38}$$

Spatially convolving equation (1) with φ_0, and integrating by parts, gives

$$\int_{\Omega_0} \frac{\partial \mathbf{u}(\mathbf{z}+\mathbf{x}, t)}{\partial t} \varphi_0(\mathbf{z}) \, d\mathbf{z} - \int_{\Omega_0} \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x}, t)) \cdot \nabla \varphi_0(\mathbf{z}) \, d\mathbf{z} = 0. \tag{39}$$

Adding

$$\mathbf{R}_3(\mathbf{x}, t) = \int_{\Omega_0} \left(\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} - \frac{\partial \mathbf{u}(\mathbf{z}+\mathbf{x}, t)}{\partial t} \right) \varphi_0(\mathbf{z}) \, \mathbf{dz} \tag{40}$$

to (39) gives

$$\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} = \frac{1}{A} \left\{ \int_{\Omega_0} \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x}, t)) \cdot \nabla \varphi_0(\mathbf{z}) \, d\mathbf{z} + \mathbf{R}_3(\mathbf{x}, t) \right\}. \tag{41}$$

Substituting (41) into (38), we have

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t+\Delta t) - \mathbf{u}(\mathbf{x}, t) - \frac{\Delta t}{2A} \left(\int_{\Omega_0} \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x}, t)) \cdot \nabla \varphi_0(\mathbf{z}) \, d\mathbf{z} + \mathbf{R}_3(\mathbf{x}, t) \right) \\ - \frac{\Delta t}{2A} \left(\int_{\Omega_0} \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x}, t+\Delta t)) \cdot \nabla \varphi_0(\mathbf{z}) \, d\mathbf{z} + \mathbf{R}_3(\mathbf{x}, t+\Delta t) \right) = O(\Delta t^3). \end{aligned} \tag{42}$$

Recalling the integration by parts and using Taylor's expansion in t, equations (41) and (28), the second flux integration term in (42) can be written,

$$\begin{aligned} &\int_{\Omega_0} \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x}, t+\Delta t)) \cdot \nabla \varphi_0(\mathbf{z}) \, d\mathbf{z} \\ &\qquad = -\int_{\Omega_0} \nabla \cdot \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x}, t+\Delta t)) \varphi_0(\mathbf{z}) \, d\mathbf{z} \\ &\qquad = -\int_{\Omega_0} \nabla \cdot \mathbf{f} \left(\mathbf{u}(\mathbf{z}+\mathbf{x}, t) + \frac{\partial \mathbf{u}(\mathbf{z}+\mathbf{x}, t)}{\partial t}\Delta t + O(\Delta t^2) \right) \varphi_0(\mathbf{z}) \, d\mathbf{z} \\ &\qquad = -\int_{\Omega_0} \nabla \cdot \mathbf{f} \left(\mathbf{w}(\mathbf{z}+\mathbf{x}, t) + \mathbf{R}_3(\mathbf{z}+x, t)\frac{\Delta t}{A} + O(\Delta t^2) \right) \varphi_0(\mathbf{z}) \, d\mathbf{z}. \end{aligned} \tag{43}$$

As the mesh is uniform and $\varphi_0(\mathbf{z})$ is an even function in the symmetric domain Ω_0, then from (40)

$$\mathbf{R}_3(\mathbf{x},t) = \int_{\Omega_0} \left(- \bigtriangledown \left(\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}\right) \cdot \mathbf{z} + O(h^2) \right) \varphi_0(\mathbf{z})\, d\mathbf{z}. \tag{44}$$

Since $- \bigtriangledown (\partial \mathbf{u}(\mathbf{x},t)/\partial t) \cdot \mathbf{z}$ is an odd function on the symmetric domain Ω_0,

$$\int_{\Omega_0} - \bigtriangledown \left(\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}\right) \cdot \mathbf{z} \varphi_0(\mathbf{z})\, d\mathbf{z} = 0\,. \tag{45}$$

Therefore $\mathbf{R}_3(x,t) = O(h^4)$. Similarly, $\nabla \mathbf{R}_3(\mathbf{x},t) = O(h^4)$. With $A = O(h^2)$, (43) becomes

$$\begin{aligned} \int_{\Omega_0} \mathbf{f}(\mathbf{u}(\mathbf{z}+\mathbf{x}, t+\Delta t)) \cdot \nabla \varphi_0(\mathbf{z})\, d\mathbf{z} &= - \int_{\Omega_0} \nabla \cdot \mathbf{f}(\mathbf{w}(\mathbf{z}+\mathbf{x},t)) \varphi_0(\mathbf{z})\, d\mathbf{z} + O(h^2\Delta t^2 + h^4 \Delta t) \\ &= \int_{\Omega_0} \mathbf{f}(\mathbf{w}(\mathbf{z}+\mathbf{x},t)) \cdot \nabla \varphi_0(\mathbf{z})\, d\mathbf{z} + O(h^2\Delta t^2 + h^4 \Delta t). \end{aligned} \tag{46}$$

Substituting equations (28) and (46) into (42) yields

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t+\Delta t) - \frac{1}{2}\mathbf{u}(\mathbf{x},t) - \frac{1}{2}\mathbf{w}(\mathbf{x},t) \\ - \frac{\Delta t}{2A} \left\{ \int_{\Omega_0} \mathbf{f}(\mathbf{w}(\mathbf{z}+\mathbf{x},t)) \cdot \nabla \varphi_0(\mathbf{z})\, d\mathbf{z} + O(h^2\Delta t^2 + h^4) \right\} = 0\,. \end{aligned} \tag{47}$$

Applying (47) to the nodal coordinate $\mathbf{X}_i$, and adding and subtracting common terms in analogy to the procedure in (33) gives

$$\begin{aligned} \mathbf{u}_i^{n+1} = \frac{1}{2}\mathbf{u}_i^n + \frac{1}{2}\mathbf{w}_i^n - \frac{\Delta t}{2A} \left\{ \varepsilon \int_{\Omega_i} \nabla \mathbf{w}_I^n \cdot \nabla \varphi_i\, d\mathbf{x} - \int_{\Omega_i} \mathbf{f}(\mathbf{w}^n) \cdot \nabla \varphi_i\, d\mathbf{x} + O(h^2\Delta t^2 + h^4) \right\} \\ + \frac{\Delta t}{2A} \left\{ \varepsilon \int_{\Omega_i} \nabla \mathbf{w}^n \cdot \nabla \varphi_i\, d\mathbf{x} - \varepsilon \int_{\Omega_i} \nabla (\mathbf{w}^n - \mathbf{w}_I^n) \cdot \nabla \varphi_i\, d\mathbf{x} \right\}. \end{aligned} \tag{48}$$

The same estimates hold for the last two terms in (48) as hold in (34) and (35), giving the desired result (30). □

The following Lemma, referred to as a **superapproximation estimate** plays a key role in the error estimate for Theorem 3.1. Its proof is similar to that given in [16].

Lemma 3.3. *If* $\mathbf{\Phi} \in ((W^{1,\infty}(\Omega\star))^m)^2$, *then*

$$\left\| \int_{\Omega\star} \mathbf{\Phi} \cdot (\mathbf{u} - \mathbf{u}_I) \cdot \nabla v\, d\mathbf{x} \right\| \le C h^2 \|\mathbf{u}\|_{3,\Omega\star} \|v\|_{0,\Omega\star} \qquad \forall v \in S_0^h,$$

where $\Omega\star$ *is a bounded domain and* $S_0^h = \{v \mid v \in H_0^1(\Omega\star), v|_T \in \mathbb{P}^1(T), \forall T\}$.

We now prove Theorem 3.1. To deal with the nonlinearity of the flux function $\mathbf{f}(\mathbf{u})$, we assume the *a priori* that $\|\mathbf{u}_h^n\| \le 2M_0$ implies $\|\nabla \mathbf{u}_h^n\| \le C_0$ and $\|\nabla \mathbf{w}_h^n\| \le C_0$, where the constant C_0 is to be determined, then sets $\mathcal{T}_{1,i}$ and $\hat{\mathcal{T}}_{1,i}$ are empty. We will prove this assumption in Step 3 (below) by induction.

Define the nodal differences $\mathbf{U}_i^n = \mathbf{u}_i^n - \mathbf{u}_{hi}^n$ and $\mathbf{W}_i^n = \mathbf{w}_i^n - \mathbf{w}_{hi}^n$. Let

$$\mathbf{U}^n = \sum_i \mathbf{U}_i^n \varphi_i, \mathbf{W}^n = \sum_i \mathbf{W}_i^n \varphi_i$$

denote the respective piecewise linear functions interpolating the nodal differences. We employ Lemma 3.2, equations (9), (21)-(23), and (25) to obtain

$$\begin{aligned} \mathbf{W}_i^n &= \mathbf{U}_i^n - \frac{\Delta t}{A}\left\{\varepsilon \int_{\Omega_i} \nabla \mathbf{U}^n \cdot \nabla \varphi_i \, d\mathbf{x} - \int_{\Omega_i} (\mathbf{f}(\mathbf{u}^n) - \mathbf{f}(\mathbf{u}_h^n)) \cdot \nabla \varphi_i \, d\mathbf{x} + O(h^4)\right\} \\ &=: \mathbf{U}_i^n + \mathbf{\Theta}_i^n, \end{aligned} \tag{49}$$

$$\begin{aligned} \mathbf{U}_i^{n+1} &= \frac{1}{2}\mathbf{U}_i^n + \frac{1}{2}\mathbf{W}_i^n - \frac{\Delta t}{2A}\left\{\varepsilon \int_{\Omega_i} \nabla \mathbf{W}^n \cdot \nabla \varphi_i \, d\mathbf{x} - \int_{\Omega_i} (\mathbf{f}(\mathbf{w}^n) - \mathbf{f}(\mathbf{w}_h^n)) \cdot \nabla \varphi_i \, d\mathbf{x}\right\} \\ &\quad - \frac{\Delta t}{2A} O(h^2 \Delta t^2 + h^4) \\ &=: \frac{1}{2}\mathbf{U}_i^n + \frac{1}{2}\mathbf{W}_i^n + \frac{1}{2}\mathbf{\Lambda}_i^n. \end{aligned} \tag{50}$$

Noting that $((\mathbf{\Theta}_i^n)^T \cdot \mathbf{\Lambda}_i^n)^T = (\mathbf{\Theta}_i^n)^T \cdot \mathbf{\Lambda}_i^n$, we have

$$\begin{aligned} &(\mathbf{\Theta}_i^n)^T \cdot \mathbf{U}_i^n + (\mathbf{\Lambda}_i^n)^T \cdot \mathbf{U}_i^n + (\mathbf{\Theta}_i^n)^T \cdot \mathbf{\Lambda}_i^n \\ &\quad = 2\frac{(\mathbf{\Theta}_i^n + \mathbf{\Lambda}_i^n)^T}{2} \cdot \mathbf{U}_i^n + \frac{(\mathbf{\Theta}_i^n + \mathbf{\Lambda}_i^n)^T \cdot (\mathbf{\Theta}_i^n + \mathbf{\Lambda}_i^n)}{4} - \frac{(\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)^T \cdot (\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)}{4} \\ &\quad = 2(\mathbf{U}_i^{n+1} - \mathbf{U}_i^n)^T \cdot \mathbf{U}_i^n + (\mathbf{U}_i^{n+1} - \mathbf{U}_i^n)^T \cdot (\mathbf{U}_i^{n+1} - \mathbf{U}_i^n) - \frac{(\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)^T \cdot (\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)}{4} \\ &\quad = (\mathbf{U}_i^{n+1} - \mathbf{U}_i^n)^T \cdot (\mathbf{U}_i^{n+1} + \mathbf{U}_i^n) - \frac{(\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)^T \cdot (\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)}{4} \\ &\quad = (\mathbf{U}_i^{n+1})^T \cdot \mathbf{U}_i^{n+1} - (\mathbf{U}_i^n)^T \cdot \mathbf{U}_i^n - \frac{(\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)^T \cdot (\mathbf{\Theta}_i^n - \mathbf{\Lambda}_i^n)}{4} \\ &\quad = \|\mathbf{U}_i^{n+1}\|^2 - \|\mathbf{U}_i^n\|^2 - \frac{1}{4}\|\mathbf{\Lambda}_i^n - \mathbf{\Theta}_i^n\|^2. \end{aligned} \tag{51}$$

Slight re-arrangement, and use of (49), gives

$$\sum_i \left\{\|\mathbf{U}_i^{n+1}\|^2 - \|\mathbf{U}_i^n\|^2\right\} A = \sum_i \left\{(\mathbf{\Theta}_i^n)^T \cdot \mathbf{U}_i^n + (\mathbf{\Lambda}_i^n)^T \cdot \mathbf{W}_i^n + \frac{1}{4}\|\mathbf{\Lambda}_i^n - \mathbf{\Theta}_i^n\|^2\right\} A \tag{52}$$

We use energy estimates to analyze the three terms on the right hand side of (52). We treat the third terms in Step 1, and the first and second terms in Step 2.

Step 1. From the definition of $\mathbf{\Lambda}_i^n$ and $\mathbf{\Theta}_i^n$, the last term in (52) is

$$\begin{aligned} \mathbf{\Lambda}_i^n - \mathbf{\Theta}_i^n = &- \frac{\Delta t}{A}\left\{\varepsilon \int_{\Omega_i} \nabla(\mathbf{W}^n - \mathbf{U}^n) \cdot \nabla \varphi_i \, d\mathbf{x}\right\} \\ &+ \frac{\Delta t}{A}\left\{\int_{\Omega_i} (\mathbf{f}(\mathbf{w}^n) - \mathbf{f}(\mathbf{w}_h^n) - \mathbf{f}(\mathbf{u}^n) + \mathbf{f}(\mathbf{u}_h^n)) \cdot \nabla \varphi_i \, d\mathbf{x} + O(h^2 \Delta t^2 + h^4)\right\}. \end{aligned} \tag{53}$$

By the Schwarz inequality, we have

$$\left\| \varepsilon \int_{\Omega_i} \nabla(\mathbf{W}^n - \mathbf{U}^n) \cdot \nabla \varphi_i \, d\mathbf{x} \right\| \leq C h^2 (|\mathbf{W}^n|_{1,\Omega_i} + |\mathbf{U}^n|_{1,\Omega_i}), \tag{54}$$

and

$$
\begin{aligned}
&\Big\|\int_{\Omega_i}(\mathbf{f}(\mathbf{w}^n)-\mathbf{f}(\mathbf{w}_h^n)-\mathbf{f}(\mathbf{u}^n)+\mathbf{f}(\mathbf{u}_h^n))\cdot\nabla\varphi_i\,d\mathbf{x}\Big\|\\
&\quad\le\frac{C}{h}\int_{\Omega_i}\|\mathbf{f}(\mathbf{w}^n)-\mathbf{f}(\mathbf{w}_h^n)-\mathbf{f}(\mathbf{u}^n)+\mathbf{f}(\mathbf{u}_h^n)\|d\mathbf{x}\\
&\quad=\frac{C}{h}\int_{\Omega_i}\Big\|\int_0^1\mathbf{f}'(\tau\mathbf{w}^n+(1-\tau)\mathbf{w}_h^n)\,d\tau\cdot(\mathbf{w}^n-\mathbf{w}_h^n)\\
&\qquad-\int_0^1\mathbf{f}'(\tau\mathbf{u}^n+(1-\tau)\mathbf{u}_h^n)\,d\tau\cdot(\mathbf{u}^n-\mathbf{u}_h^n)\Big\|d\mathbf{x}\\
&\quad\le\frac{C}{h}\int_{\Omega_i}\Big\|\int_0^1\mathbf{f}'(\tau\mathbf{w}^n+(1-\tau)\mathbf{w}_h^n)\,d\tau\cdot(\mathbf{w}^n-\mathbf{w}_h^n-\mathbf{u}^n+\mathbf{u}_h^n)\Big\|d\mathbf{x}\\
&\qquad+\frac{C}{h}\int_{\Omega_i}\Big\|\int_0^1\Big\{\mathbf{f}'(\tau\mathbf{w}^n+(1-\tau)\mathbf{w}_h^n)-\mathbf{f}'(\tau\mathbf{u}^n+(1-\tau)\mathbf{u}_h^n)\Big\}d\tau\cdot(\mathbf{u}^n-\mathbf{u}_h^n)\Big\|d\mathbf{x}\\
&\quad\le\frac{C}{h}\int_{\Omega_i}\Big\|\int_0^1\mathbf{f}'(\tau\mathbf{w}^n+(1-\tau)\mathbf{w}_h^n)d\tau\cdot(\mathbf{w}^n-\mathbf{w}_I^n-\mathbf{u}^n+\mathbf{u}_I^n)\Big\|d\mathbf{x}\\
&\qquad+\frac{C}{h}\int_{\Omega_i}\Big\|\int_0^1\mathbf{f}'(\tau\mathbf{w}^n+(1-\tau)\mathbf{w}_h^n)\,d\tau\cdot(\mathbf{w}_I^n-\mathbf{w}_h^n-\mathbf{u}_I^n+\mathbf{u}_h^n)\Big\|d\mathbf{x}\\
&\qquad+\frac{C}{h}\int_{\Omega_i}\|\mathbf{f}'(\mathbf{w}^n)-\mathbf{f}'(\mathbf{u}^n)\|\|\mathbf{u}^n-\mathbf{u}_h^n\|d\mathbf{x}\\
&\qquad+(1-\tau)\frac{C}{h}\int_{\Omega_i}\int_0^1\|\mathbf{f}''\|\|\mathbf{w}^n-\mathbf{w}_h^n\|\,d\tau\cdot\|\mathbf{u}^n-\mathbf{u}_h^n\|d\mathbf{x}\\
&\qquad+(1-\tau)\frac{C}{h}\int_{\Omega_i}\int_0^1\|\mathbf{f}''\|\|\mathbf{u}^n-\mathbf{u}_h^n\|d\tau\cdot\|\mathbf{u}^n-\mathbf{u}_h^n\|d\mathbf{x}\\
&\quad=:\sum_{l=1}^5 I_l,
\end{aligned}
\tag{55}
$$

where the $I_l, 1\le l\le 5$ are defined by the 5 terms in the last inequality in (55). In (55), $\mathbf{f}'$ was defined at the beginning of this section and $\mathbf{f}''$ is to be understood as the following: by the intermediate value theorem for continuous derivatives and Taylor expansion, there exist $\tau,\tau_1,\tau_2\in(0,1)$ such that

$$
(\mathbf{f}'(\tau\mathbf{w}^n+(1-\tau)\mathbf{w}_h^n))_{ij}=(\mathbf{f}'(\mathbf{w}^n))_{ij}+(1-\tau)D(\mathbf{f}'(\tau_1\mathbf{w}^n+(1-\tau_1)\mathbf{w}_h^n))_{ij}\cdot(\mathbf{w}_h^n-\mathbf{w}^n)_{ij},\tag{56}
$$

$$
(\mathbf{f}'(\tau\mathbf{u}^n+(1-\tau)\mathbf{u}_h^n))_{ij}=(\mathbf{f}'(\mathbf{u}^n))_{ij}+(1-\tau)D(\mathbf{f}'(\tau_2\mathbf{u}^n+(1-\tau_2)\mathbf{u}_h^n))_{ij}\cdot(\mathbf{u}_h^n-\mathbf{u}^n)_{ij}.\tag{57}
$$

From (28), we have

$$
\mathbf{w}^n-\mathbf{u}^n=-\frac{\Delta t}{A}\int_{\Omega_0}\nabla\cdot\mathbf{f}(\mathbf{u}^n(\mathbf{z}+\mathbf{x},t))\varphi_0(\mathbf{z})d\mathbf{z},\tag{58}
$$

giving

$$
\|\mathbf{w}^n-\mathbf{u}^n\|\le C\Delta t,\quad\text{and}\quad\|\mathbf{w}^n-\mathbf{u}^n\|_{0,\Omega_i}\le Ch\Delta t,\tag{59}
$$

with similar inequalities holding for the spatial derivatives of $\mathbf{w}^n - \mathbf{u}^n$. Using the Schwarz inequality and the interpolation inequality in Sobolev spaces [1], we have

$$I_1 \le \frac{C}{h}\left(\int_{\Omega_i} d\mathbf{x}\right)^{\frac{1}{2}} \|\mathbf{w}^n - \mathbf{w}_I^n - \mathbf{u}^n + \mathbf{u}_I^n\|_{0,\Omega_i} \le Ch^2\|\mathbf{w}^n - \mathbf{u}^n\|_{2,\Omega_i} \le Ch^3\Delta t. \tag{60}$$

To bound I_2, we note

$$\begin{aligned} I_2 &= \frac{C}{h}\sum_{T\in\mathcal{T}_i}\int_T \Big\| \int_0^1 \mathbf{f}'(\tau\mathbf{w}^n + (1-\tau)\mathbf{w}_h^n)\, d\tau \cdot (\mathbf{W}^n - \mathbf{U}^n)\Big\| d\mathbf{x} \\ &= \frac{C}{h}\sum_{T\in\mathcal{T}_i}\int_T \Big\| \int_0^1 \mathbf{f}'(\tau\mathbf{w}^n + (1-\tau)\mathbf{w}_h^n) d\tau \cdot \sum_{k=1}^{3} \mathbf{\Theta}_k^n \varphi_k\Big\| d\mathbf{x}. \end{aligned} \tag{61}$$

To proceed further with I_2, we need to evaluate $\mathbf{\Theta}_k^n$, the set of braced terms on the right hand side of (49). For the first term in $\mathbf{\Theta}_k^n$,

$$\left\| \frac{\varepsilon\Delta t}{A}\int_{\Omega_i} \nabla\mathbf{U}^n \cdot \nabla\varphi_i d\mathbf{x}\right\| \le \frac{C\varepsilon\Delta t}{A}\|\nabla\mathbf{U}^n\|_{0,\Omega_i} \le C\Delta t|\mathbf{U}^n|_{1,\Omega_i}. \tag{62}$$

For the second term in $\mathbf{\Theta}_k^n$, by Taylor's expansion,

$$(\mathbf{f}(\mathbf{u}^n) - \mathbf{f}(\mathbf{u}_h^n))_{ij} = (\mathbf{f}'(\mathbf{u}^n)\cdot(\mathbf{u}^n - \mathbf{u}_h^n))_{ij} + (\mathbf{u}^n - \mathbf{u}_h^n)^T \cdot D^2\mathbf{f}_{ij} \cdot (\mathbf{u}^n - \mathbf{u}_h^n), \tag{63}$$

where $D^2\mathbf{f}_{ij}$ is the exact remainder term in the Taylor expansion. Using Lemma 3.3,

$$\begin{aligned} &\left\| \frac{\Delta t}{A}\int_{\Omega_i} \mathbf{f}'(\mathbf{u}^n)\cdot(\mathbf{u}^n - \mathbf{u}_h^n)\cdot\nabla\varphi_i d\mathbf{x}\right\| \\ &\quad\le \left\| \frac{\Delta t}{A}\int_{\Omega_i} \mathbf{f}'(\mathbf{u}^n)\cdot(\mathbf{u}^n - \mathbf{u}_I^n)\cdot\nabla\varphi_i d\mathbf{x}\right\| + \left\| \frac{\Delta t}{A}\int_{\Omega_i} \mathbf{f}'(\mathbf{u}^n)\cdot(\mathbf{u}_I^n - \mathbf{u}_h^n)\cdot\nabla\varphi_i d\mathbf{x}\right\| \\ &\quad\le \frac{C\Delta t}{A}h^2\|\mathbf{u}^n\|_{3,\Omega_i}\|\varphi_i\|_{0,\Omega_i} + \frac{C\Delta t}{A}\|\mathbf{U}^n\|_{0,\Omega_i} \\ &\quad\le Ch\Delta t\|\mathbf{u}^n\|_{3,\Omega_i} + \frac{C\Delta t}{A}\|\mathbf{U}^n\|_{0,\Omega_i}. \end{aligned} \tag{64}$$

The contribution of the second term in right hand side (63) to $\mathbf{\Theta}_k^n$ is bounded by

$$\begin{aligned} \left\| \frac{\Delta t}{A}\int_{\Omega_i} \frac{1}{2}(\mathbf{u}^n - \mathbf{u}_h^n)^T\cdot(D^2\mathbf{f}_{ij})\cdot(\mathbf{u}^n - \mathbf{u}_h^n)\cdot\nabla\varphi_i d\mathbf{x}\right\| &\le \frac{C\Delta t}{h^3}\|\mathbf{u}^n - \mathbf{u}_h^n\|_{0,\Omega_i}^2 \\ &\le \frac{C\Delta t}{h^3}\|\mathbf{U}^n\|_{0,\Omega_i}^2 + Ch\Delta t|\mathbf{u}^n|_{2,\Omega_i}^2. \end{aligned} \tag{65}$$

Thus (62), (64) and (65) imply

$$\begin{aligned} \|\mathbf{\Theta}_i^n\| \le{}& C\Delta t|\mathbf{U}^n|_{1,\Omega_i} + \frac{C\Delta t}{h^2}\|\mathbf{U}^n\|_{0,\Omega_i} \\ &+ \frac{C\Delta t}{h^3}\|\mathbf{U}^n\|_{0,\Omega_i}^2 + Ch\Delta t\|\mathbf{u}^n\|_{3,\Omega_i} + Ch\Delta t|\mathbf{u}^n|_{2,\Omega_i}^2 + Ch^2\Delta t. \end{aligned}$$

Then, from (61),

$$I_2 \le Ch\Delta t|\mathbf{U}^n|_{1,\Omega_i} + \frac{C\Delta t}{h}\|\mathbf{U}^n\|_{0,\Omega_i} + \frac{C\Delta t}{h^2}\|\mathbf{U}^n\|^2_{0,\Omega_i} + Ch^2\Delta t\|\mathbf{u}^n\|_{3,\Omega_i} + Ch^2\Delta t|\mathbf{u}^n|^2_{2,\Omega_i} + Ch^3\Delta t.$$

Using $\|\mathbf{w}^n - \mathbf{u}^n\| \le Ch\Delta t$ and

$$\begin{aligned}\|\mathbf{u}^n - \mathbf{u}_h^n\|_{0,\Omega_i} &\le \|\mathbf{u}^n - \mathbf{u}_I^n\|_{0,\Omega_i} + \|\mathbf{u}_I^n - \mathbf{u}_h^n\|_{0,\Omega_i} \\ &= \|\mathbf{u}^n - u_I^n\|_{0,\Omega_i} + \|\mathbf{U}^n\|_{0,\Omega_i} \\ &\le Ch^2\|\mathbf{u}^n\|_{2,\Omega_i} + \|\mathbf{U}^n\|_{0,\Omega_i},\end{aligned} \tag{66}$$

the bound on I_3 is

$$I_3 \le \frac{C}{h}\|\mathbf{w}^n - \mathbf{u}^n\|_{0,\Omega_i} \cdot \|\mathbf{u}^n - \mathbf{u}_h^n\|_{0,\Omega_i} \le C\Delta t(h^2|\mathbf{u}^n|_{2,\Omega_i} + \|\mathbf{U}^n\|_{0,\Omega_i}). \tag{67}$$

Estimates for I_4 and I_5 follow in analogous fashion

$$I_4 \le \frac{C}{h}\|\mathbf{w}^n - \mathbf{w}_h^n\|_{0,\Omega_i} \cdot \|\mathbf{u}^n - \mathbf{u}_h^n\|_{0,\Omega_i} \le C\max\|\mathbf{w}^n - \mathbf{w}_h^n\| \cdot (h^2|\mathbf{u}^n|_{2,\Omega_i} + \|\mathbf{U}^n\|_{0,\Omega_i}), \tag{68}$$

$$I_5 \le C\max\|\mathbf{u}^n - \mathbf{u}_h^n\| \cdot (h^2|\mathbf{u}^n|_{2,\Omega_i} + \|\mathbf{U}^n\|_{0,\Omega_i}). \tag{69}$$

Combing (54) and the bounds developed for $I_1 \to I_5$,

$$\begin{aligned}\|\mathbf{\Lambda}_i^n - \mathbf{\Theta}_i^n\| \le{}& C(\Delta t + \frac{\Delta t^2}{h})|\mathbf{U}^n|_{1,\Omega_i} \\ &+ C\frac{\Delta t}{h^2}(\frac{\Delta t}{h} + \max\|\mathbf{u}^n - \mathbf{u}_h^n\| + \max\|\mathbf{w}^n - \mathbf{w}_h^n\|) \cdot \|\mathbf{U}^n\|_{0,\Omega_i} \\ &+ C\frac{\Delta t^2}{h^4}\|\mathbf{U}^n\|^2_{0,\Omega_i} + C\Delta t|\mathbf{W}^n|_{1,\Omega_i} + C\Delta t^2\|\mathbf{u}^n\|_{3,\Omega_i} + C\Delta t^2|\mathbf{u}^n|^2_{2,\Omega_i} \\ &+ C\Delta t(\Delta t + \max\|\mathbf{u}^n - \mathbf{u}_h^n\| + \max\|\mathbf{w}^n - \mathbf{w}_h^n\|)|\mathbf{u}^n|_{2,\Omega_i} + C\Delta t(\Delta t^2 + h^2).\end{aligned} \tag{70}$$

By the inverse inequality, we have

$$\begin{aligned}\|\mathbf{\Lambda}_i^n - \mathbf{\Theta}_i^n\| \le C\Big\{&\frac{\Delta t}{h}(1 + \frac{\Delta t}{h^2} + \frac{\max\|\mathbf{u}^n - \mathbf{u}_h^n\|}{h} + \frac{\max\|\mathbf{w}^n - \mathbf{w}_h^n\|}{h})\|\mathbf{U}^n\|_{0,\Omega_i} \\ &+ \frac{\Delta t^2}{h^4}\|\mathbf{U}^n\|^2_{0,\Omega_i} + \frac{\Delta t}{h}\|\mathbf{W}^n\|_{0,\Omega_i} + \Delta t^2\|\mathbf{u}^n\|_{3,\Omega_i} + C\Delta t^2|\mathbf{u}^n|^2_{2,\Omega_i} \\ &+ \Delta t(\Delta t + \max\|\mathbf{u}^n - \mathbf{u}_h^n\| + \max\|\mathbf{w}^n - \mathbf{w}_h^n\|)\,|\mathbf{u}^n|_{2,\Omega_i} + \Delta t(\Delta t^2 + h^2)\Big\}.\end{aligned} \tag{71}$$

By assumption (e), $\Delta t^3 \le Ch^4$ and we conclude

$$\begin{aligned}\frac{1}{4}\sum_i \|\mathbf{\Lambda}_i^n - \mathbf{\Theta}_i^n\|^2 A \le C\Big\{&\Delta t(1 + h^{-\frac{2}{3}}\max\|\mathbf{u}^n - \mathbf{u}_h^n\|^2 + h^{-\frac{2}{3}}\max\|\mathbf{w}^n - \mathbf{w}_h^n\|^2)\|\mathbf{U}^n\|^2_{0,\Omega} \\ &+ \Delta t^2\|\mathbf{W}^n\|^2_{0,\Omega} + h^2\Delta t^2(\Delta t^2 + \max\|\mathbf{u}^n - \mathbf{u}_h^n\|^2 + \max\|\mathbf{w}^n - \mathbf{w}_h^n\|^2)\Big\}.\end{aligned} \tag{72}$$

Step 2. We analyze the first and second terms on the RSH of (52). Let $\mathbf{U}_i^n = (U_1^n, U_2^n, ...U_m^n)_i^T$. For the first term in $\sum_i (\mathbf{\Theta}_i^n)^T \mathbf{U}_i^n A$, by the inverse inequality, we have

$$\begin{aligned}
\Big\| -\sum_i \frac{\Delta t}{A} \varepsilon \int_{\Omega_i} (\bigtriangledown \mathbf{U}^n \cdot \bigtriangledown \varphi_i)^T d\mathbf{x}\ \mathbf{U}_i^n A \Big\| &= \Big\| -\varepsilon \Delta t \sum_i \int_{\Omega_i} \sum_{j=1}^m (U_j)_i^n (U_{jx_1}^n, U_{jx_2}^n) \cdot (\varphi_{ix_1}, \varphi_{ix_2})^T d\mathbf{x} \Big\| \\
&= \Big\| -\varepsilon \Delta t \sum_{j=1}^m \sum_T \int_T (U_{jx_1}^n, U_{jx_2}^n) \cdot \sum_i (U_j)_i^n (\varphi_{ix_1}, \varphi_{ix_2})^T d\mathbf{x} \Big\| \\
&= \Big\| -\varepsilon \Delta t \sum_{j=1}^m \int_\Omega (U_{jx_1}^n, U_{jx_2}^n) \cdot (U_{jx_1}^n, U_{jx_2}^n)^T d\mathbf{x} \Big\| \\
&= \varepsilon \Delta t \int_\Omega \| \bigtriangledown \mathbf{U}^n \|^2 d\mathbf{x} \\
&\le C \Delta t \| U^n \|_{0,\Omega}^2.
\end{aligned} \tag{73}$$

Let $\hat{f}_{j1} = f_{j1}(\mathbf{u}^n) - f_{j1}(\mathbf{u}_h^n)$ and $\hat{f}_{j2} = f_{j2}(\mathbf{u}^n) - f_{j2}(\mathbf{u}_h^n)$, then

$$\begin{aligned}
\sum_i \frac{\Delta t}{A} \int_{\Omega_i} ((\mathbf{f}(\mathbf{u}^n) - \mathbf{f}(\mathbf{u}_h^n)) \cdot \bigtriangledown \varphi_i)^T d\mathbf{x} \mathbf{U}_i^n A &= \Delta t \sum_{j=1}^m \sum_T \int_T \sum_i (U_j)_i^n (\hat{f}_{j1}, \hat{f}_{j2}) \cdot (\varphi_{ix_1}, \varphi_{ix_2})^T d\mathbf{x} \\
&= \Delta t \sum_{j=1}^m \sum_T \int_T (\hat{f}_{j1}, \hat{f}_{j2}) \cdot \sum_i (U_j)_i^n (\varphi_{ix1}, \varphi_{ix2})^T d\mathbf{x} \\
&= \Delta t \sum_{j=1}^m \sum_T \int_T (\hat{f}_{j1}, \hat{f}_{j2}) \cdot (U_{jx_1}^n, U_{jx_2}^n)^T d\mathbf{x}.
\end{aligned} \tag{74}$$

By Taylor expansion, using the symbol Df, there are $\tau_j, \tau_{1j} \in (0,1), j = 1, 2, ..., m$, we have

$$\begin{aligned}
&\sum_{j=1}^m \int_\Omega (\hat{f}_{j1}, \hat{f}_{j2}) \cdot (U_{jx_1}^n, U_{jx_2}^n)^T d\mathbf{x} \\
&\quad = \sum_{j=1}^m \int_\Omega (Df_{j1}(\tau_j \mathbf{u}^n + (1-\tau_j)\mathbf{u}_h^n), Df_{j2}(\tau_{1j}\mathbf{u}^n + (1-\tau_{1j})\mathbf{u}_h^n)) \cdot (\mathbf{u}^n - \mathbf{u}_I^n) \cdot \bigtriangledown U_j^n \, d\mathbf{x} \\
&\quad\quad + \sum_{j=1}^m \int_\Omega (Df_{j1}(\tau_j \mathbf{u}^n + (1-\tau_j)\mathbf{u}_h^n), Df_{j2}(\tau_{1j}\mathbf{u}^n + (1-\tau_{1j})\mathbf{u}_h^n)) \cdot \mathbf{U}^n \cdot \bigtriangledown U_j^n \, d\mathbf{x} \\
&\quad =: \sum_{l=1}^2 \hat{I}_l \ .
\end{aligned} \tag{75}$$

From Lemma 3.3, we have the estimate for $\hat{I}_1$,

$$\hat{I}_1 \le C h^2 \| \mathbf{u}^n \|_{3,\Omega} \| \mathbf{U}^n \|_{0,\Omega}. \tag{76}$$

We analyze $\hat{I}_2$ by integrating by parts,

$$\begin{aligned}
\hat{I}_2 &= \sum_{j=1}^{m}\int_{\Omega}(Df_{j1}(\tau_j\mathbf{u}^n+(1-\tau_j)\mathbf{u}_h^n), Df_{j2}(\tau_{1j}\mathbf{u}^n+(1-\tau_{1j})\mathbf{u}_h^n))\cdot\mathbf{U}^n\cdot\bigtriangledown U_j^n d\mathbf{x}\\
&= \sum_{j=1}^{m}\sum_{i}\int_{\Omega_i}(Df_{j1}(\tau_j\mathbf{u}^n+(1-\tau_j)\mathbf{u}_h^n)\cdot\mathbf{U}^n, Df_{j2}(\tau_{1j}\mathbf{u}^n+(1-\tau_{1j})\mathbf{u}_h^n)\cdot\mathbf{U}^n)\cdot(U_j^n)_i\bigtriangledown\varphi_i d\mathbf{x}\\
&= \sum_{j=1}^{m}\sum_{i}\sum_{\partial T_i\subset\partial\Omega_i}\int_{\partial T_i}(Df_{j1}(\tau_j\mathbf{u}^n+(1-\tau_j)\mathbf{u}_h^n)\cdot\mathbf{U}^n, Df_{j2}(\tau_{1j}\mathbf{u}^n+(1-\tau_{1j})\mathbf{u}_h^n)\cdot\mathbf{U}^n)\cdot\mathbf{n}_i(U_j^n)_i\varphi_i ds\\
&\quad-\sum_{j=1}^{m}\int_{\Omega}\nabla\cdot(Df_{j1}(\tau_j\mathbf{u}^n+(1-\tau_j)\mathbf{u}_h^n)\cdot\mathbf{U}^n, Df_{j2}(\tau_{1j}\mathbf{u}^n+(1-\tau_{1j})\mathbf{u}_h^n)\cdot\mathbf{U}^n)U_j^n d\mathbf{x}\\
&= -\sum_{j=1}^{m}\int_{\Omega}(Df_{j1}(\tau_j\mathbf{u}^n+(1-\tau_j)\mathbf{u}_h^n), Df_{j2}(\tau_{1j}\mathbf{u}^n+(1-\tau_{1j})\mathbf{u}_h^n))\cdot\nabla\mathbf{U}^n U_j^n d\mathbf{x}\\
&\quad-\sum_{j=1}^{m}\int_{\Omega}\nabla\cdot(Df_{j1}(\tau_j\mathbf{u}^n+(1-\tau_j)\mathbf{u}_h^n), Df_{j2}(\tau_{1j}\mathbf{u}^n+(1-\tau_{1j})\mathbf{u}_h^n))\cdot\mathbf{U}^n U_j^n d\mathbf{x}.
\end{aligned}\tag{77}$$

Therefore

$$\begin{aligned}
2\hat{I}_2 &= -\sum_{j=1}^{m}\int_{\Omega}\nabla\cdot(Df_{j1}(\tau_j\mathbf{u}^n+(1-\tau_j))\mathbf{u}_h^n), Df_{j2}(\tau_{1j}\mathbf{u}^n+(1-\tau_{1j})\mathbf{u}_h^n))\cdot\mathbf{U}^n U_j^n d\mathbf{x}\\
&\quad-\int_{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{2}(\frac{\partial f_{ij}}{\partial u_k}-\frac{\partial f_{kj}}{\partial u_i})U_i^n U_{kx_j}^n d\mathbf{x}\\
&= -\int_{\Omega}(\mathbf{U}^n)^T(F_{ij})_{m\times m}\mathbf{U}^n d\mathbf{x}\\
&\quad-\int_{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{2}(\frac{\partial f_{ij}}{\partial u_k}-\frac{\partial f_{kj}}{\partial u_i})U_i^n U_{kx_j}^n d\mathbf{x}\\
&=: \hat{I}_7+\hat{I}_8.
\end{aligned}\tag{78}$$

Here $F_{ij}=\frac{\partial f_{j1}}{\partial x_1\partial u_i}+\frac{\partial f_{j2}}{\partial x_2\partial u_i}$. By assumption (c), the solution $\mathbf{u}$ and the flux function $\mathbf{f}(\mathbf{u})$ are sufficiently smooth with bounded derivatives, which ensures that

$$\hat{I}_7\le C\|\mathbf{U}^n\|_{0,\Omega}^2.\tag{79}$$

Term $\hat{I}_8$ vanishes for symmetric systems, but not for the systems here. Under assumption (f) in Theorem 3.1, for all $0<\eta<h$, there exists $\mathbf{u}^*$ such that $\|\mathbf{u}-\mathbf{u}^*\|<\eta$ and

$$\sum_{i=1}^{2}\|D\mathbf{f}_i(\mathbf{u}^*)-D\mathbf{f}_i^T(\mathbf{u}^*)\|<\eta.$$

Using this, and inverse inequality, we have $\hat{\tau}_1, \hat{\tau}_2 \in (0,1)$,

$$\begin{aligned}\hat{I}_8 &\le C\sum_{i=1}^{2}\|D\mathbf{f}_i(\hat{\tau}_i\mathbf{u}+(1-\hat{\tau}_i)\mathbf{u}_h^n)-D\mathbf{f}_i^T(\hat{\tau}_i\mathbf{u}+(1-\hat{\tau}_i)\mathbf{u}_h^n)\|\,\frac{1}{h}\,\|\mathbf{U}^n\|_{0,\Omega}^2,\\ &=: \hat{I}_9\,\frac{1}{h}\,\|\mathbf{U}^n\|_{0,\Omega}^2.\end{aligned} \tag{80}$$

By continuity and the *a priori* assumption, $\|\nabla\mathbf{u}_h^n\| \le C_0$, we have

$$\|\nabla\mathbf{U}^n\| = \|\nabla(\mathbf{u}_I-\mathbf{u}_h^n)\| \le \|\nabla\mathbf{u}_h^n\| + \|\nabla\mathbf{u}_I\| \le C. \tag{81}$$

By inverse inequality and interpolation inequality,

$$\|\mathbf{U}^n\| \le Ch, \|\mathbf{u}^n-\mathbf{u}_h^n\| \le \|\mathbf{u}_h^n-\mathbf{u}_I\| + \|\mathbf{u}^n-\mathbf{u}_I\| \le Ch + Ch^{-1}\cdot h^2|\mathbf{u}|_{2,\Omega} \le Ch. \tag{82}$$

Similarly, $\|\mathbf{w}^n-\mathbf{w}_h^n\| \le Ch, \|\mathbf{u}^n-\mathbf{u}_h^n\|_{0,\Omega} \le Ch^2$. Therefore

$$\begin{aligned}\hat{I}_9 &\le C\{\sum_{i=1}^{2}\|D\mathbf{f}_i(\hat{\tau}_i\mathbf{u}+(1-\hat{\tau}_i)\mathbf{u}_h^n)-D\mathbf{f}_i(\mathbf{u})\| + \sum_{i=1}^{2}\|D\mathbf{f}_i(\mathbf{u})-D\mathbf{f}_i(\mathbf{u}^*)\| + \sum_{i=1}^{2}\|D\mathbf{f}_i(\mathbf{u}^*)-D\mathbf{f}_i^T(\mathbf{u}^*)\|\\ &\quad + \sum_{i=1}^{2}\|D\mathbf{f}_i^T(\mathbf{u}^*)-D\mathbf{f}_i^T(\mathbf{u})\| + \quad \sum_{i=1}^{2}\|D\mathbf{f}_i^T(\hat{\tau}_i\mathbf{u}+(1-\hat{\tau}_i)\mathbf{u}_h^n)-D\mathbf{f}_i^T(\mathbf{u})\|\}\\ &= 2C\sum_{i=1}^{2}\|D\mathbf{f}_i(\hat{\tau}_i\mathbf{u}+(1-\hat{\tau}_i)\mathbf{u}_h^n)-D\mathbf{f}_i(\mathbf{u})\| + 2C\sum_{i=1}^{2}\|D\mathbf{f}_i(\mathbf{u})-D\mathbf{f}_i(\mathbf{u}^*)\|\\ &\quad + C\sum_{i=1}^{2}\|D\mathbf{f}_i(\mathbf{u}^*)-D\mathbf{f}_i^T(\mathbf{u}^*)\|\\ &\le C\|\mathbf{u}^n-\mathbf{u}_h^n\| + C\eta + C\eta < Ch,\end{aligned} \tag{83}$$

which implies $\hat{I}_8 \le C\|\mathbf{U}^n\|_{0,\Omega}^2$, $\hat{I}_2 \le C\|\mathbf{U}^n\|_{0,\Omega}^2$. Moreover, we have

$$\sum_{j=1}^{m}\int_{\Omega_i}(\hat{f}_{j1},\hat{f}_{j2})(U_{jx_1}^n, U_{jx_2}^n)^T d\mathbf{x} \le C\|\mathbf{U}^n\|_{0,\Omega}^2, \tag{84}$$

$$\sum_i (O(h^4))^T U_i^n \le C\|\mathbf{U}^n\|_{0,\Omega}^2 + Ch^4, \tag{85}$$

$$\sum_i (\mathbf{\Theta}_i^n)^T\mathbf{U}_i^n A \le C\Delta t\|\mathbf{U}^n\|_{0,\Omega}^2, \tag{86}$$

$$\sum_i (\mathbf{\Lambda}_i^n)^T\mathbf{W}_i^n A \le C\Delta t\|\mathbf{W}^n\|_{0,\Omega}^2. \tag{87}$$

By (72), (86) and (87), (52), and $\Delta t \le Ch^{\frac{4}{3}}, \max\|\mathbf{u}^n - \mathbf{u}_h^n\| \le Ch, \max\|\mathbf{w}^n - \mathbf{w}_h^n\| \le Ch$, we have

$$\begin{aligned}\sum_i \{\|\mathbf{U}_i^{n+1}\|^2 - \|\mathbf{U}_i^n\|^2\}A &\le C\Delta t\{(1 + h^{-\frac{2}{3}} \max\|\mathbf{u}^n - \mathbf{u}_h^n\|^2 + h^{-\frac{2}{3}} \max\|\mathbf{w}^n - \mathbf{w}_h^n\|^2)\|\mathbf{U}^n\|_{0,\Omega}^2 \\ &\quad + (1+\Delta t)\|\mathbf{W}^n\|_{0,\Omega}^2 + h^2(\Delta t^3 + \Delta t(\max\|\mathbf{w}^n - \mathbf{w}_h^n\|^2 \\ &\quad + \max\|\mathbf{u}^n - \mathbf{u}_h^n\|^2))\} \le C\Delta t(\|\mathbf{U}^n\|_{0,\Omega}^2 + \|\mathbf{W}^n\|_{0,\Omega}^2 + h^4),\end{aligned}$$

$$\sum_i \|\Theta_i^n\|^2 A \le C\frac{\Delta t^2}{h^2}\|\mathbf{U}^n\|_0^2 + Ch^4\Delta t^2,$$

$$\|\mathbf{W}^n\|_{0,\Omega}^2 \le C\left(1 + \frac{\Delta t^2}{h^2}\right)\|\mathbf{U}^n\|_{0,\Omega}^2 + Ch^4\Delta t^2,$$

$$\sum_i \{\|\mathbf{U}_i^{n+1}\|^2 - \|\mathbf{U}_i^n\|^2\}A \le C\Delta t(\|\mathbf{U}^n\|_{0,\Omega}^2 + h^4).$$

By interpolation inequality, we get $\|\mathbf{U}^0\|_{0,\Omega}^2 = \|\mathbf{u}_I^0 - \mathbf{u}_h^0\|_{0,\Omega}^2 \le C\|\mathbf{u}_I^0 - \mathbf{u}^0\|_{0,\Omega}^2 + C\|\mathbf{u}^0 - u_h^0\|_{0,\Omega}^2 \le Ch^4$. Then

$$\|\mathbf{U}^{n+1}\|_0^2 - \|\mathbf{U}^0\|_0^2 \le Ch^4, \quad \forall n\Delta t \le T^*,$$

and $\|\mathbf{U}^{n+1}\|_{0,\Omega}^2 \le C_4^2 h^4$, which deduces $\|\mathbf{W}^{n+1}\|_{0,\Omega}^2 \le C_5^2 h^4$.

Step 3. Finally we verify the *a priori* assumption imposed earlier. Using inverse inequality, we get

$$\|\nabla\mathbf{U}^{n+1}\| \le \frac{C}{h}\|\mathbf{U}^{n+1}\| \le \frac{C}{h^2}\|\mathbf{U}^{n+1}\|_{0,\Omega} = CC_4, \quad \|\nabla\mathbf{W}^{n+1}\| \le CC_5;$$

then we define $C_0 = \max\{CC_4 + \|\nabla\mathbf{u}_I\|, CC_5 + \|\nabla\mathbf{w}_I\|\}$. The *a priori* assumption is obviously satisfied for $n = 0$. If it is satisfied for a certain n, by the inverse inequality we get $\max\|\mathbf{U}^{n+1}\| \le CC_4 h$. Since the exact solution is smooth enough, and on the bounded and closed domain Ω, we have $\|\mathbf{u}(x,t)\| \le M_0$. Take h small enough, satisfying $CC_4 h \le M_0$, then

$$\|\mathbf{u}_h^{n+1}\| = \|\mathbf{u}_I^{n+1} - U^{n+1}\| \le \|\mathbf{u}_I^{n+1}\| + \|\mathbf{U}^{n+1}\| \le M_0 + M_0 = 2M_0.$$

We defined C_0 such that

$$\|\nabla\mathbf{u}_h^{n+1}\| \le \|\nabla\mathbf{U}^{n+1}\| + \|\nabla\mathbf{u}_I^{n+1}\| \le C_0.$$

The *a priori* assumption for $\mathbf{w}_h^n$ can be verified by the same way:

$$\|\nabla\mathbf{w}_h^{n+1}\| \le \|\nabla\mathbf{W}^{n+1}\| + \|\nabla\mathbf{w}_I^{n+1}\| \le C_0.$$

This completes the proof of Theorem 3.1. □

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Error Estimate on Crank-Nicolson Scheme for Stochastic Parabolic Partial Differential Equations*

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Abstract

In this paper we study a numerical method for the partial equations. Firstly, we introduce the basic conceptions and properties of the space, operator, stochastic partial equations and the finite element method. Then we mainly describe and analyze the finite element method for a stochastic parabolic partial problem with homogeneous Dirichlet boundary conditions. The discretization with respect to space is done by piecewise linear finite elements, and in time we apply the C-N method. Optimal convergence error estimates in the L^2 and H^{-1} norms are obtained, and the rigorous prove is given.

Key words space, operator, normal, partial equations, the finite element.

1 Introduction

We study the finite element approximation of the stochastic parabolic partial differential equation

$$\mathrm{d}u + Au\mathrm{d}t = \sigma(u)\mathrm{d}W \qquad for\ 0 < t \leq T, \quad with\ u(0) = u_0, \tag{1.1}$$

in the Hilbert space H, with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, where $u(t)$ is an H-valued random process; $A : D(A) \subset H \to H$ denotes an unbounded, non-negative self-adjoint operator, such that $D(A)$ is compactly embedded into H. Here $A = -\triangle$, where $\triangle$ stands for the Laplacian operator subject to homogeneous Dirichlet boundary conditions, and $D(A) \subset L_2(D)$, where D is a bounded convex domain in $R^d, d = 1, 2, 3$, with a sufficiently smooth boundary ∂D.

The existence, uniqueness, and properties of the solutions of stochastic equations have been well studied. Curtain and Falb [4] [5] first studied the properties of such equations. Prato and Lunardi [6] [7] [8] obtained several results for the linear stochastic evolution equations using semigroups method. Gozzi [9] researched the regularity of such solutions of a second order Hamilton-Jacobi equation. Peszat and Zabczyk [10] used the Wiener process to approximate the noise. Walsh [11] gave an introduction to stochastic partial differential equations. However, numerical approximation of such stochastic equations has not been studied thoroughly.

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Before we start the numerical approximation of (1.1), we first introduce some spaces and the Wiener process.

We assume that $W(t)$ is a cylindrical Wiener process on H defined on a given stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$ with covariance operator Q. This process many be considered in terms of its Fourier series. Suppose that Q is a bounded, linear, self-adjoint, positive define operator on H, with eigenvalues $\gamma_l > 0$ and corresponding eigenfunctions e_l. Let $\beta_l, l = 1, 2, ...,$ be a sequence of real-valued independently and identically distributed Brownian motions. Then

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t)$$

is a Wiener process with covariance operator Q.

Let $L_2^0 = HS(Q^{1/2}(H), H)$ denote the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to H,

$$L_2^0 = \left\{ \psi \in L(H) : \sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2 < \infty \right\},$$

with norm $\|\psi\|_{L_2^0} = (\sum_{l=1}^{\infty} \|\psi Q^{1/2} e_l\|^2)^{1/2}$, where $L(H)$ is the space of bounded linear operators from H to H.

Let $\mathbf{E}$ denote the expectation. Let $\psi \in L_2^0$. Then $\int_0^t \psi(s)\, \mathrm{d}W(s)$ can be defined, and the following isometry property holds:

$$\mathbf{E} \left\| \int_0^t \psi(s)\, \mathrm{d}W(s) \right\|^2 = \int_0^t \|\mathbf{E}\psi(s)\|_{L_2^0}^2\, \mathrm{d}s. \tag{1.2}$$

We assume that $\sigma : H \to L_2^0$ satisfies the following global Lipschitz and growth conditions:

$$(i) \|\sigma(x) - \sigma(y)\| \leq C\|x - y\| \qquad \forall x, y \in H,$$

$$(ii) \|\sigma(x)\| \leq C\|x\| \qquad \forall x \in H.$$

Then (1.1) admits a unique mild solution which has the form

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\sigma(u(s))\, \mathrm{d}W(s), \tag{1.3}$$

where $E(t) = e^{-tA}$ is the analytic semigroup generated by $-A$. Moreover,

$$\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|^2 \leq C(1 + \mathbf{E}\|u_0\|^2). \tag{1.4}$$

Note that if $Tr(Q) = \infty$, then the identity mapping $\sigma(u) = I$ does not satisfy the condition (ii). In order to cover this important case, we introduce a modified version of (ii), i.e.,

$$(ii') \|A^{\beta-1/2}\sigma(x)\|_{L_2^0} \leq C\|x\|, \quad for\ some\ \beta \in [0, 1], \forall x \in H.$$

Then (ii) is the special case $\beta = 1$ of (ii'). If $\sigma(\cdot) = I$, the condition (ii') reduces to $\|A^{(\beta-1)/2}\|_{L_2^0} \leq C$.

The numerical approximation for (1.1) started with the work by Grecksch and Kloeden [12] and Gyöngy and Nualart [13]. Then Allen, Novosel, and Zhang [14] used the finite element on some linear

stochastic partial differential equations. Benth and Gjerde [15], Davie and Gaines [16] studied the convergence rates for such finite element approximations of stochastic partial differential equations. Du and Zhang [17] and Hausenblas [20] [21] made further contribution to the numerical approximation of some linear stochastic partial differential equations. Gyöngy [18] [19] researched the Lattice approximations for such equations. Other authors who works on this field include Kloeden and Shott [22], Lord and Rougemont [23], Printems [32], Shardlow [25], Theting [26] [27], and Yan [28] [29].

In this paper, we will consider error estimates for approximations of (1.1) based on the finite element method in space and the C-N method in time.

Let $\dot{H}^s = \dot{H}^s(D) = D(A^{s/2})$ with norm $|v|_s = \|A^{s/2}v\|$ for any $s \in R$. For any Hilbert space H, we define

$$L_2(\Omega; H) = \{v : \mathbf{E}\|v\|_H^2 = \int_\Omega \|v(\omega)\|_H^2 \, \mathrm{d}\mathbf{P}(\omega) < \infty\},$$

with norm $\|v\|_{L_2(\Omega;H)} = (\mathbf{E}\|v\|_H^2)^{1/2}$.

Let k be a time step and $t_n = nk$ with $n \geq 1$. We define the C-N scheme

$$\frac{U^n - U^{n-1}}{k} + A_h \frac{U^n + U^{n-1}}{2} = \frac{1}{k}\int_{t_{n-1}}^{t_n} P_n\sigma\left(\frac{U^n + U^{n-1}}{2}\right) \mathrm{d}W(s), \ n \geq 1, \tag{1.5}$$

$$U^0 = P_h u_0.$$

with $r(\lambda) = 1/1 + \lambda$, we can rewrite it in the form

$$\begin{aligned} U^n &= \frac{1 - \frac{kA_h}{2}}{1 + \frac{kA_h}{2}} \times U^{n-1} + \int_{t_{n-1}}^{t_n} \frac{1}{1 + \frac{kA_h}{2}} P_n\sigma\left(\frac{U^n + U^{n-1}}{2}\right) \mathrm{W}(s) \\ &= [1 - 2r(\frac{2}{kA_h})] \times U^{n-1} + \int_{t_{n-1}}^{t_n} r(\frac{kA_h}{2}) P_n\sigma\left(\frac{U^n + U^{n-1}}{2}\right) \mathrm{W}(s), \end{aligned}$$

where $E_{kh}^n = [1 - 2r(\frac{kA_h}{2})]^n, U^0 = P_h u_0$, then we have

$$\begin{aligned} U^n &= [1 - 2r(\frac{kA_h}{2})]^n \times U_0 \\ &\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} [1 - 2r(\frac{kA_h}{2})]^{n-j} r(\frac{kA_h}{2}) P_h\sigma\left(\frac{U^n + U^{n-1}}{2}\right) \mathrm{W}(s) \\ &= E_{kh}^n P_h U_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E_{kh}^{n-j} r(\frac{kA_h}{2}) P_h\sigma\left(\frac{U^n + U^{n-1}}{2}\right) \mathrm{W}(s). \end{aligned} \tag{1.6}$$

In this paper, the following theorem is our main result.

Theorem 1.1 *Let U^n and $u(t_n)$ be the solutions of (1.6) and (1.1), respectively. Assume that σ satisfies (i) and (ii'). Assume that $u_0 \in L_2(\Omega; \dot{H}^\beta)$, $0 \leq \beta \leq 1$. Then there exists a constant $C = C(T)$ such that, for $t_n \in [0, T]$ and $0 \leq \gamma < \beta \leq 1$,*

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \leq C(k^{\frac{\gamma}{2}} + h^\beta)(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0<s<t} \|u(s)\|_{L_2(\Omega;H)}) \tag{1.7}$$

In particular, if σ satisfies (i) and (ii), then we have, for $u_0 \in L_2(\Omega; \dot{H}^1)$ and $0 \leq \gamma \leq 1$,

$$\|U^n - u(t_n)\|_{L_2(\Omega;H)} \leq C(k^{\frac{\gamma}{2}} + h)(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \sup_{0<s<t} \|u(s)\|_{L_2(\Omega;H)}) \tag{1.8}$$

The paper is organized as follow: In Section 2, some notations and preliminaries are given. In Section 3, we complete the proof of Theorem 1.1 using lemmas which are mentioned in Section 2. And Section 4 is the conclusion.

2 Theorem and lemma

In this section, we introduce some lemmas which will be used in the proof of our main theorem. For later use, we collect some results in the next two lemmas; see Thomée [30] or Pazy [31].

Lemma 2.1 *For any $\mu, \nu \in \boldsymbol{R}$ and $l \geq 0$, there is a $C > 0$ such that*

$$|D_t^l E(t)v|_\nu \leq Ct^{-(\nu-\mu)/2-l}|v|_\mu \qquad for\ t > 0,\ 2l + \nu \geq \mu,$$

and

$$\int_0^t s^\mu |D_t^l E(t)v|_\nu^2 \,\mathrm{d}s \leq C|v|_{2l+\nu-\mu-1}^2 \qquad for\ t \geq 0,\ \mu \geq 0.$$

Lemma 2.2 *For any $\mu \geq 0, 0 \leq \nu \leq 1$, there is a $C > 0$ such that*

$$||A^\mu E(t)|| \leq Ct^{-\mu}, for\ t \geq 0$$

and

$$||A^{-\nu}(I - E(t))|| \leq Ct^\nu, for\ t \geq 0$$

By these lemmas, we can get the following.

Lemma 2.3 *Let $\boldsymbol{E}$ denote the expectation. $E(t)$ is the mild solution of* (1.1)*.Then we have*

$$\boldsymbol{E}\left\|\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (E(t_n - t_j) - E(t_n - s))\sigma(u(s))\,\mathrm{d}W(s)\right\|^2 \leq Ck^\beta$$

Proof. Firstly we know that

$$\begin{aligned}
&\mathbf{E}\left\|\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (E(t_n - t_j) - E(t_n - s))\sigma(u(s))\,\mathrm{d}W(s)\right\|^2 \\
= &\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathbf{E}\left\|(E(t_n - t_j) - E(t_n - s))A^{1-\beta/2}A^{\beta-1/2}\sigma(u(s))\right\|_{L_2^0}^2 \,\mathrm{d}s \\
\leq &\ C\left(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|(E(t_n - t_j) - E(t_n - s))A^{1-\beta/2}\|^2 \,\mathrm{d}s\right) \sup_{0\leq s\leq T} \mathbf{E}\|u(s)\|^2
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|(E(t_n - t_j) - E(t_n - s))A^{1-\beta/2}\|^2 \,\mathrm{d}s \\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|A^{1/2}E(t_n - t_j)A^{-\beta/2}(I - E(t_n - s))\|^2 \,\mathrm{d}s \\
&\leq Ck^\beta \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|A^{1/2}E(t_n - t_j)\|^2 \,\mathrm{d}s \\
&= Ck^\beta \left(\sum_{j=1}^n \int_{t_{j-1}}^{t_j} k\|A^{1/2}E(t_n - t_j)\|^2\right) \leq Ck^\beta.
\end{aligned}$$

So we have

$$\mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-t_j)-E(t_n-s))\sigma(u(s))\,\mathrm{d}W(s)\right\|^2\le Ck^{\beta}.$$

We also need regularity in time of the solution of (1.1); see Printems [32], Proposition 3.4.

Lemma 2.4 *Let $u(t)$ be the mild solution of (1.1). Assume that $\sigma(\cdot)=I$. If $\|A^{\beta-1/2}\|_{L_2^0}<\infty$, for some $\beta\in[0,1]$,then we have, for fixed $t\in[0,T]$,*

$$\|u(t)\|_{L_2(\Omega;H^\beta)}\le C(\|u_0\|_{L_2(\Omega;H^\beta)}+\|A^{\frac{\beta-1}{2}}\|_{L_2^0}),u_0\in L_2(\Omega;H^\beta)$$

In particular, if $W(t)$ is an H-valued Wiener process with covariance operator Q, $Tr(Q)<\infty$, then we have

$$\|u(t)\|_{L_2(\Omega;H^1)}\le C(\|u_0\|_{L_2(\Omega;H^1)}+Tr(Q)^{\frac{1}{2}}),u_0\in L_2(\Omega;H^1)$$

Lemma 2.5 *Assume that σ satisfies (i) and (ii'). Let $u(t)$ be the mild solution of (1.1). For $0\le\gamma\le\beta\le1$,*

$$E\|u(t_2)-u(t_1)\|^2\le C(t_2-t_1)^{\gamma}E|u_0|_{\gamma}^2+C(t_2-t_1)^{\gamma}\sup_{0\le s\le T}E\|u(s)\|^2$$

For the coefficient, we have the following conclusion.

Theorem 2.1 *Let $F_n=E_{kh}^n r(\frac{kA_h}{2})P_h-E(t_n)=[1-2r(kA_h)]^n r(\frac{kA_h}{2})P_h-e^{-t_nA}$, then*

$$\|F_n\nu\|\le C(k^{\frac{\beta}{2}}+h^{\beta})|v|_{\beta},\ for\ \nu\in\dot{H}^{\beta},\ 0\le\beta\le1 \tag{2.1}$$

$$(k\sum_{j=1}^{n}\|F_j\nu\|^2)^{1/2}\le C(k^{\beta/2}+h^{\beta})|\nu|_{\beta},\ for\ \nu\in\dot{H}^{\beta-1},\ 0\le\beta\le1. \tag{2.2}$$

Proof. Define

$$u(t_n)=u^n=E(t_n)\nu,\ U^n=E_{kh}^n r(\frac{kA_h}{2})P_h\nu=[1-2r(kA_h)]^n r(\frac{kA_h}{2})P_h\nu,$$
$$e^n=U^n\nu-u(t_n)\nu=F_n\nu.$$

By $\partial_t e^n=(e^n-e^{n-1})/k$, we have

$$G_h\partial_t e^n+e^n=\rho^n+G_h\tau^n, \tag{2.3}$$

where $\rho^n=(G_h-G)u_t(t_n),\ \tau^n=u_t(t_n)-\partial_t u^n$. Taking the inner product of (2.3) with e^n, we have

$$(G_h\partial_t e^n,e^n)+(e^n,e^n)=(\rho^n,e^n)+(G_h\tau^n e^n), \tag{2.4}$$

By summation on n, using the inequality $(\rho^n,e^n)\le\frac{1}{2}(\|\rho^n\|^2+\|e^n\|^2)$. Noting that $G_he^0=0$, we have

$$(G_he^n,e^n)+k\sum_{j=1}^{n}\|e^j\|^2\le Ck\sum_{j=1}^{n}\|\rho^j\|^2+Ck\sum_{j=1}^{n}\|G\tau^j\|^2+Ck\sum_{j=1}^{n}\|(G_h-G)\tau^j\|^2.$$

Using Lemma 2.1, we have, since $\rho^j = \rho(s) + \int_s^{t_j} \rho_t(\tau)\,\mathrm{d}\tau$,

$$\begin{aligned}
k\sum_{j=1}^{n}||\rho^j||^2 &= k||\rho||^2 + \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}||\rho^j||^2 ds \\
&\le k||\rho||^2 + 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}(||\rho(s)||^2 + ||\int_{t_{j-1}}^{t_j}\rho_t(\tau)d\tau||^2)ds \\
&\le k||\rho||^2 + 2\int_{t_1}^{t_n}||\rho(s)||^2 ds + 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}((t_j - s)\int_{t_s}^{t_j}||\rho_t(\tau)||^2 d\tau)ds \\
&\le k||\rho||^2 + 2\int_{t_1}^{t_n}||\rho(s)||^2 ds + 2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\tau||\rho_t(\tau)||^2 d\tau \\
&\le Ck||u||^2 + Ch^2\int_0^{t_n}|u(s)|_1^2 ds + Ck\int_0^{t_n}\tau||u_t(\tau)||^2 d\tau \\
&\le C(k+h^2)||v||^2
\end{aligned}$$

and, by Taylor's formula,

$$\begin{aligned}
k\sum_{j=1}^{n}||(G_h - G)\tau^j||^2 &\le Ckh^2|r^1|_{-1}^2 + Ckh^2\sum_{j=2}^{n}|r^j|_{-1}^2 \\
&= Ckh^2|u_t(k) - \frac{1}{k}\int_0^k u_t(\tau)d\tau|_{-1}^2 + Ckh^2\sum_{j=2}^{n}|\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_{tt}(s)ds|_{-1}^2 \\
&\le Ch^2||v||^2 + Ch^2\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}s^2|u_{tt}(s)|_{-1}^2 ds \\
&\le Ch^2||v||^2
\end{aligned}$$

and

$$\begin{aligned}
k\sum_{j=1}^{n}||G\tau^j||^2 &= k\sum_{j=1}^{n}||\frac{1}{k}\int_{t_{j-1}}^{t_j}(s - t_{j-1})u_t(s)ds||^2 \\
&\le k\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(s - t_{j-1})||u_t(s)||^2 ds \\
&\le Ck\int_0^{t_n}s||u_t(s)||^2 ds \le k||v||^2
\end{aligned}$$

Therefore we have

$$(G_h e^n, e^n)^{\frac{1}{2}} + (k\sum_{j=1}^{n}||e^j||^2)^{\frac{1}{2}} \le C(k^{\frac{1}{2}} + h)||v||$$

Then

$$(k\sum_{j=1}^{n}||F_j v||^2)^{\frac{1}{2}} \le C(k^{\frac{1}{2}} + h)||v||.$$

By the proof we have the following lemma:

Lemma 2.6 *If* $F_n = E_{kh}^n r(\frac{kA_h}{2})P_h - E(t_n)$, *then*

$$||F_n v|| \le C(k^{\frac{\beta}{2}} + h^\beta)|v|_\beta, for\ v \in \dot{H}^\beta,\ 0 \le \beta \le 1$$

and

$$k\sum_{j=1}^{n} \left\|F_{n-j}A^{(1-\beta)/2}\right\|^2 \le C(k^\beta + h^{2\beta})$$

Proof. Here we only prove the latter in detail.

$$\begin{aligned}
&k\sum_{j=1}^{n} \left\|F_{n-j}A^{(1-\beta)/2}\right\|^2 \\
&= k\sum_{j=1}^{n} \left\|(E_{kh}^{n-j} r(\frac{kA_h}{2})P_h - E(t_n - t_j))A^{1-\beta/2}\right\|^2 \\
&= k\sum_{j=1}^{n} \left(\sup_{\nu\neq 0} \frac{\|(E_{kh}^{n-j} r(\frac{kA_h}{2})P_h - E(t_n - t_j))A^{1-\beta/2}\nu\|}{\|\nu\|}\right)^2 \\
&= \sup_{\nu\neq 0} \frac{k\sum_{j=1}^{n} \|(E_{kh}^{n-j} r(\frac{kA_h}{2})P_h - E(t_n - t_j))A^{1-\beta/2}\nu\|^2}{\|\nu\|^2} \\
&\le \sup_{\nu\neq 0} \frac{C(k^\beta + h^{2\beta})|A^{1-\beta/2}\nu|_{\beta-1}^2}{\|\nu\|^2} \\
&\le C(k^\beta + h^{2\beta})
\end{aligned}$$

3 Proofs of Teorem 1.1

By the definition of the mild solution of (1.1), with $E(t) = e^{-tA}$,

$$u(t_n) = E(t_n)u_0 + \int_0^{t_n} E(t_n - s)\sigma(u(s))\,\mathrm{d}W(s).$$

Defining $e^n = U^n - u(t_n)$ and $F_n = E_{kh}^n r(\frac{kA_h}{2})P_h - E(t_n)$, then

$$\begin{aligned}
e^n &= F_n u_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E_{kh}^{n-j} r(\frac{kA_h}{2})P_h \left(\sigma\left(\frac{U^j + U^{j-1}}{2}\right) - \sigma\left(\frac{u(t_j) + u(t_{j-1})}{2}\right)\right) \mathrm{d}W(s) \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} E_{kh}^{n-j} r(\frac{kA_h}{2})P_h \left(\sigma\left(\frac{u(t_j) + u(t_{j-1})}{2}\right) - \sigma\left(\frac{u(s) + u(s)}{2}\right)\right) \mathrm{d}W(s) \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left((E_{kh}^{n-j} r(\frac{kA_h}{2})P_h - E(t_n - t_j))\sigma(u(s))\right) \mathrm{d}W(s) \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (E(t_n - t_j) - E(t_n - s))\sigma(u(s))\,\mathrm{d}W(s) \\
&= \sum_{j=1}^{n} I_j
\end{aligned}$$

Thus

$$\|e^n\|_{L_2(\Omega;H)} \le C\sum_{j=1}^{5}\|I_j\|_{L_2(\Omega;H)}.$$

For I_1, let $v = u_0$,

$$\|I_1\| = \|F_n u_0\| \le C(k^{\beta/2}+h^\beta)|u_0|_\beta,$$

then $\|I_1\|_{L_2(\Omega;H)} \le C(k^{\beta/2}+h^\beta)\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}$.

For I_2, we have, by isometry, the stability of $r(\lambda)$, and the Lipschitz condition (i),

$$\begin{aligned}
\|I_2\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} E_{kh}^{n-j} r(\frac{kA_h}{2})P_h\left(\sigma\left(\frac{U^j+U^{j-1}}{2}\right)-\sigma\left(\frac{u(t_j)+u(t_{j-1})}{2}\right)\right)\mathrm{d}W(s)\right\|^2\\
&= k\sum_{j=1}^{n}\mathbf{E}\left\|E_{kh}^{n-j} r(\frac{kA_h}{2})P_h\left(\sigma\left(\frac{U^j+U^{j-1}}{2}\right)-\sigma\left(\frac{u(t_j)+u(t_{j-1})}{2}\right)\right)\right\|^2_{L_2^0}\\
&\le k\sum_{j=1}^{n}\mathbf{E}\left\|E_{kh}^{n-j} r(\frac{kA_h}{2})P_h\right\|^2\times\mathbf{E}\left\|\sigma\left(\frac{U^j+U^{j-1}}{2}\right)-\sigma\left(\frac{u(t_j)+u(t_{j-1})}{2}\right)\right\|^2_{L_2^0}\\
&\le Ck\sum_{j=1}^{n}\mathbf{E}\left\|\frac{U^j+U^{j-1}}{2}-\frac{u(t_j)+u(t_{j-1})}{2}\right\|^2\\
&= C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|\frac{e^j+e^{j-1}}{2}\right\|^2\mathrm{d}s\\
&\le Ck\sum_{j=1}^{n}\|e^j\|^2
\end{aligned}$$

For I_3, we have, by Lemma 2.5, for $0\le\gamma<\beta\le 1$,

$$\begin{aligned}
\|I_3\|^2_{L_2(\Omega;H)} &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|E_{kh}^{n-j} r(\frac{kA_h}{2})P_h\left(\sigma\left(\frac{u(t_j)+u(t_{j-1})}{2}\right)-\sigma\left(\frac{u(s)+u(s)}{2}\right)\right)\right\|^2_{L_2^0}\mathrm{d}s\\
&\le C\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|\frac{u(t_j)+u(t_{j-1})}{2}-\frac{u(s)+u(s)}{2}\right\|^2\mathrm{d}s\\
&\le C\left(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(t_j-s)^\gamma+(t_{j-1}-s)^\gamma\,\mathrm{d}s\right)\left(\mathbf{E}|u_0|^2_\gamma+\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right)\\
&\le Ck^\gamma\left(\mathbf{E}|u_0|_\gamma+\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2\right).
\end{aligned}$$

For I_4, we have,

$$\begin{aligned}
\|I_4\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E_{kh}^{n-j} r(\frac{kA_h}{2})P_h-E(t_n-t_j))\sigma(u(s))\,\mathrm{d}W(s)\right\|^2\\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\left\|(E_{kh}^{n-j} r(\frac{kA_h}{2})P_h-E(t_n-t_j))A^{1-\beta/2}A^{\beta-1/2}\sigma(u(s))\right\|^2\\
&\le C\left(k\sum_{j=1}^{n}\left\|(E_{kh}^{n-j} r(\frac{kA_h}{2})P_h-E(t_n-t_j))A^{1-\beta/2}\right\|^2\right)\sup_{0\le s\le T}\mathbf{E}\|u(s)\|^2
\end{aligned}$$

By the lemma 2.6, we have,

$$\|I_4\|^2_{L_2(\Omega;H)} \leq C(k^\beta + h^{2\beta}) \sup_{0\leq s\leq T} \mathbf{E}\|u(s)\|^2.$$

For I_5, by lemma 2.3, we have,

$$\begin{aligned} \|I_5\|^2_{L_2(\Omega;H)} &= \mathbf{E}\left\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-t_j)-E(t_n-s))\sigma(u(s))\,\mathrm{d}W(s)\right\|^2 \\ &\leq Ck^\beta \sup_{0\leq s\leq T}\mathbf{E}\|u(s)\|^2. \end{aligned}$$

Together these estimates show, for $0 \leq \gamma < \beta \leq 1$,

$$\begin{aligned} \mathbf{E}\|e^n\|^2 &\leq C(k^\gamma + h^{2\beta})\mathbf{E}|u_0|^2_\beta + Ck\sum_{j=1}^{n}\mathbf{E}\|e^j\|^2 \\ &+ C(k^\gamma + h^{2\beta}) \sup_{0\leq s\leq T}\mathbf{E}\|u(s)\|^2. \end{aligned}$$

which implies that

$$\|e^n\|^2_{L_2(\Omega;H)} \leq C(k^{\gamma/2} + h^{2\beta})\mathbf{E}|u_0|_{L_2(\Omega;H^\beta)} + \sup_{0\leq s\leq T}\|u(s)\|_{L_2(\Omega;H)}. \tag{3.1}$$

4 Conclusion

In this paper, we first obtain results on the order of convergence of a discretization in time by an implicit Crank-Nicolson scheme of a stochastic parabolic equation driven by nuclear or space-time white noise in the multidimensional case. The noise is approximated by using the generalized L_2-projection operator. The proof of our main theorem is based on appropriate nonsmooth data error estimates for the corresponding deterministic parabolic problem.

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EXTENDED CESÁRO OPERATORS ON ZYGMUND SPACES IN THE UNIT BALL

ZHONG-SHAN FANG AND ZE-HUA ZHOU*

ABSTRACT. Let g be a holomorphic function of the unit ball B in the n-dimensional space, and denote by T_g and I_g the induced extended Cesáro operator and another integral operator. The boundedness and compactness of T_g and I_g acting on the Zygmund spaces in the unit ball are discussed and necessary and sufficient conditions are given in this paper.

1. INTRODUCTION

Let $f(z)$ be a holomorphic function on the unit disc D with $Taylor$ expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the classical Cesáro operator acting on f is

$$\mathcal{C}[f](z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^{j} a_k \right) z^j.$$

In the past few years, many authors focused on the boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions . It is well known that the operator $\mathcal{C}$ is bounded on the usual Hardy spaces $H^p(D)$ for $0 < p < \infty$ and Bergman space, we recommend the interested readers refer to [10, 12, 8, 2, 13]. But the operator $\mathcal{C}$ is not always bounded, in [16], Shi and Ren gave a sufficient and necessary condition for the operator $\mathcal{C}$ to be bounded on mixed norm spaces in the unit disc. Recently, Siskakis and Zhao in [14] obtained sufficient and necessary conditions for Volterra type operator, which is a generalization of $\mathcal{C}$, to be bounded or compact between $BMOA$ spaces in the unit disc. It is a natural question to ask what are the conditions for higher dimensional case.

Let dv be the $Lebesgue$ measure on the unit ball B of C^n normalized so that $v(B) = 1$,and $dv_\beta = c_\beta(1-|z|^2)^\beta dv$,where c_β is a normalizing constant so that dv_β is a probability measure. The class of all holomorphic functions on B is defined by $H(B)$. For $f \in H(B)$ we write

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

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A little calculation shows $\mathcal{C}[f](z) = \frac{1}{z}\int_0^z f(t)(\log\frac{1}{1-t})'dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesáro operator (also called Volterra-type operator or Riemann-Stieltijes type operator) T_g on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz)Rg(tz)\frac{dt}{t}.$$

It is easy to show that T_g take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesáro operator is bounded or compact.

Motivated by [16], Hu and Zhang [6, 7, 19] gave some sufficient and necessary conditions for the extended $\mathcal{C}$ to be bounded and compact on mixed norm spaces, Bloch space as well as Dirichlet space in the unit ball.

Another natural integral operator is defined as follows:

$$I_g(f)(z) = \int_0^1 Rf(tz)g(tz)\frac{dt}{t}.$$

The importance of them comes from the fact that

$$T_g(f) + I_g(f) = M_g f - f(0)g(0) \tag{1}$$

where the multiplication operator is defined by

$$M_g(f)(z) = g(z)f(z), f \in H(B), z \in B.$$

Now we introduce some spaces first. Let H^∞ denote the space of all bounded holomorphic functions on the unit ball, equipped with the norm $||f||_\infty = \sup_{z\in B}|f(z)|$.

The Bloch space $\mathcal{B}$ is defined as the space of holomorphic functions such that

$$||f||_{\mathcal{B}} = \sup\{(1-|z|^2)|Rf(z)| : z \in B\} < \infty.$$

It is easy to check that if $f \in \mathcal{B}$ then

$$|f(z)| \le C\log\frac{2}{1-|z|^2}\|f\|_{\mathcal{B}}. \tag{2}$$

We define weighted Bloch space $\mathcal{B}_{log}$ as the space of holomorphic functions $f \in H(B)$ such that

$$||f||_{\mathcal{B}_{log}} = \sup\{(1-|z|^2)|Rf(z)|\log\frac{2}{1-|z|^2} : z \in B\} < \infty.$$

The Zygmund space $\mathcal{Z}$ [20] in the unit ball consists of those functions whose first order partial derivatives are in the Bloch space.

It is well known that (Theorem 7.11 in [20]) $f \in \mathcal{Z}$ if and only if $Rf \in \mathcal{B}$, and $\mathcal{Z}$ is a Banach space with the norm

$$||f|| = |f(0)| + ||Rf||_{\mathcal{B}}. \tag{3}$$

The purpose of this paper is to discuss the boundedness and compactness of extended Cesáro operator T_g and another integral operator I_g on the Zygmund space in the unit ball.

2. Some Lemmas

In the following, we will use the symbol C to denote a finite positive number which does not depend on variable z and f.

In order to prove the main results, we will give some Lemmas first.

Lemma 1. *Assume $f \in \mathcal{Z}$, then we have*

$$||f||_\infty \le C||f||$$

Proof. Since $f \in \mathcal{Z}$ implies that $Rf \in \mathcal{B}$, it follows from (2) that

$$|Rf(z)| \le C \log \frac{2}{1-|z|^2} \|Rf\|_{\mathcal{B}} \le C \log \frac{2}{1-|z|^2} ||f||. \tag{4}$$

Furthermore by $\lim\limits_{|z|\to 1}(1-|z|^2)\log \frac{2}{1-|z|^2} = 0$ we have

$$(1-|z|^2)|Rf(z)| \le C(1-|z|^2)\log \frac{2}{1-|z|^2}||f|| < \infty, \tag{5}$$

so $f \in \mathcal{B}$. It follows from Theorem 2.2 in [20] that

$$Rf(z) = \int_B \frac{Rf(z)dv_\beta(w)}{(1- < z, w >)^{n+1+\beta}}$$

where β is a sufficiently large positive constant. Since $Rf(0) = 0$,

$$f(z) - f(0) \;=\; \int_0^1 \frac{Rf(tz)}{t}dt = \int_B Rf(w)L(z,w)dv_\beta(w)$$

where the kernel

$$L(z,w) = \int_0^1 (\frac{1}{(1-t < z, w >)^{n+1+\beta}} - 1)\frac{dt}{t}$$

satisfies

$$|L(z,w)| \le \frac{C}{|1- < z, w > |^{n+\beta}}$$

for all z and w in B. Note that $t^{1/2}\log \frac{2}{t} \le \frac{2}{e}\cdot(1-\log 2)$ for all $t \in (0,1]$, then

$$\begin{aligned} |f(z)-f(0)| &= C\int_B \frac{(1-|w|^2)|Rf(w)|dv_{\beta-1}(w)}{|1- < z, w > |^{n+\beta}} \\ &\le C\int_B \frac{(1-|w|^2)\log \frac{2}{1-|w|^2}||f||dv_{\beta-1}(w)}{|1- < z, w > |^{n+\beta}} \\ &\le C\int_B \frac{(1-|w|^2)^{1-1/2}||f||dv_{\beta-1}(w)}{|1- < z, w > |^{n+\beta}} \\ &\le C||f||. \end{aligned}$$

The last inequality holds since $\int_B \frac{(1-|w|^2)^t dv(w)}{|1-<z,w>|^{n+1+t+c}}$ is bounded for $c < 0$. This completes the proof of Lemma 1.

By Lemma 1, Montel theorem and the definition of compact operator, the following lemma follows.

Lemma 2. *Assume that $g \in H(B)$. Then T_g (or I_g) : $\mathcal{Z} \to \mathcal{Z}$ is compact if and only if T_g (or I_g) is bounded and for any bounded sequence $(f_k)_{k\in N}$ in $\mathcal{Z}$ which converges to zero uniformly on $\overline{B}$ as $k \to \infty$, $||T_g f_k|| \to 0$(or $||I_g f_k|| \to 0$) as $k \to \infty$.*

Lemma 3. *If $(f_k)_{k\in N}$ is a bounded sequence in $\mathcal{Z}$ which converges to zero uniformly on compact subsets of B as $k \to \infty$, then $\lim\limits_{k\to\infty} \sup\limits_{z\in B} |f_k(z)| = 0$.*

proof. Assume $||f_k|| \le M$. For any given $\epsilon > 0$, there exists $0 < \eta < 1$ such that $\frac{\sqrt{1-\eta}}{\eta} < \epsilon$. Note that $t^{1/2} \log \frac{2}{t} \le \frac{2}{e} \cdot (1 - \log 2)$ for all $t \in (0, 1]$, then when $\eta < |z| < 1$, it follows from (4) that

$$\begin{aligned} |f_k(z) - f_k(\frac{\eta}{|z|}z)| &= \left| \int_{\frac{\eta}{|z|}}^1 Rf_k(tz)\frac{dt}{t} \right| \le C \int_{\frac{\eta}{|z|}}^1 \log \frac{2}{1-|tz|^2} ||f_k|| \frac{dt}{t} \\ &\le C\frac{|z|}{\eta} \int_{\frac{\eta}{|z|}}^1 \frac{||f_k||dt}{(1-|tz|^2)^{1/2}} \le C\frac{M}{\eta} \int_{\frac{\eta}{|z|}}^1 \frac{|z|dt}{(1-t|z|)^{1/2}} \\ &\le 2CM\frac{(1-\eta)^{1/2}}{\eta} < C\epsilon. \end{aligned}$$

So we get $\sup\limits_{\eta<|z|<1} |f_k(z)| \le C\epsilon + \sup\limits_{|w|=\eta} |f_k(w)|$. Thus, we have

$$\lim_{k\to\infty} \sup_{z\in B} |f_k(z)| \le \lim_{k\to\infty} (\sup_{|z|\le\eta} |f_k(z)| + \sup_{\eta<|z|<1} |f_k(z)|) \le C\epsilon.$$

Now we finish the proof of this lemma.

Lemma 4. *Let $g \in H(B)$, then*

$$R[T_g f](z) = f(z)Rg(z)$$

for any $f \in H(B)$ and $z \in B$.

Proof. Suppose the holomorphic function fRg has the *Taylor* expansion

$$(fRg)(z) = \sum_{|\alpha|\ge 1} a_\alpha z^\alpha.$$

Then we have

$$R(T_g f)(z) = R\int_0^1 f(tz)R(tz)\frac{dt}{t} = R\int_0^1 \sum_{|\alpha|\ge 1} a_\alpha (tz)^\alpha \frac{dt}{t}$$

$$= R[\sum_{|\alpha|\ge 1} \frac{a_\alpha z^\alpha}{|\alpha|}] = \sum_{|\alpha|\ge 1} a_\alpha z^\alpha = (fRg)(z).$$

3. Main Theorems

Theorem 1. Suppose $g \in H(B)$, then the following conditions are all equivalent:

(a) T_g is bounded on $\mathcal{Z}$;

(b) T_g is compact on $\mathcal{Z}$;

(c) $g \in \mathcal{Z}$.

Proof. $b \Longrightarrow a$ is obvious. For $a \Longrightarrow c$ we just take the test function given by $f(z) \equiv 1$.

We are going to prove $c \Longrightarrow b$. Now assume that $g \in \mathcal{Z}$ and that $(f_k)_{k\in N}$ is a sequence in $\mathcal{Z}$ such that $\sup_{k\in N} ||f_k|| \leq M$ and that $f_k \to 0$ uniformly on on $\overline{B}$ as $k \to \infty$. Now note that $T_g g_k(0) = 0$ and for every $\epsilon > 0$, there is a $\delta \in (0,1)$, such that

$$(1-|z|^2)(\ln \frac{2}{1-|z|^2})^2 < \epsilon$$

whenever $\delta < |z| < 1$. Let $K = \{z \in B : |z| \leq \delta\}$, it follows from Lemma 4 and (4) that

$$\begin{aligned}
||T_g f_k|| &= \sup_{z\in B}(1-|z|^2)\,|R(R(T_g f_k))| \\
&= \sup_{z\in B}(1-|z|^2)|Rf_k \cdot Rg + f_k \cdot R(Rg)| \\
&\leq \sup_{z\in B}(1-|z|^2)(|Rf_k \cdot Rg| + |f_k \cdot R(Rg)|) \\
&\leq \sup_{z\in K}(1-|z|^2)|Rf_k \cdot Rg| + \sup_{z\in B-K}(1-|z|^2)(|Rf_k \cdot Rg| \\
&\quad + \sup_{z\in B}(1-|z|^2)|f_k \cdot R(Rg)| \\
&\leq C||g|| \sup_{z\in K}(1-|z|^2)|Rf_k(z)| \log \frac{2}{1-|z|^2} \\
&\quad + C||f_k|| \cdot ||g|| \sup_{z\in B-K}(1-|z|^2)(\log \frac{2}{1-|z|^2})^2 + ||g|| \cdot \sup_{z\in B}|f_k(z)|.
\end{aligned}$$

With the uniform convergence of f_k to 0 and the Cauchy estimate, the conclusion follows by letting $k \to \infty$.

Theorem 2. Suppose $g \in H(B)$, $I_g : \mathcal{Z} \to \mathcal{Z}$. Then I_g is bounded if and only if $g \in H^\infty \cap \mathcal{B}_{log}$.

Proof. First we assume that $g \in H^\infty \cap \mathcal{B}_{log}$. Notice that $I_g f(0) = 0$ and $R(I_g f) = fRg$, it follows from (4) that

$$\begin{aligned}
(1-|z|^2)|RR(I_g f)(z)| &= (1-|z|^2)|R(Rf(z) \cdot g(z))| \\
&= (1-|z|^2)|R(Rf)(z) \cdot g(z) + Rf(z) \cdot Rg(z)| \\
&\leq \|Rf(z)\|_{\mathcal{B}}\|g\|_\infty + |Rf(z)|(1-|z^2|)|Rg(z)| \\
&\leq C||f|| \cdot ||g||_\infty + C||f||(1-|z|^2)|Rg(z)| \log \frac{2}{1-|z|^2} \\
&\leq C||f|| \cdot ||g||_\infty + C||f|| \cdot ||g||_{\mathcal{B}_{log}}.
\end{aligned}$$

The boundedness of I_g follows.

Conversely, assume that I_g is bounded, then there is a positive constant C such that

$$||I_g f|| \leq C||f|| \tag{6}$$

for every $f \in \mathcal{Z}$. Setting

$$h_a(z) = (\log \frac{2}{1-|a|^2})^{-1}(< z, a > -1)[(1 + \log \frac{2}{1- < z, a >})^2 + 1]$$

for $a \in B$ such that $|a| \geq \sqrt{1-2/e}$, then

$$Rh_a(z) =< z, a > (\log \frac{2}{1- < z, a >})^2 (\log \frac{2}{1-|a|^2})^{-1}$$

and

$$RRh_a(z) = \{< z, a > (\log \frac{2}{1- < z, a >})^2 + \frac{2 < z, a >^2}{1- < z, a >} \log \frac{2}{1- < z, a >}\} (\log \frac{2}{1-|a|^2})^{-1}$$

It is easy to check that $M = \sup_{\sqrt{1-2/e} \leq |a| < 1} ||h_a|| < \infty$. Therefore, we have that

$$\begin{aligned}
\infty &> \|I_g\|\|h_a\| \geq ||I_g h_a|| \\
&\geq \sup_{z \in B}(1-|z|^2)|RRh_a(z) \cdot g(z) + Rh_a(z) \cdot Rg(z)| \\
&\geq (1-|a|^2)|\frac{2|a|^4}{1-|a|^2} g(a) + |a|^2 \log \frac{2}{1-|a|^2} g(a) + |a|^2 Rg(a) \log \frac{2}{1-|a|^2}| \\
&\geq -\{2|a|^4 + |a|^2 \frac{2}{e}(1 - \log 2)\}|g(a)| + |a|^2(1-|a|^2)|Rg(a)| \log \frac{2}{1-|a|^2} \\
(7) \quad &\geq -(2 + \frac{2}{e}(1 - \log 2))|a|^2 + |a|^2(1-|a|^2)|Rg(a)| \log \frac{2}{1-|a|^2}.
\end{aligned}$$

Next let

$$f_a(z) = h_a(z) - \int_0^1 < z, a > \log \frac{2}{1 - t < z, a >} dt$$

then

$$Rf_a(z) =< z, a > \{(\log \frac{2}{1- < z, a >})^2 (\log \frac{2}{1-|a|^2})^{-1} - \log \frac{2}{1- < z, a >}\}$$

$$RRf_a(z) = RRh_a(z) - < z, a > \log \frac{2}{1- < z, a >} - \frac{< z, a >^2}{1- < z, a >}$$

and consequently $N = \sup_{\sqrt{1-2/e} \leq |a| < 1} ||f_a|| < \infty$. Note that $Rf_a(a) = 0$ and $RRf_a(a) = \frac{|a|^4}{1-|a|^2}$, we have

$$\begin{aligned}
\infty &> ||I_g|| \cdot ||f_a|| \geq ||I_g f_a|| \\
&\geq \sup_{z \in B}(1-|z|^2)|RRf_a(z) \cdot g(z) + Rf_a(z) \cdot Rg(z)| \\
(8) \quad &\geq (1-|a|^2)|RRf_a(a)g(a) + Rf_a(a)Rg(a)| = |a|^4 |g(a)|.
\end{aligned}$$

From the maximum modulus theorem, we get $g \in H^\infty$. So it follows from (7) and (8) that

$$(9) \qquad \sup_{\sqrt{1-2/e} \leq |a| < 1} (1-|a|^2)|Rg(a)| \log \frac{2}{1-|a|^2} < \infty.$$

On the other hand, we have

$$\begin{aligned}
&\sup_{|a|\le\sqrt{1-2/e}}(1-|a|^2)|Rg(a)|\log\frac{2}{1-|a|^2}\\
&\le\frac{2}{e}\cdot(1-\log 2)\max_{|a|=\sqrt{1-2/e}}|Rg(a)|\\
&\le\sup_{\sqrt{1-2/e}\le|a|<1}(1-|a|^2)|Rg(a)|\log\frac{2}{1-|a|^2}<+\infty. \qquad (10)
\end{aligned}$$

Combining (9) and (10), we finish the proof of Theorem 2.

Corollary The multiplication operator $M_g : \mathcal{Z} \to \mathcal{Z}$ is bounded if and only if $g \in \mathcal{Z}$.

Proof. If M_g is bounded on $\mathcal{Z}$, then setting the test function $f \equiv 1$, we have $M_g f = g \in \mathcal{Z}$.

Conversely, if $g \in \mathcal{Z}$, from Lemma 1 and (5), it is easy to see that $g \in H^\infty \cap \mathcal{B}_{log}$, so by Theorems 1 and 2, both T_g and I_g are bounded, it follows from (1) that M_g is also bounded.

Theorem 3. Suppose $g \in H(B)$, $I_g : \mathcal{Z} \to \mathcal{Z}$. Then I_g is compact if and only if $g = 0$.

Proof. The sufficiency is obvious. We just need to prove the necessity. Suppose that I_g is compact, for any given sequence $(z_k)_{k\in N}$ in B such that $|z_k| \to 1$ as $k \to \infty$, if we can show $g(z_k) \to 0$ as $k \to \infty$, then by the maximum modulus theorem we have $g \equiv 0$. In fact, setting

$$f_k(z) = h_{z_k}(z) - (\log\frac{2}{1-|z_k|})^{-2}\int_0^1 < z, z_k > (\log\frac{2}{1-t< z, z_k >})^3 dt.$$

Using the same way as in Theorem 2, we can show $\sup_{k\in N}||f_k|| \le C$ and f_k converges to 0 uniformly on compact subsets of B. Since I_g is compact, we have $||I_g f_k|| \to 0$ as $k \to \infty$. Note that $Rf_k(z_k) = 0$ and $RRf_k(z_k) = -\frac{|z_k|^4}{1-|z_k|^2}$, it follows that

$$\begin{aligned}
|z_k|^4|g(z_k)| &\le \sup_{z\in B}(1-|z|^2)|RRf_k(z)\cdot g(z) + Rf_k(z)\cdot Rg(z)|\\
&\le \sup_{z\in B}(1-|z|^2)|RR(I_g f_k)(z)| \le ||I_g f_k|| \to 0
\end{aligned}$$

as $k \to \infty$. This ends the proof of Theorem 3.

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Linear Combination of Laplace and Gumbel Random Variables

by

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Abstract: The distribution of linear combinations of random variables arises explicitly in many areas of the sciences, engineering and medicine. This has increased the need to have available the widest possible range of statistical results on linear combinations of random variables. In this note, the exact distribution of the linear combination $\alpha X + \beta Y$ is derived when X and Y are independent Laplace and Gumbel random variables. A computer program is provided for the associated percentile points.

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Keywords and Phrases: Gumbel distribution; Laplace distribution; Linear combination of random variables; Maple.

1 Introduction

The distribution of the linear combination $\alpha X + \beta Y$ has been studied by several authors when X and Y are independent random variables and come from the same family. For instance, see Fisher (1935) and Chapman (1950) for Student's t family, Christopeit and Helmes (1979) for normal family, Davies (1980) and Farebrother (1984) for chi-squared family, Ali and Obaidullah (1982) for exponential family, Moschopoulos (1985) and Provost (1989) for gamma family, Dobson *et al.* (1991) for Poisson family, Pham-Gia and Turkkan (1993) and Pham and Turkkan (1994) for beta family, Kamgar-Parsi *et al.* (1995) and Albert (2002) for uniform family, Hitezenko (1998) and Hu and Lin (2001) for Rayleigh family, and Witkovský (2001) for inverted gamma family.

However, there is relatively little work of the above kind when X and Y belong to different families. In applications, it is quite possible that X and Y could arise from different but similar distributions. Two such distributions are the Gumbel and Laplace distributions specified by the probability density functions (pdfs)

$$f_X(x) = \exp\left(-\frac{x-\mu}{\sigma}\right)\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\} \tag{1}$$

and

$$f_Y(y) = \frac{\lambda}{2}\exp\left\{-\lambda \mid y-\theta \mid\right\}, \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \mu < \infty$, $-\infty < \theta < \infty$, $\sigma > 0$ and $\lambda > 0$.

The Gumbel distribution given by (1) is perhaps the most widely applied statistical distribution for problems in engineering. It is also known as the extreme value distribution of type I. Some

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of its recent application areas in engineering include: flood frequency analysis, network engineering, nuclear engineering, offshore engineering, risk-based engineering, space engineering, software reliability engineering, structural engineering, and wind engineering. A recent book by Kotz and Nadarajah (2000), which describes this distribution, lists over fifty applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds and track race records (to mention just a few).

The Laplace distribution given by (2) has found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. They are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data.

The aim of this note is to study the exact distribution of $\alpha X+\beta Y$ when X and Y are independent random variables distributed according to (1) and (2), respectively. We assume without loss of generality that $\alpha > 0$. The results of this note are organized as follows. Section 2 derives explicit expressions for the pdf and the cdf (cumulative distribution function) of $\alpha X + \beta Y$. Moment properties of $\alpha X + \beta Y$, including characteristic functions, moments, factorial moments, skewness and kurtosis, are considered in Section 3. A computer program for the percentile points of $\alpha X+\beta Y$ is given in Section 4.

The calculations of this note involve several special functions, including the exponential integral defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\exp(t)}{t} dt,$$

the complementary error function defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt,$$

the complementary error function defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt,$$

the complementary incomplete gamma function defined by

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt,$$

the incomplete gamma function defined by

$$\gamma(a,x) = \int_0^x t^{a-1} \exp\left(-t\right) dt$$

and the complementary incomplete gamma function defined by

$$\Gamma(a,x) = \int_x^\infty t^{a-1} \exp\left(-t\right) dt.$$

The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Exact Distribution of $Z = \alpha X + \beta Y$

Theorem 1 derives explicit expressions for the pdf and the cdf of $\alpha X+\beta Y$ in terms of the incomplete gamma functions.

Theorem 1 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. The cdf of $Z = \alpha X + \beta Y$ can be expressed as*

$$\begin{aligned} F(z) &= \frac{\alpha\lambda\sigma}{2 \mid \beta \mid}\left[\exp\left(\frac{\lambda x}{\mid \beta \mid}\right)\Gamma\left(-\frac{\lambda\alpha\sigma}{\mid \beta \mid}, \exp\left(\frac{x}{\alpha\sigma}\right)\right)\right. \\ &\quad \left. + \exp\left(-\frac{\lambda x}{\mid \beta \mid}\right)\gamma\left(\frac{\lambda\alpha\sigma}{\mid \beta \mid}, \exp\left(\frac{x}{\alpha\sigma}\right)\right)\right] \end{aligned} \tag{3}$$

for $-\infty < z < \infty$, where $x = \beta\theta + \mu\alpha - z$. The corresponding pdf is:

$$\begin{aligned} f(z) &= \frac{\lambda}{2 \mid \beta \mid}\exp\left(\frac{2x}{\alpha\sigma}\right)\left[\exp\left(\frac{\lambda x}{\mid \beta \mid}\right)\Gamma\left(-\frac{\lambda\alpha\sigma}{\mid \beta \mid} - 1, \exp\left(\frac{x}{\alpha\sigma}\right)\right)\right. \\ &\quad \left. + \exp\left(-\frac{\lambda x}{\mid \beta \mid}\right)\gamma\left(\frac{\lambda\alpha\sigma}{\mid \beta \mid} - 1, \exp\left(\frac{x}{\alpha\sigma}\right)\right)\right] \end{aligned} \tag{4}$$

for $-\infty < z < \infty$.

Proof: One can write

$$\begin{aligned} \Pr(\alpha X + \beta Y \le z) &= \Pr\left(X \le \frac{z - \beta Y}{\alpha}\right) \\ &= \int_{-\infty}^{\infty} F_X\left(\frac{z - \beta y}{\alpha}\right) f_Y(y) dy \\ &= \frac{\lambda}{2}\int_{-\infty}^{\infty} \exp\{-\lambda \mid y - \theta \mid\} \exp\left\{-\exp\left(\frac{\mu\alpha + \beta y - z}{\alpha\sigma}\right)\right\} dy \\ &= \frac{\lambda}{2}\left[\int_{-\infty}^{\theta} \exp\{-\lambda(\theta - y)\} \exp\left\{-\exp\left(\frac{\mu\alpha + \beta y - z}{\alpha\sigma}\right)\right\} dy\right. \\ &\quad \left. + \int_{\theta}^{\infty} \exp\{-\lambda(y - \theta)\} \exp\left\{-\exp\left(\frac{\mu\alpha + \beta y - z}{\alpha\sigma}\right)\right\} dy\right] \\ &= \frac{\alpha\lambda\sigma}{2 \mid \beta \mid}\left[\exp\left(\frac{\lambda x}{\mid \beta \mid}\right)\int_{\exp\{x/(\alpha\sigma)\}}^{\infty} u^{-(\lambda\alpha\sigma)/\mid\beta\mid - 1}\exp(-u) du\right. \\ &\quad \left. + \exp\left(-\frac{\lambda x}{\mid \beta \mid}\right)\int_{0}^{\exp\{x/(\alpha\sigma)\}} u^{(\lambda\alpha\sigma)/\mid\beta\mid - 1}\exp(-u) du\right], \end{aligned} \tag{5}$$

where the last step follows by substituting $u = \exp\{(\beta y + \alpha\mu - z)/(\alpha\sigma)\}$. The result in (3) follows from (5) by using the definitions of the incomplete gamma functions defined in Section 1. The result in (4) is the derivative of (3). ■

Using special properties of the incomplete gamma functions, one can reduce (3) to elementary forms when $(\lambda\alpha\sigma)/ \mid \beta \mid$ takes integer or half-integer values. This is illustrated in the corollaries below.

Corollary 1 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. If $\lambda\alpha\sigma/\mid\beta\mid = n+1/2$ then (3) can be reduced to*

$$F(z) = \frac{\alpha\lambda\sigma}{2\mid\beta\mid}\left[\exp\left(\frac{\lambda x}{\mid\beta\mid}\right)I_1(n)+\exp\left(-\frac{\lambda x}{\mid\beta\mid}\right)I_2(n)\right] \tag{6}$$

for $-\infty < z < \infty$, where

$$I_1(n) = \frac{(-1)^{n+1}\sqrt{\pi}}{(1/2)_{n+1}}\mathrm{erfc}\,(\sqrt{y})-y^{-(n+1/2)}\exp(-y)\sum_{k=0}^{n}\frac{y^k}{(-1/2-n)_{k+1}}$$

and

$$I_2(n) = \Gamma\left(n+\frac{1}{2}\right)\mathrm{erf}\,(\sqrt{y})-(-1)^{n-1}\exp(-y)\sqrt{y}\sum_{k=0}^{n-1}\left(\frac{1}{2}-n\right)_{n-k-1}(-y)^k,$$

where $y=\exp\{x/(\alpha\sigma)\}$ and $x=\beta\theta+\mu\alpha-z$. If $-\lambda\alpha\sigma/\mid\beta\mid = n+1/2$ then (3) can be reduced to (6) with

$$I_1(n) = \Gamma\left(n+\frac{1}{2}\right)\mathrm{erfc}\,(\sqrt{y})+(-1)^{n-1}\exp(-y)\sqrt{y}\sum_{k=0}^{n-1}\left(\frac{1}{2}-n\right)_{n-k-1}(-y)^k$$

and

$$I_2(n) = \Gamma\left(-n-\frac{1}{2}\right)-\frac{(-1)^{n+1}\sqrt{\pi}}{(1/2)_{n+1}}\mathrm{erfc}\,(\sqrt{y})+y^{-(n+1/2)}\exp(-y)\sum_{k=0}^{n}\frac{y^k}{(-1/2-n)_{k+1}}.$$

Corollary 2 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. If $\lambda\alpha\sigma/\mid\beta\mid = n$ then (3) can be reduced to*

$$F(z) = \frac{\alpha\lambda\sigma}{2\mid\beta\mid}\left[\exp\left(\frac{\lambda x}{\mid\beta\mid}\right)I_1(n)+\exp\left(-\frac{\lambda x}{\mid\beta\mid}\right)I_2(n)\right] \tag{7}$$

for $-\infty < z < \infty$, where

$$I_1(n) = \frac{(-1)^{n-1}}{n!}\left[\mathrm{Ei}(y)-\frac{1}{2}\left\{\log(-y)-\log\left(-\frac{1}{y}\right)\right\}+\log(y)\right]-\exp(-y)\sum_{k=1}^{n}\frac{y^{k-n-1}}{(-n)_k}$$

and

$$I_2(n) = (n-1)!-(n-1)!\exp(-y)\sum_{k=0}^{n-1}\frac{y^k}{k!},$$

where $y=\exp\{x/(\alpha\sigma)\}$ and $x=\beta\theta+\mu\alpha-z$. If $-\lambda\alpha\sigma/\mid\beta\mid = n$ then (3) can be reduced to (7) with

$$I_1(n) = (n-1)!\exp(-y)\sum_{k=0}^{n-1}\frac{y^k}{k!}$$

and

$$I_2(n) = \Gamma(-n) - \frac{(-1)^{n-1}}{n!}\left[\mathrm{Ei}(y) - \frac{1}{2}\left\{\log(-y) - \log\left(-\frac{1}{y}\right)\right\} - \log(y)\right] - \exp(-y)\sum_{k=1}^{n}\frac{y^{k-n-1}}{(-n)_k}.$$

The following corollaries provide the pdfs and the cdfs for the sum and the difference of the Gumbel and Laplace random variables.

Corollary 3 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. Then, the cdf and the pdf of $Z = X + Y$ can be expressed as*

$$F(z) = \frac{\lambda\sigma}{2}\left[\exp(\lambda x)\,\Gamma\left(-\lambda\sigma, \exp\left(\frac{x}{\sigma}\right)\right) + \exp(-\lambda x)\,\gamma\left(\lambda\sigma, \exp\left(\frac{x}{\sigma}\right)\right)\right]$$

and

$$f(z) = \frac{\lambda}{2}\exp\left(\frac{2x}{\sigma}\right)\left[\exp(\lambda x)\,\Gamma\left(-\lambda\sigma - 1, \exp\left(\frac{x}{\sigma}\right)\right) + \exp(-\lambda x)\,\gamma\left(\lambda\sigma - 1, \exp\left(\frac{x}{\sigma}\right)\right)\right]$$

for $-\infty < z < \infty$, where $x = \theta + \mu - z$.

Corollary 4 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. Then, the cdf and the pdf of $Z = X - Y$ can be expressed as*

$$F(z) = \frac{\lambda\sigma}{2}\left[\exp(\lambda x)\,\Gamma\left(-\lambda\sigma, \exp\left(\frac{x}{\sigma}\right)\right) + \exp(-\lambda x)\,\gamma\left(\lambda\sigma, \exp\left(\frac{x}{\sigma}\right)\right)\right]$$

and

$$f(z) = \frac{\lambda}{2}\exp\left(\frac{2x}{\sigma}\right)\left[\exp(\lambda x)\,\Gamma\left(-\lambda\sigma - 1, \exp\left(\frac{x}{\sigma}\right)\right) + \exp(-\lambda x)\,\gamma\left(\lambda\sigma - 1, \exp\left(\frac{x}{\sigma}\right)\right)\right]$$

for $-\infty < z < \infty$, where $x = -\theta + \mu - z$.

[Figure 1 about here.]

Figure 1 illustrates possible shapes of (4) for selected values of α, β and σ. The four curves in each plot correspond to selected values of σ. The effect of the parameters is evident.

3 Moment Properties of $Z = \alpha X + \beta Y$

The moment properties of $Z = \alpha X + \beta Y$ can be derived by knowing the same for X and Y since

$$E(Z^n) = \sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}E\left(X^k\right)E\left(Y^{n-k}\right).$$

It is well known (see, for example, Johnson *et al.* (1995)) that

$$E(X^n) = \sum_{k=0}^{n}\binom{n}{k}\mu^{n-k}\sigma^k(-1)^k\Gamma^{(k)}(1)$$

and

$$E\left(Y^n\right) = \lambda \sum_{k=0}^{n} \binom{n}{k} \theta^{n-k} \lambda^{-k-1} k! I\{k \text{ even}\},$$

where $\Gamma^{(k)}(1)$ denotes the kth derivative of $\Gamma(x)$ at $x = 1$ and $I\{\}$ denotes the indicator function. Thus, the first four moments of Z can be calculated as

$$E\left(Z\right) = \alpha\left\{\mu - \sigma\Gamma^{'}(1)\right\} + \beta\theta,$$

$$E\left(Z^2\right) = \alpha^2\left\{\mu^2 - 2\mu\sigma\Gamma^{'}(1) + \sigma^2\Gamma^{''}(1)\right\} + \beta^2\left(\theta^2 + 2\lambda^{-2}\right) + 2\theta\alpha\beta\left\{\mu - \sigma\Gamma^{'}(1)\right\},$$

$$\begin{aligned} E\left(Z^3\right) = {} & \alpha^3\left\{\mu^3 - 3\mu^2\sigma\Gamma^{'}(1) + 3\mu\sigma^2\Gamma^{''}(1) - \sigma^3\Gamma^{'''}(1)\right\} + \beta^3\theta\left(\theta^2 + 6\lambda^{-2}\right) \\ & + 3\alpha\beta^2\left(\theta^2 + 2\lambda^{-2}\right)\left\{\mu - \sigma\Gamma^{'}(1)\right\} + 3\theta\alpha^2\beta\left\{\mu^2 - 2\mu\sigma\Gamma^{'}(1) + \sigma^2\Gamma^{''}(1)\right\} \end{aligned}$$

and

$$\begin{aligned} E\left(Z^4\right) = {} & \alpha^4\left\{\mu^4 - 4\mu^3\sigma\Gamma^{'}(1) + 6\mu^2\sigma^2\Gamma^{''}(1) - 4\mu\sigma^3\Gamma^{'''}(1) + \sigma^4\Gamma^{''''}(1)\right\} \\ & + \beta^4\left(\theta^4 + 12\theta^2\lambda^{-2} + 24\lambda^{-4}\right) + 4\alpha\beta^3\theta\left(\theta^2 + 6\lambda^{-2}\right)\left\{\mu - \sigma\Gamma^{'}(1)\right\} \\ & + 4\theta\alpha^3\beta\left\{\mu^3 - 3\mu^2\sigma\Gamma^{'}(1) + 3\mu\sigma^2\Gamma^{''}(1) - \sigma^3\Gamma^{'''}(1)\right\} \\ & + 6\alpha^2\beta^2\left(\theta^2 + 2\lambda^{-2}\right)\left\{\mu^2 - 2\mu\sigma\Gamma^{'}(1) + \sigma^2\Gamma^{''}(1)\right\}. \end{aligned}$$

Note that $\Gamma^{'}(1) = -C$, $\Gamma^{''}(1) = C^2 + \pi^2/6$, $\Gamma^{'''}(1) = -C^3 - \pi^2 C/2 - 2\eta(3)$ and $\Gamma^{''''}(1) = C^4 + \pi^2 C^2 + 8\eta(3)C + 3\pi^2/20$, where C is the Euler's constant and $\eta(\cdot)$ denotes the zeta function defined by $\zeta(x) = \sum_{k=1}^{\infty} k^{-x}$. The factorial moments, skewness and the kurtosis can be calculated by using the relationships that

$$E\left[(Z)_n\right] = E\left[Z(Z-1)\cdots(Z-n+1)\right],$$

$$Skewness(Z) = \frac{E\left(Z^3\right) - 3E(Z)E\left(Z^2\right) + 2E^3(Z)}{\left\{E\left(Z^2\right) - E^2\left(Z\right)\right\}^{3/2}},$$

and

$$Kurtosis(Z) = \frac{E\left(Z^4\right) - 4E(Z)E\left(Z^3\right) + 6E\left(Z^2\right)E^2(Z) - 3E^4(Z)}{\left\{E\left(Z^2\right) - E^2\left(Z\right)\right\}^2}.$$

Finally, using the facts that the characteristic functions (chfs) of X and Y are

$$E\left[\exp(itX)\right] = \Gamma\left(1 - i\sigma t\right)\exp(i\mu t)$$

and

$$E\left[\exp(itY)\right] = \frac{\lambda^2 \exp(i\theta t)}{\lambda^2 + t^2},$$

where $i = \sqrt{-1}$, the chf of Z can be expressed as

$$E\left[\exp(itZ)\right] = \Gamma\left(1 - i\sigma\alpha t\right)\exp(i\mu\alpha t)\frac{\lambda^2 \exp(i\theta\beta t)}{\lambda^2 + \beta^2 t^2}.$$

4 Percentiles of $Z = \alpha X + \beta Y$

In this section, we provide a computer program for the percentage points z_p associated with the cdf of $Z = \alpha X + \beta Y$. The value of z_p is obtained by solving the equation

$$\frac{\alpha\lambda\sigma}{2 \mid \beta \mid}\left[\exp\left(\frac{\lambda x_p}{\mid \beta \mid}\right)\Gamma\left(-\frac{\lambda\alpha\sigma}{\mid \beta \mid}, \exp\left(\frac{x_p}{\alpha\sigma}\right)\right) + \exp\left(-\frac{\lambda x_p}{\mid \beta \mid}\right)\gamma\left(\frac{\lambda\alpha\sigma}{\mid \beta \mid}, \exp\left(\frac{x_p}{\alpha\sigma}\right)\right)\right] = p, \quad (8)$$

where $x_p = \beta\theta + \mu\alpha - z_p$. Evidently, this involves computation of the incomplete gamma functions and routines for this are widely available. We used the functions GAMMA $(\cdot)$ and GAMMA $(\cdot, \cdot)$ in the algebraic manipulation package, MAPLE.

```
percent:=proc(alpha,beta,mu,sigma,theta,lambda,p)
local x,z,tt1,tt2,tt3,tt4,tt,ff;
x:=beta*theta+mu*alpha-z;
tt1:=exp(lambda*x/abs(beta));
tt2:=exp(-lambda*x/abs(beta));
tt3:=GAMMA(-lambda*alpha*sigma/abs(beta),exp(x/(alpha*sigma)));
tt4:=GAMMA(lambda*alpha*sigma/abs(beta),exp(x/(alpha*sigma)));
tt4:=tt4-GAMMA(lambda*alpha*sigma/abs(beta));
ff:=alpha*lambda*sigma*(tt1*tt3+tt2*tt4)/(2*abs(beta));
tt:=fsolve(ff=p,z=-1000..1000);
end proc;
```

The above is a listing of a MAPLE procedure for solving (8). For given alpha, beta, mu, sigma, theta, lambda and p, the call to percent(alpha,beta,mu,sigma,theta,lambda,p) will return the value of z_p.

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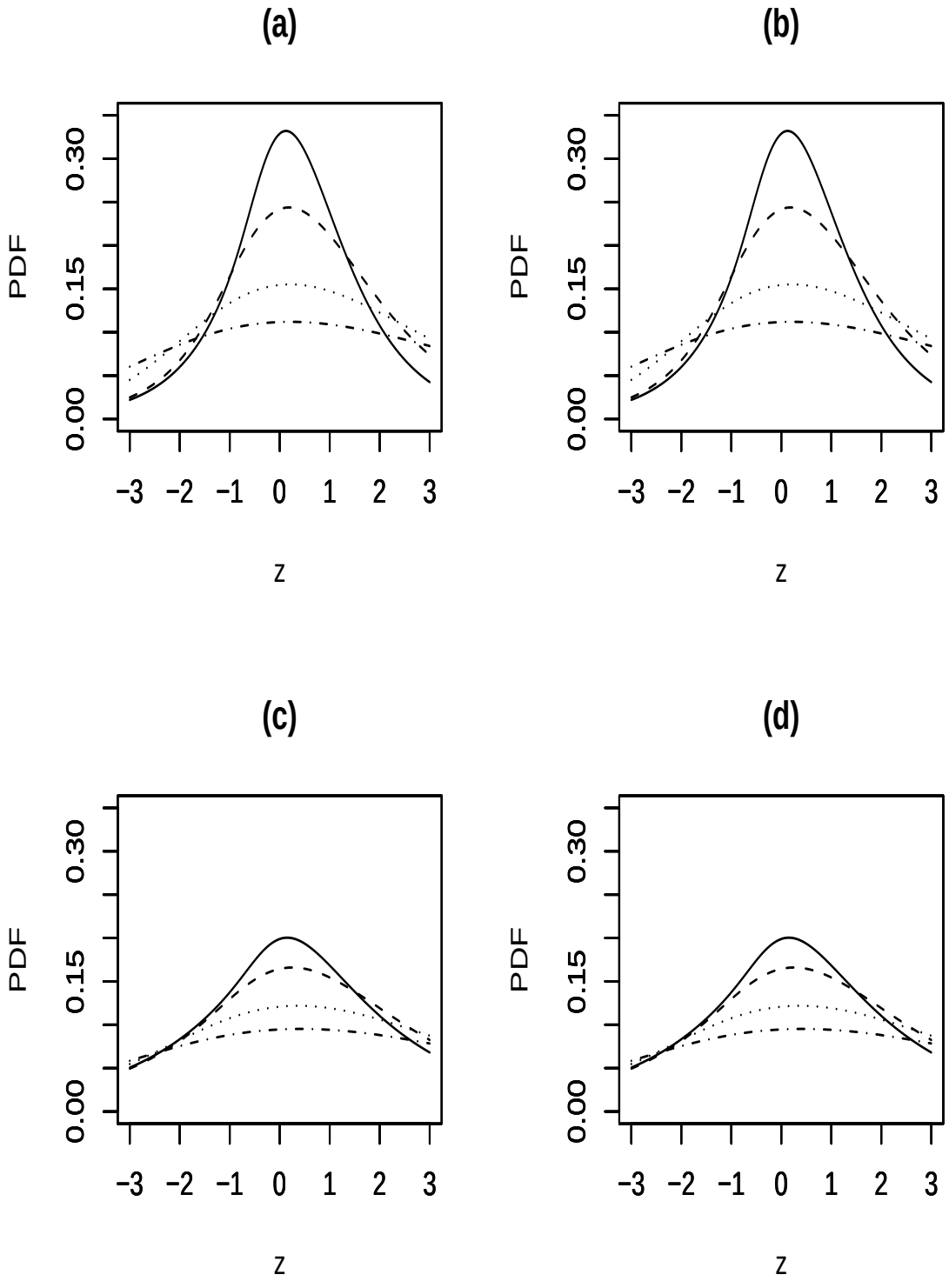


Figure 1. Plots of the pdf (4) for $\mu = 0$, $\theta = 0$, $\lambda = 1$, $\sigma = 0.5, 1, 2, 3$ and (a): $\alpha = 1$ and $\beta = 1$; (b): $\alpha = 1$ and $\beta = -1$; (c): $\alpha = 1$ and $\beta = 2$; and, (d): $\alpha = 1$ and $\beta = -2$. The four curves in each plot from the top to the bottom correspond to increasing values of σ.

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ON SOME NEW DOUBLE LACUNARY SEQUENCES SPACES VIA ORLICZ FUNCTION

EKREM SAVAŞ

ABSTRACT. In this paper we define and study two concepts which arise from the notions of invariant means and lacunary sequences namely: double lacunary strong σ- convergence defined by Orlicz function and uniform (θ, σ)-statistical convergence and establish natural characterization for the underline sequence spaces.

1. Introduction and Background

Let l_∞ be the Banach space of bounded $x = (x_k)$ with the usual norm $||x|| = \sup_n |x_n|$. A sequence $x \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [2] proved that

$$\hat{c} = \{x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + \cdots + x_{m+n}}{m+1}.$$

The following space of strongly almost convergent sequence was introduced by Maddox in [3]

$$[\hat{c}] = \{x \in l_\infty : \lim_m t_{m,n}(|x - Le|) \text{ exists uniformly in } n \text{ for some } L \in c\}$$

where $e = (1, 1, \ldots)$.

Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean provided that

i $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all k,
ii $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and
iii $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in l_\infty$.

For certain class of mapping σ every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$.

The space $[V_\sigma]$ is of strongly σ-convergent sequence was introduced by Mursaleen [6] as follows: A sequence $x = (x_k)$ is said to be strongly σ-convergent if there exists a number L such that

$$\frac{1}{k} \sum_{i=1}^{k} |x_{\sigma^i(m)} - L| \to 0 \tag{1.1}$$

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as $k \to \infty$ uniformly in m. We will denote $[V_\sigma]$ as the set of all strongly σ-convergent sequences. When (1.1) holds we write $[V_\sigma]-\lim x = L$. If we let $\sigma(m) = m+1$, then $[V_\sigma] = [\hat{c}]$. In 1900 Pringsheim presented the following definition for the convergence of double sequences.

Definition 1.1 (Pringsheim, [7]). *A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as "P-convergent".*

We shall denote the space of all P-convergent sequences by $c^{''}$. By a bounded double sequence we shall mean there exists a positive number K such that $|x_{k,l}| < K$ for all (k, l), and denote such bounded by $||x||_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty$. We shall also denote the set of all bounded double sequences by $l^{''}_{\infty}$. We also note in contrast to the case for single sequence, a P-convergent double sequence need not be bounded.

Quite recently Savaş and Patterson [10] defined the following sequence spaces by using Orlicz function.

Definition 1.2. *Let M be an Orlicz function,which is defined in* [8], *$p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers, and let*

$$[V^{''}_\sigma, M]_p = \left\{ x = (x_{k,l}) : P - \lim_{p,q} \frac{1}{pq} \sum_{k,l=1,1}^{p,q} \left(M \left(\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho} \right) \right)^{p_{k,l}} = 0, \right.$$

$$\left. \text{uniformly in } (m,n), \text{ for some } \rho > 0, \text{ and some } L > 0 \right\},$$

and

$$[V^{''}_\sigma, M]^0_p = \left\{ x = (x_{k,l}) : P - \lim_{p,q} \frac{1}{pq} \sum_{k,l=1,1}^{p,q} \left(M \left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho} \right) \right)^{p_{k,l}} = 0, \right.$$

$$\left. \text{uniformly in } (m,n), \text{ for some } \rho > 0 \right\},$$

If $M(x) = x$ then $[V^{''}_\sigma, M]_p$,and $[V^{''}_\sigma, M]^0_p$, reduces to $[V^{''}_\sigma]_p$, and $[V^{''}_\sigma]^0_p$ respectively. When $p_{k,l} = 1$ for all k and l, $[V^{''}_\sigma]_p$,and $[V^{''}_\sigma]^0_p$ reduces to $[V^{''}_\sigma]$,and $[V^{''}_\sigma]^0$ respectively.

Before we enter the motivation for this paper and the presentation of the main results we give some known definitions.

Definition 1.3. *(*[12]*). The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called* **double lacunary** *if there exist two increasing of integers such that*

$$k_0 = 0, h_r = k_r - k_{k-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$, $\theta_{r,s}$ is determine by $I_{r,s} = \{(k,l) : k_{r-1} < k \le k_r \& l_{s-1} < l \le l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r \bar{q}_s$.

It is quite natural to expect that the sets of sequences that are double lacunary strong double σ-summable to zero, lacunary strong double σ-summable and lacunary strong double σ-bounded can be defined by combining the concepts of Orlicz

function, and σ-mean. We now ready to present the multidimensional sequence spaces.

Definition 1.4. *Let $\theta_{r,s}$ be a double lacunary sequence and let M be an Orlicz function, and $p = (p_{k,l})$ be any factorable double sequence of strictly positive real numbers. Then we write*

$$[N_{\theta_{r,s}}, M, p]^{\sigma} = \{x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}| - L}{\rho}\right)\right]^{p_{k,l}} = 0$$

$$\text{uniformly in } (m,n), \text{ for some } \rho > 0 \text{ and some } L > 0\},$$

$$[N_{\theta_{r,s}}, M, p]_0^{\sigma} = \{x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} = 0$$

$$\text{uniformly in } (m,n), \text{ for some } \rho > 0\},$$

and

$$[N_{\theta_{r,s}}, M, p]_{\infty}^{\sigma} = \{x = (x_{k,l}) : \sup_{r,s,m,n} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{x_{\sigma^{k,l}(m,n)}}{\rho}\right)\right]^{p_{k,l}} < \infty$$

$$\text{for some } \rho > 0\}.$$

We shall denote $[N_{\theta_{r,s}}, M, p]^{\sigma}$, $[N_{\theta_{r,s}}, M, p]_0^{\sigma}$, and $[N_{\theta_{r,s}}, M, p]_{\infty}^{\sigma}$ as $[N_{\theta_{r,s}}, M]^{\sigma}$, $[N_{\theta_{r,s}}, M,]_0^{\sigma}$, and $[N_{\theta_{r,s}}, M,]_{\infty}^{\sigma}$ respectively when $p_{k,l} = 1$ for all k and l. If x is in $[N_{\theta_{r,s}}, M]$, we shall say that x is double lacunary strongly σ-convergent with respect to the Orlicz function M. Also note if $M(x) = x$, $p_{k,l} = 1$ for all k and l, then $[N_{\theta_{r,s}}, M, p]^{\sigma} = [N_{\theta_{r,s}}]^{\sigma}$ and $[N_{\theta_{r,s}}, M, p]_0^{\sigma} = [N_{\theta_{r,s}}]_0^{\sigma}$ which are defined as follows:

$$[N_{\theta_{r,s}}]^{\sigma} = \{x = (x_{k,l}) : P\text{–}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{\sigma^k(m),\sigma^l(n)} - L| = 0, \text{ uniformly in } (m,n)$$

$$\text{and some } L > 0\}.$$

and

$$[N_{\theta_{r,s}}]_0^{\sigma} = \{x = (x_{k,l}) : P\text{–}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{\sigma^k(m),\sigma^l(n)}| = 0, \text{ uniformly in } (m,n)\}.$$

If we take $\sigma(m) = m + 1$, and $\sigma(n) = n + 1$ we can get $[\hat{N}_{\theta_{r,s}}, M, p]$, $[\hat{N}_{\theta_{r,s}}, M, p]_0$, and $[\hat{N}_{\theta_{r,s}}, M, p]_{\infty}$.

2. Main Results

With these new concepts we can now consider the following theorem. The proof of the theorem is standard and so we omitted.

Theorem 2.1. *Let the sequence $(p_{k,l})$ be bounded then $[N_{\theta_{r,s}}, M, p]^{\sigma}$, $[N_{\theta_{r,s}}, M, p]_0^{\sigma}$, and $[N_{\theta_{r,s}}, M, p]_{\infty}^{\sigma}$ are linear spaces over the set of complex numbers.*

Definition 2.1. *An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ for all $u \geq 0$. The Δ_2-condition is equivalent to the satisfaction of the following inequality $M(lu) \leq K(l)M(u)$ for all values of u and for $l \geq 1$.*

Theorem 2.2. *Let M be an Orlicz function. If $\beta = \lim_{t\to\infty} \frac{M(t\backslash\rho)}{t} \geq 1$, then $[N_{\theta_{r,s}}, M]^{\sigma} = [N_{\theta_{r,s}}]^{\sigma}$.*

Proof. Let $x \in [N_{\theta_{r,s}}]^{\sigma}$, then

$$T_{m,n}^{r,s} = P - \lim_{r,s} \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} |x_{\sigma^k(m),\sigma^l(n)} - L| = 0, \text{ uniformly in } (m,n).$$

Let $\epsilon > 0$ be given and choose $0 < \delta < 1$ such that $M(u) < \epsilon$ for every $0 \leq u \leq \delta$. We can write for each (m,n)

$$\begin{aligned}\sum_{(k,l)\in I_{r,s}} M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right) &= \sum_{(k,l)\in I_{r,s}\&|x_{\sigma^k(m),\sigma^l(n)}-L|\leq\delta} M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right) \\ &+ \sum_{(k,l)\in I_{r,s}\&|x_{\sigma^{k,l}(m,n)}-L|>\delta} M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right).\end{aligned}$$

It is clear that:

$$\sum_{(k,l)\in I_{r,s}\&|x_{\sigma^k(m),\sigma^l(n)}-L|\leq\delta} M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right) < \epsilon(h_{rs}).$$

On the other hand, we use the fact that

$$|x_{\sigma^k(m),\sigma^l(n)} - L| < 1 + \left[\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right]$$

where $[h]$ denotes the integer part of h. Since M is an Orlicz function we have

$$M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right) \geq M(1).$$

Now, let us consider the second part where the sum is taken over $|x_{\sigma^k(m),\sigma^l(n)}-L| > \delta$. Thus

$$\begin{aligned}\sum_{(k,l)\in I_{r,s}\&|x_{\sigma^k(m),\sigma^l(n)}-L|>\delta} M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right) &\leq \sum_{(k,l)\in I_{r,s}} M\left(1 + \left[\frac{|x_{\sigma^k(m),\sigma^l(n)} - L|}{\rho}\right]\right) \\ &\leq 2M(1)\frac{1}{\delta}(h_{r,s})T_{m,n}^{r,s}.\end{aligned}$$

Therefore

$$\sum_{(k,l)\in I_{r,s}} M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}-L|}{\rho}\right) \leq \epsilon(h_{r,s}) + 2M(1)\frac{1}{\delta}(h_{r,s})T^{r,s}_{m,n}$$

for every (m,n). Hence $x \in [N_{\theta_{r,s}}, M]$. Observe that in this part of the proof we did not need $\beta \geq 1$. Let $\beta \geq 1$ and $x \in [N_{\theta_{r,s}}, M]^\sigma$. Since $\beta \geq 1$ we have $M(t) \geq \beta(t)$ for all $t \geq 0$. It follows that $x_{k,l} \to L[N_{\theta_{r,s}}, M]^\sigma$ implies $x_{k,l} \to L[N_{\theta_{r,s}}]^\sigma$. This implies $[N_{\theta_{r,s}}, M]^\sigma = [N_{\theta_{r,s}}]^\sigma$. □

Theorem 2.3. *Let $\theta_{r,s} = \{k_r, l_s\}$ be a double lacunary sequence with $\liminf_r q_r > 1$, and $\liminf_s \bar{q}_s > 1$ then for any Orlicz function M, $[V_\sigma^{''}, M]_p \subset [N_{\theta_{r,s}}, M, p]^\sigma$.*

Proof. It is sufficient to show that $[V_\sigma^{''}, M]_p^0 \subset [N_{\theta_{r,s}}, M, p]_0^\sigma$. The general inclusion follows by linearity. Suppose $\liminf_r q_r > 1$ and $\liminf_s \bar{q}_s > 1$; then there exists $\delta > 0$ such that $q_r > 1+\delta$ and $\bar{q}_s > 1+\delta$. This implies $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$ and $\frac{\bar{h}_s}{l_s} \geq \frac{\delta}{1+\delta}$. Then for $x \in [V_\sigma^{''}, M]_p^0$, we can write for each m and n

$$\begin{aligned}
B_{r,s} &= \frac{1}{h_{rs}} \sum_{(k,l)\in I_{r,s}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} \\
&= \frac{1}{h_{rs}} \sum_{k=1}^{k_r}\sum_{l=1}^{l_s} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} \\
&- \frac{1}{h_{rs}} \sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} \\
&- \frac{1}{h_{rs}} \sum_{k=k_r+1}^{k_r-1}\sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} \\
&- \frac{1}{h_{rs}} \sum_{l=l_s+1}^{l_s-1}\sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} \\
&= \frac{k_r k_s}{h_{rs}}\left(\frac{1}{k_r l_s}\sum_{k=1}^{k_r}\sum_{l=1}^{l_s} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}}\right) \\
&- \frac{k_{r-1} l_{s-1}}{h_{rs}}\left(\frac{1}{k_{r-1} l_{s-1}}\sum_{k=1}^{k_{r-1}}\sum_{l=1}^{l_{s-1}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}}\right) \\
&- \frac{1}{h_r}\sum_{k=k_{r-1}+1}^{k_r} \frac{l_s-1}{h_s}\frac{1}{l_s-1}\sum_{l=1}^{l_s-1} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} \\
&- \frac{1}{h_s}\sum_{l=l_{s-1}+1}^{l_s} \frac{k_{r-1}}{h_r}\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}}.
\end{aligned}$$

Since $x \in [V_\sigma^{''}, M]_p$ the last two terms trends to zero uniformly in (m,n) in the Pringsheim sense, thus for each m and n

$$B_{r,s} = \frac{k_r k_s}{h_{rs}} \left(\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M \left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho} \right) \right]^{p_{k,l}} \right)$$
$$- \frac{k_{r-1} l_{s-1}}{h_{rs}} \left(\frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M \left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho} \right) \right]^{p_{k,l}} \right) + o(1).$$

Since $h_{rs} = k_r l_s - k_{r-1} l_{s-1}$ we are granted for each m and n the following:

$$\frac{k_r l_s}{h_{rs}} \leq \frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1} l_{s-1}}{h_{rs}} \leq \frac{1}{\delta}.$$

The terms

$$\frac{1}{k_r l_s} \sum_{k=1}^{k_r} \sum_{l=1}^{l_s} \left[M \left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho} \right) \right]^{p_{k,l}}$$

and

$$\frac{1}{k_{r-1} l_{s-1}} \sum_{k=1}^{k_{r-1}} \sum_{l=1}^{l_{s-1}} \left[M \left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho} \right) \right]^{p_{k,l}}$$

are both Pringsheim null sequences. Thus $B_{r,s}$ is a Pringsheim null sequence for all m and n. Therefore x is in $[N_{\theta_{r,s}}, M, P]^{\sigma}$. This completes the proof of this theorem. □

Theorem 2.4. *Let $\theta_{r,s} = \{k, l\}$ be a double lacunary sequence with $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$ then for any Orlicz function M, $[N_{\theta_{r,s}}, M, p]^{\sigma} \subset [V_{\sigma}^{''}, M]_p$.*

Proof. Since $\limsup_r q_r < \infty$ and $\limsup_s \bar{q}_s < \infty$ there exists $H > 0$ such that $q_r < H$ and $\bar{q}_s < H$ for all r and s. Let $x \in [N_{\theta_{r,s}}, M, p]^{\sigma}$ and $\epsilon > 0$. Also there exist $r_0 > 0$ and $s_0 > 0$ such that for every $i \geq r_0$ and $j \geq s_0$ and for all m and n,

$$A_{i,j} = \frac{1}{h_{ij}} \sum_{(k,l) \in I_{i,j}} \left[M \left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho} \right) \right]^{p_{k,l}} < \epsilon.$$

Let $M^{'} = \max\{A_{i,j} : 1 \leq i \leq r_0 \text{ and } 1 \leq j \leq s_0\}$, and p and q be such that $k_{r-1} < p \leq k_r$ and $l_{s-1} < q \leq l_s$. Thus we obtain the following:

$$\begin{aligned}
\frac{1}{pq}\sum_{k,l=1,1}^{p,q}\left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} &\leq \frac{1}{k_{r-1}l_{s-1}}\sum_{k,l=1,1}^{k_r l_s}\left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}} \\
&\leq \frac{1}{k_{r-1}l_{s-1}}\sum_{p,u=1,1}^{r,s}\left(\sum_{k,l\in I_{p,u}}\left[M\left(\frac{|x_{\sigma^k(m),\sigma^l(n)}|}{\rho}\right)\right]^{p_{k,l}}\right) \\
&= \frac{1}{k_{r-1}l_{s-1}}\sum_{p,u=1,1}^{r_0,s_0} h_{p,u}A_{p,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leq r)\cup(s_0<u\leq s)} h_{p,u}A_{p,u} \\
&\leq \frac{M^{'}}{k_{r-1}l_{s-1}}\sum_{p,u=1,1}^{r_0,s_0} h_{p,u} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leq r)\cup(s_0<u\leq s)} h_{p,u}A_{p,u} \\
&\leq \frac{M^{'}k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leq r)\cup(s_0<u\leq s)} A_{p,u}h_{p,u} \\
&\leq \frac{M^{'}k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \left(\sup_{p\geq r_0\cup u\geq s_0} A_{p,u}\right)\frac{1}{k_{r-1}l_{s-1}}\sum_{(r_0<p\leq r)\cup(s_0<u\leq s)} h_{p,u} \\
&\leq \frac{M^{'}k_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \epsilon\sum_{(r_0<p\leq r)\cup(s_0<u\leq s)} h_{p,u} \\
&\leq \frac{M^{'}l_{r_0}l_{s_0}r_0s_0}{k_{r-1}l_{s-1}} + \epsilon H^2.
\end{aligned}$$

Since k_r and l_s both approaches infinity as both p and q approaches infinity.Therefore $x \in [V_\sigma^{''}, M]_p$. □

The following is an immediate consequence of Theorem 2.3 and Theorem 2.4

Theorem 2.5. *Let $\theta_{r,s} = \{k,l\}$ be a double lacunary sequence with $1 < \liminf_{r,s} q_{rs} \leq \limsup_{r,s} q_{r,s} < \infty$, then for any Orlicz function M, $[N_{\theta_{r,s}}, M, p]^\sigma = [V_\sigma^{''}, M]_p$.*

3. Double statistical convergence

In this section we discuss uniformly double lacunary statistical convergence.

The following definition was presented by Mursaleen and Edely in [4]: A real double sequence $x = (x_{k,l})$ is said to be statistically convergent to L, provided that for each $\epsilon > 0$

$$P - \lim_{m,n}\frac{1}{mn}|\{(k,l) : k \leq m \text{ and } l \leq n, |x_{k,l} - L| \geq \epsilon\}| = 0$$

where the vertical bars indicate the numbers of elements in the enclosed set.

We now present uniformly lacunary (θ,σ)- statistical analogues for double sequence $x = (x_{k,l})$ as follows:

Definition 3.1. *A double sequence $x = (x_{k,l})$ is said to be uniformly $S_{(\theta,\sigma)}^{''}$-convergent or uniformly (θ,σ)-statistical convergent to L, provided that for every*

$\epsilon > 0$

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \max_{m,n} |\{(k,l) \in I_{r,s} : |x_{\sigma^k(m),\sigma^l(n)}| > \epsilon\}| = 0.$$

In this case we write $S''_{(\theta,\sigma)} - \lim x = L$ or $x_{k,l} \to L(S''_{(\theta,\sigma)})$ and $S''_{(\theta,\sigma)} = \{x : \exists L \in \mathbb{R}, S''_{(\theta,\sigma)} - \lim x = L\}$.

Finally we conclude this paper by stating a theorem which establish an inclusion between $[N_{\theta_{r,s}}, M]^\sigma$ and $S''_{(\theta,\sigma)}$. We omit the proof since this can be proved by using the techniques similar to those used in Theorem 2.1. of Savas and Patterson [12].

Theorem 3.1. *For any Orlicz function M, $[N_{\theta_{r,s}}, M]^\sigma \subset S''_{(\theta,\sigma)}$.*

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Approximate Method for Solving a Neutron Transport Equation

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Abstract

An algorithm based on the variational form of the integral identity method is used for the determining the numerical solution of a boundary value problem that accompanies a stationary transport equation. The numerical example proves the accuracy and computational efficiency of the proposed method.

AMS: 35J99, 65N99

Key words: Integral-differential equation, integral identity method, variational method,

1 Introduction

The neutron transport equation has wide applications in physics, geophysics and astrophysics. This integral-differential equation models the transport of neutral particles in a scattering fission and absorption events with no self-interactions. It is used in radiation shielding and the reactor calculations, as well as in radiative transfer of stellar and planetary atmospheres and it also describes dispersion of light. The resolution of the problems dealing with transport phenomena is the subject of several works:[1]- [15]. The following methods are proposed by these papers: Fourier transform, Laplace transform, the least squares, the finite element, Monte Carlo, truncated series of Chebyshev polynomials. The differences schemes have a significant importance in the solution of the problems from the mathematical physics field. From these, we use for the solving an integral-differential equation, a variational form of the integral identity method, [6]. Unlike the Galerkin method, this algorithm allows the using of the discontinuous basis functions for the constructing of the approximate solution.

2 Problem formulation

The main problem in the nuclear reactor theory is to find the neutrons distribution, therefore the neutron flux φ. The number of neutrons that pass in the unit time through a perpendicular unit area on a given direction Ox defines the scalar φ. By convention, the positive flux leaves a closed surface and a negative flux enters in a closed surface.

Let us consider the neutron transport stationary theory in a plan-parallel geometry. In this case, the flux verifies the integral-differential equation

$$\mu \frac{\partial \varphi(x,\mu)}{\partial x} + \sigma \varphi(x,\mu) = \frac{\sigma_s}{2} \int_{-1}^{1} \varphi(x,\mu')d\mu' + f(x,\mu) \tag{1}$$

$$\forall (x,\mu) \in D_1 \times D_2 = [0,H] \times [-1,1], \quad D_2 = D_2' \cup D_2'' = [-1,0] \cup [0,1]$$

and the following boundary conditions:

$$\begin{aligned} &\varphi(0,\mu) = 0 \quad \text{if} \quad \mu > 0 \\ &\varphi(H,\mu) = 0 \quad \text{if} \quad \mu < 0 \end{aligned} \tag{2}$$

where

- $\varphi(x, \mu)$ is the flux in the point x corresponding to the neutrons, which migrate in a direction that makes an angle α with the x – axis and $\mu = \cos \alpha$;
- σ and σ_s are the constants (isotropic region), which depend on the material properties of the region, such that $0 < \sigma_s < \sigma$;
- $f(x, \mu)$ is a given radioactive source function. We assume f as a even function with respect to μ.

Now, we split the equation (1) in two equations, using the following notations:

$$\varphi^+ = \varphi(x, \mu) \text{ if } \mu > 0; \quad \varphi^- = \varphi(x, \mu) \text{ if } \mu < 0 \tag{3}$$

Substituting $\mu'' = -\mu'$, we get

$$\int_{-1}^{0} \varphi(x, \mu')d\mu' = \int_{0}^{1} \varphi(x, -\mu'')d\mu'' = \int_{0}^{1} \varphi^- d\mu''.$$

and the conditions (2) become

$$\varphi^+(0, \mu) = 0, \ \varphi^-(H, \mu) = 0 \tag{4}$$

In view of (3), the equation (1) can be written in the form

$$\begin{aligned} \mu \frac{\partial \varphi^+}{\partial x} + \sigma \varphi^+ &= \frac{\sigma_s}{2} \int_0^1 (\varphi^+ + \varphi^-)d\mu' + f^+ \\ -\mu \frac{\partial \varphi^-}{\partial x} + \sigma \varphi^- &= \frac{\sigma_s}{2} \int_0^1 (\varphi^+ + \varphi^-)d\mu' + f^- \end{aligned} \tag{5}$$

Adding and subtracting the equations (5) and introducing the notations:

$$\begin{aligned} u &= \frac{1}{2}\left(\varphi^+ + \varphi^-\right), \quad v = \frac{1}{2}\left(\varphi^+ - \varphi^-\right) \\ g &= \frac{1}{2}\left(f^+ + f^-\right), \quad r = \frac{1}{2}\left(f^+ - f^-\right) = 0 \end{aligned} \tag{6}$$

we obtain the following system

$$\begin{aligned} &\mu \frac{\partial v}{\partial x} + \sigma u = \sigma_s \int_0^1 u d\mu + g \qquad (a) \\ &\mu \frac{\partial u}{\partial x} + \sigma v = 0 \qquad (b) \end{aligned} \tag{7}$$

The corresponding boundary conditions will be of the form

$$\begin{aligned} u + v &= 0 \quad \text{for} \quad x = 0, \\ u - v &= 0 \quad \text{for} \quad x = H. \end{aligned} \tag{8}$$

Now, we find v from the second equation of (7) and using the first equation, we rewrite the problem (7) - (8) in the following form

$$-\frac{\mu^2}{\sigma}\frac{\partial^2 u}{\partial x^2}+\sigma u=\sigma_s\int_0^1 u d\mu+g \tag{9}$$

$$\left(u-\frac{\mu}{\sigma}\frac{\partial u}{\partial x}\right)\Big|x=0=\left(u+\frac{\mu}{\sigma}\frac{\partial u}{\partial x}\right)\Big|x=H=0 \tag{10}$$

Let us now suppose that the functions u and g belong to the Hilbert space $L_2(D)$, where $D = D_1 \times D_2''$ and the norm is defined by the formula

$$\|u\|=\left(\int_0^H\int_0^1 u^2(x,\mu)dxd\mu\right)^{1/2} \tag{11}$$

Theorem

If the solution of the problem (9) – (10), $u \in L_2(D)$ and the function $g \in L_2(D)$, then

$$\|u\|\le\frac{1}{\sigma-\sigma_s}\|g\| \tag{12}$$

Proof

Multiplying (9) by u and integrating with respect to x and μ, we obtain

$$-\frac{1}{\sigma}\iint_D \mu^2\frac{\partial^2 u}{\partial x^2}u\,dxd\mu+\sigma\iint_D u^2dxd\mu=\sigma_s\int_0^H dx\left(\int_0^1 u(x,\mu)d\mu\right)\left(\int_0^1 u(x,\mu')d\mu'\right)+\iint_D g\,u\,dxd\mu \tag{13}$$

In view of the boundary conditions (10) and using the integration by parts we have

$$-\iint_D \mu^2\frac{\partial^2 u}{\partial x^2}u\,dxd\mu=-\sigma\int_0^1\mu\big(-u^2(H,\mu)-u^2(0,\mu)\big)d\mu+\iint_D\mu^2\left(\frac{\partial u}{\partial x}\right)^2dxd\mu\ge 0$$

On the other hand, we get from the Cauchy-Schwarz inequality

$$\left(\int_0^1 u(x,\mu)d\mu\right)^2\le\int_0^1 u^2(x,\mu)d\mu$$

With the help of these inequalities, (13) can be written in the form

$$\sigma\int_0^H\int_0^1 u^2dxd\mu\le\sigma_s\int_0^H\int_0^1 u^2dxd\mu+\int_0^H\int_0^1 g\,u\,dxd\mu$$

Finally, we arrive at inequality (12).

In order to get a numerical solution of the problem (9)-(10), we consider on x axis two points systems:

- a principal system: $\{x_k\}=\Delta_1'$, $k\in\{0,1,\ldots,N\}$, with $x_0=0$, $x_N=H$ and $h=x_{k+1}-x_k$;
- a secondary system, $\{x_{k+1/2}\}=\Delta_1''$, $k\in\{0,1,2,\ldots,N-1\}$, which verifies the inequalities:

$$x_{k-1/2}<x_k<x_{k+1/2},\quad \text{where}\quad x_{k+1/2}=(x_k+x_{k+1})/2$$

and $0 = x_0 < x_{1/2} < < x_{N-1/2} < x_N = H$. Besides, let $\Delta_2 = \{\mu_l\}, l \in \{0,1,..., L\}$ be a partition of the interval $D_2'' = [0,1]$ with the step $\tau = \mu_{l+1} - \mu_l$, $l \in \{0,1,..., L-1\}$.

If we now consider $H = 1$, $\sigma = 1$ and $\sigma_s = 1/2$, we get for every fixed value $\mu_l \in \Delta_2$ a boundary problem for a one-dimensional diffusion equation

$$-\mu_l^2 \frac{d^2 u(x,\mu_l)}{dx^2} + u(x,\mu_l) = f_1(x,\mu_l) \tag{14}$$

where

$$f_1(x,\mu_l) = S(x) + g(x,\mu_l),\ S(x) = \frac{1}{2}\int_0^1 u(x,\mu)d\mu$$

and

$$\begin{aligned} \left(u(x,\mu_l) - \mu_l \frac{du(x,\mu_l)}{dx}\right)\Big|_{x=0} &= 0 \\ \left(u(x,\mu_l) + \mu_l \frac{du(x,\mu_l)}{dx}\right)\Big|_{x=1} &= 0 \end{aligned} \tag{15}$$

Integrating (14) with respect to x on the intervals: $(x_{k-1/2}, x_{k+1/2})$, we obtain

$$-J_{k+1/2} + J_{k-1/2} + \int_{x_{k-1/2}}^{x_{k+1/2}} (u - f_1)dx = 0 \tag{16}$$

where

$$J_{k\pm1/2} = J(x_{k\pm1/2}),\quad J(x,\mu_l) = \mu_l^2 \frac{du(x,\mu_l)}{dx}.$$

We find $J_{k-1/2}$, integrating (14) on the interval $(x_{k-1/2}, x)$, where $x \in (x_{k-1}, x_k)$. We get

$$\mu_l^2 \frac{du(x,\mu_l)}{dx} = J_{k-1/2} + \int_{x_{k-1/2}}^{x} (u(\xi,\mu_l) - f_1(\xi,\mu_l))d\xi \tag{17}$$

Then, dividing (17) by μ_l^2 and integrating on (x_{k-1}, x_k) we have

$$u_k - u_{k-1} = J_{k-1/2}\int_{x_{k-1}}^{x_k} \frac{dx}{\mu_l^2} + \int_{x_{k-1}}^{x_k} \frac{dx}{\mu_l^2} \int_{x_{k-1/2}}^{x} (u - f_1)d\xi \tag{18}$$

where $u_i = u(x_i, \mu_l)$.

Finally, we get

$$J_{k-1/2} = \frac{\mu_l^2}{h}\left[u_k - u_{k-1} - \int_{x_{k-1}}^{x_k} \frac{dx}{\mu_l^2} \int_{x_{k-1/2}}^{x} (u - f_1)d\xi\right] \tag{19}$$

This is an *exact* relation in the finite differences, because the differential flux $J_{k-1/2}$ is determined with respect to the known values of u: u_{k-1}, $u_{k-1/2}$, u_k. In a similar manner, we find $J_{k+1/2}$, replacing k by $k+1$.

Consequently, the equality (16) becomes:

$$\mu_l^2\left(\frac{u_k-u_{k+1}}{h}+\frac{u_k-u_{k-1}}{h}\right)+\int_{x_{k-1/2}}^{x_{k+1/2}}(u-f_1)d\xi=-\frac{1}{h}\int_{x_k}^{x_{k+1}}dx\int_{x_{k+1/2}}^{x}(u-f_1)d\xi+\frac{1}{h}\int_{x_{k-1}}^{x_k}dx\int_{x_{k-1/2}}^{x}(u-f_1)d\xi \tag{20}$$

where $k\in\{1, 2,\ldots, N\text{-}1\}$. This is *the fundamental identity of the equation in the finite differences.*

In order to obtain a form of the integral identity (20) that is similarly with the Galerkin's equation, we introduce the functions

$$\begin{aligned}\psi(x)&=u(x,\mu_l)-f_1(x,\mu_l)\\ \rho_k(x)&=(x-x_k)/h\,.\end{aligned} \tag{21}$$

Appling the method of integration by parts we get

$$-\frac{1}{h}\int_{x_k}^{x_{k+1}}d\left(\int_{x_k}^{x}d\alpha\right)\int_{x_{k+1/2}}^{x}\psi(\xi)d\xi=-\frac{1}{h}\left[h\int_{x_{k+1/2}}^{x_{k+1}}\psi(\xi)d\xi-\int_{x_k}^{x_{k+1}}(x-x_k)\psi(x)dx\right]=$$

$$=-\int_{x_{k+1/2}}^{x_{k+1}}\psi(\xi)d\xi+\int_{x_k}^{x_{k+1}}\rho_k(x)\psi(x)dx \tag{22}$$

Analogously, if we denote

$$\tilde{\rho}_k(x)=\frac{x_k-x}{h} \tag{23}$$

the equation (20) is now of the form

$$\mu_l^2\left(\frac{u_k-u_{k+1}}{h}+\frac{u_k-u_{k-1}}{h}\right)+\int_{x_{k-1}}^{x_k}(1-\tilde{\rho}_k(x))\psi(x)dx+\int_{x_k}^{x_{k+1}}(1-\rho_k(x))\psi(x)dx=0 \tag{24}$$

$$k\in\{1, 2,\ldots, N\text{-}1\}.$$

It should be observed that the integral from the left-hand side of (20) was decomposed in the intervals: $(x_{k-1/2}, x_k)\cup(x_k, x_{k+1/2})$.

Let us introduce the functions:

$$Q_k(x)=\begin{cases}\dfrac{1-\tilde{\rho}_k}{\sqrt{h}}=\dfrac{x-x_{k-1}}{h\sqrt{h}}, & x\in[x_{k-1}, x_k]\\ \dfrac{1-\rho_k}{\sqrt{h}}=\dfrac{x_{k+1}-x}{h\sqrt{h}}, & x\in[x_k, x_{k+1}]\\ 0, & x\notin[x_{k-1}, x_{k+1}]\end{cases} \tag{25}$$

where

$$Q_k(x_k)=\frac{1}{\sqrt{h}}\,.$$

Let us now assume that the function $u(x,\mu_l)\in L_2([0, 1])$, which is a Hilbert space with the scalar product defined by formula

$$(w, v)=\int_0^1 w(x)v(x)dx \tag{26}$$

Then, using (25)-(26), the equations (24) can be rewritten in the form

$$\mu_l^2\left(\frac{u(x_k,\mu_l)-u(x_{k+1},\mu_l)}{h}+\frac{u(x_k,\mu_l)-u(x_{k-1},\mu_l)}{h}\right)+(u,\sqrt{h}Q_k)=(f_1,\sqrt{h}Q_k),\ k=1,...,N-1 \tag{27}$$

On the other hand, we observe that

$$\left(\mu_l^2\frac{du(x,\mu_l)}{dx},\frac{dQ_k}{dx}\right)=\mu_l^2\int_{x_{k-1}}^{x_k}\frac{du(x,\mu_l)}{dx}\cdot\frac{1}{h\sqrt{h}}dx-\mu_l^2\int_{x_k}^{x_{k+1}}\frac{du(x,\mu_l)}{dx}\cdot\frac{1}{h\sqrt{h}}dx=$$

$$=\mu_l^2\left(\frac{u_k-u_{k-1}}{h\sqrt{h}}-\frac{u_{k+1}-u_k}{h\sqrt{h}}\right), \tag{28}$$

and (27) can be rewritten in the following form:

$$\left(\mu_l^2\frac{du}{dx},\frac{dQ_k}{dx}\right)+(u,Q_k)=(f_1,Q_k),\ k\in\{1,2,..,N-1\} \tag{29}$$

This system allows us to consider the integral identity method as a variational method and the equations (29) may be used for obtaining the approximate solutions using a sequence of coordinate functions. It should be noted that the equations (29) coincide with the relations obtained by Galerkin method, where $Q_k(x)$ are the coordinate functions. Then, the solution of the system (29) can be defined in the following way

$$\tilde{u}(x,\mu_l)=\sum_{k=1}^{N-1}a_k(\mu_l)Q_k(x) \tag{30}$$

where: $a_k(\mu_l)=2\beta_k\mu_l^2$.

We shall now determine the coefficients β_k from the condition that (30) be a solution of the system (29). Also, the boundary conditions must be satisfied. A matrix equation is obtained

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{\Gamma} \tag{31}$$

where

- matrix **A** is of the form

$$\begin{bmatrix} a_{1,1} & a_{1,2} & 0 & \Lambda & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \Lambda & 0 & 0 & 0 \\ \mathrm{M} & \mathrm{M} & \mathrm{M} & \Lambda & \mathrm{M} & \mathrm{M} & \mathrm{M} \\ 0 & 0 & 0 & \Lambda & a_{N-2,N-3} & a_{N-2,N-2} & a_{N-2,N-1} \\ 0 & 0 & 0 & \Lambda & 0 & a_{N-1,N-2} & a_{N-1,N-1} \end{bmatrix}$$

with

$$a_{00}=\frac{2\mu_l^3}{h}+\frac{2\mu_l^4}{h^2}+\frac{6\mu_l^2-1}{9}=a_{NN},\quad a_{k,k-1}=-\frac{2\mu_l^4}{h^2}+\frac{6\mu_l^2-1}{18},\ k\in\{2,...,N-1\},$$

$$a_{01}=-\frac{2\mu_l^4}{h^2}+\frac{6\mu_l^2-1}{18}=a_{N,N-1}\qquad a_{k,k}=\frac{4\mu_l^4}{h^2}+\frac{2(6\mu_l^2-1)}{9},\ k\in\{1,...,N-1\}$$

$$a_{k,k+1}=a_{k,k-1},\ k\in\{1,...,N-2\}\qquad \gamma_0=\int_0^{x_1}g(x)Q_0(x)dx,$$

$$\gamma_k = (g, Q_k) = \int_{x_{k-1}}^{x_{k+1}} g(x) Q_k(x)\,dx, \qquad \gamma_N = \int_{x_{N-1}}^{x_n} g(x) Q_N(x)\,dx .$$

Solving the matrix equation (31), we can find the values of coefficients β_k, i.e. we can construct by (30) the solution u of (14)-(15). Let us now consider: $x_k = y_0 < y_1 < \ldots < y_m = x_{k+1}$ for every interval $(x_k,\ x_{k+1})$, $k = 0, 1, \ldots, N-1$. Using the functions $Q_k(x)$, we can find the values of approximate solution: $\tilde{u}_{k,j} = \tilde{u}(y_j, \mu_l)$, $k \in \{0,1,2,\ldots,N\text{-}1\}$, $j \in \{1,2,\ldots,m\}$.

In order to get $\tilde{v}_{k,j}$, we use $\tilde{u}_{k,j}$ and the numerical derivative for equation (7*b*):

$$\begin{aligned}
\tilde{v}_{k,j} &= -\frac{\tilde{u}_{k,j+1} - \tilde{u}_{k,j-1}}{2(h/m)} \cdot \mu_l, \quad k \in \{0,1,2,\ldots,N\}, \quad j \in \{1,2,\ldots,m-1\} \\
\tilde{v}_{k,n} &= -\frac{\tilde{u}_{k+1,1} - \tilde{u}_{k,m-1}}{2(h/m)} \cdot \mu_l, \ k \in \{0,1,\ldots,N-2\}, \\
\tilde{v}_{0,1} &= -\frac{\tilde{u}_{0,1} \cdot \mu_l}{h/m}, \qquad \tilde{v}_{N-1,m} = \frac{\tilde{u}_{N-1,m-1}}{h/m} \cdot \mu_l
\end{aligned} \tag{32}$$

where

$$\tilde{u}_{k,j} = 2\mu_l^2 \left(\beta_k \frac{x_{k+1} - y_j}{h\sqrt{h}} + \beta_{k+1} \frac{y_j - x_k}{h\sqrt{h}} \right),$$

$k \in \{0,1,2,\ldots,N-1\}$, $j \in \{0,1,\ldots,m\}$, $\beta_0 = \beta_N = 0$

According to the continuity of function u we get: $\tilde{u}_{k-1,m} = \tilde{u}_{k,0}$, $k \in \{1,2,\ldots,N-1\}$.

Finally, the values of φ obtained by this algorithm will be of the form

$$\begin{aligned}
\tilde{\varphi}^{+}_{k,j} &= \tilde{u}_{k,j} + \tilde{v}_{k,j} \ \text{ for } \ \mu_l > 0 \\
\tilde{\varphi}^{-}_{k,j} &= \tilde{u}_{k,j} - \tilde{v}_{k,j} \ \text{ for } \ \mu_l < 0
\end{aligned}, \qquad k \in \{1,2,\ldots,N\}, j \in \{0,1,2,\ldots,m\}. \tag{33}$$

3 Numerical example

Let us consider the boundary problem (14)-(15), where (14) is of the form

$$-\mu_l^2 \frac{d^2 u(x,\mu_l)}{dx^2} + u(x,\mu_l) = f_1(x,\mu_l) \tag{14}$$

with $f_1(x,\mu_l) = g(x,\mu_l) + \frac{1}{2}\int_0^1 u(x,\mu)\,d\mu$.

Here g is an even function with respect to μ and a periodical function with respect to x

$$g(x,\mu) = -\left(2\pi^2\mu^4 + \frac{6\mu^2 - 1}{12}\right)\cos(2\pi x) + \frac{6\mu^2 - 1}{12} \tag{34}$$

We break up the closed interval D_1 into $N = 8$ segments of length $h = 1/8$. For $D_2'' = [0, 1]$ we have $L = 4$. Some computational results will illustrate the application of above algorithm. For (34) we get

$$\gamma_k = \int_{x_{k-1}}^{x_{k+1}} g(x)Q_k(x)dx = A_1 \frac{\cos(2k\pi h)\sin^2(\pi h)}{\pi^2} + B_1 h^2, \qquad k \in \{1,..,N-1\}$$

where

$$\gamma_0 = \gamma_8 = A_1 \frac{\sin^2(\pi h)}{2\pi^2} + B_1 \frac{h^2}{2},$$

$$A_1 = \frac{1}{h\sqrt{h}}\left(\frac{1-6\mu^2}{12} - 2\mu^4\pi^2\right), \quad B_1 = \frac{6\mu^2-1}{12h\sqrt{h}}, \quad k \in \{1,2,...,7\}.$$

Figure 1 and Figure 2 present the variations of the function g defined by (34) with respect to μ and x_i. In view of (6), the function g coincides with the source function f.

Figures 3 and 4 show the variations of the solution of the problem (14)-(15) for different values of μ_l. Finally, figures 5 and 6 show the variations of the function φ and the variations of the error $er = u - ue$, where $ue(x,\mu_l) = \mu_l^2 \sin^2(\pi x)$, with respect to μ_l.

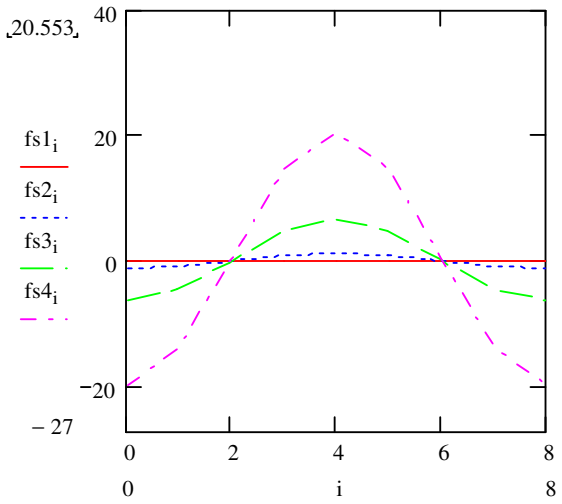
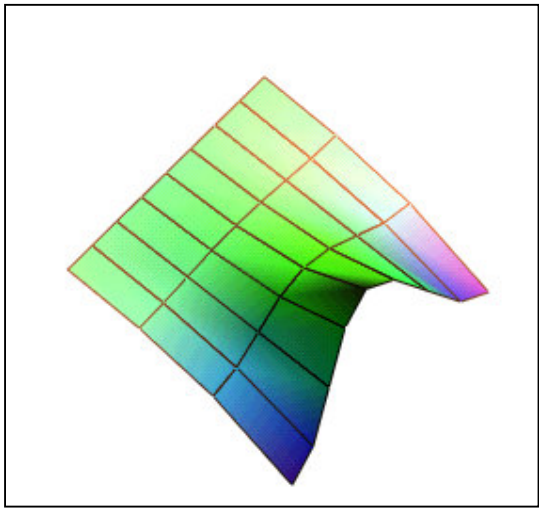


Fig. 1. Dependence of source function f on μ and x_i, $i = 0,1,\ldots,8$

$\mu = 1/4$ (fs1), $\mu = 1/2$ (fs2), $\mu = 3/4$ (fs3), $\mu = 1$(fs4)



fsm

Fig. 2. The surface $z = f(x, \mu)$
$x \in [0, 1]$ and $\mu \in [0, 1]$

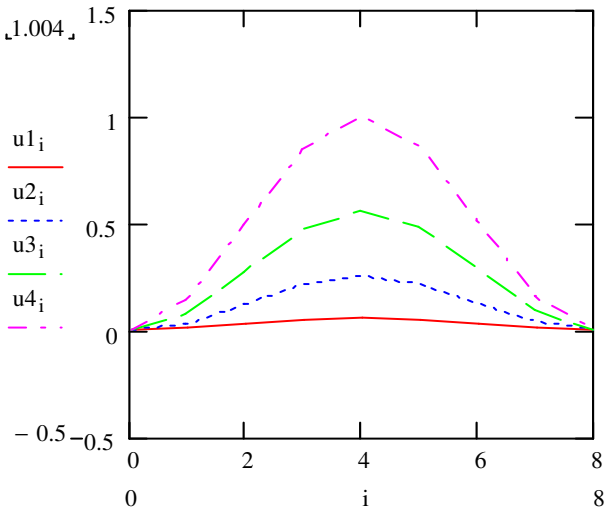
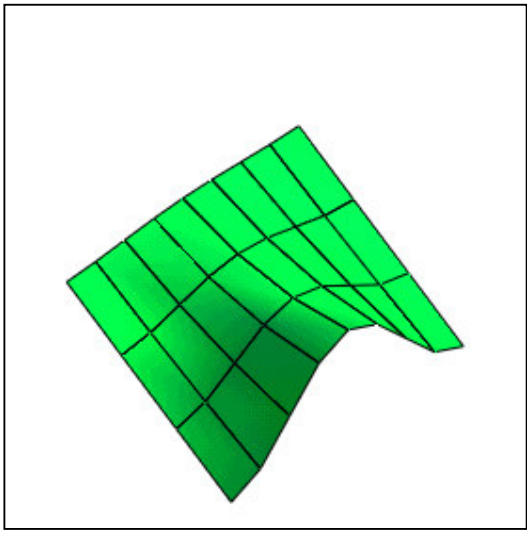


Fig. 3. Dependence of u on μ and x_i, $i = 0,1,\ldots,8$

$\mu = 1/4$ (u1), $\mu = 1/2$ (u2), $\mu = 3/4$ (u3), $\mu = 1$ (u4)



um

Fig. 4. The surface $u = u(x, \mu)$
$x \in [0, 1]$ and $\mu \in [0, 1]$

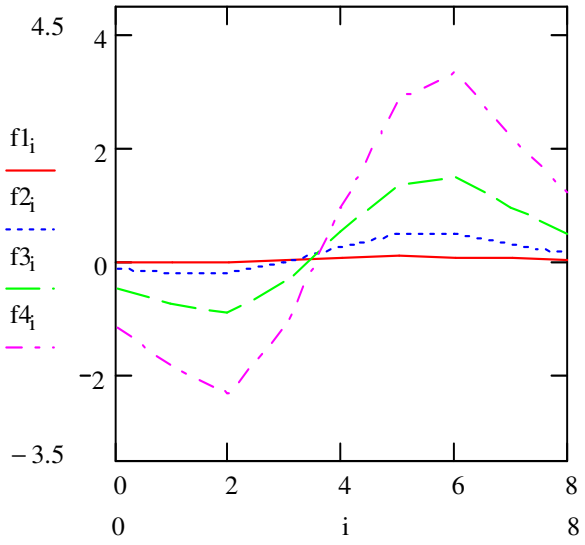


Fig 5. Dependence of φ on μ and x_i, $i = 0,1,\ldots,8$

$\mu = 1/4$(f1), $\mu = 1/2$(f2), $\mu = 3/4$(f3), $\mu = 1$(f4)

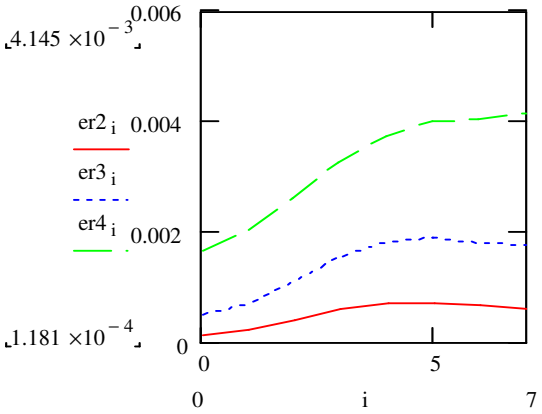


Fig. 6. Dependence of *er* on μ and x_i, $i = 0,1,\ldots,7$
$\mu = 1/2$ (er2), $\mu = 3/4$ (er3), $\mu = 1$(er4)

4 Conclusions

♦ In this paper we present an algorithm that use integral identity method in the solving a stationary transport equation..

♦ Splitting the surface $\varphi = \varphi(x, \mu)$ in two regions: φ^+ that correspond to $\mu > 0$ and φ^- that correspond to $\mu < 0$, we replace the solving of the boundary problem (1)-(2) with a boundary problem for one-dimensional diffusion equation.

♦ We proved that the exact solution of this equation obtained for every fixed μ_l is bounded by the source function f in the norm of the Hilbert space $L_2(D)$.

♦It follows from the figure 3 that the maximum values of u are obtained if the source there is on the Ox axis ($\mu = 1$) and minimum values are obtained if the source there is on a perpendicular direction on Ox.

♦Figure 5 shows that the neutron flux enters in the corresponding surfaces of $x \in [0, 1/2]$ and leaves the corresponding surfaces of $x \in [1/2, 1]$ for the source function (34).

♦It should be observed from the figure 6, that the maximum values of the errors are obtained for $\mu = 1$ and these are of the order 0.0042.

♦ This splitting numerical method can be used and if the coefficients σ and σ_s of the transport equation (1) are the discontinuous functions. In this case, we shall choose the points x_k such that these to coincide with the discontinuous points of $\sigma(x)$ and $\sigma_s(x)$.

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Weighted Composition Operators between the Little Logarithmic Bloch Space and the α-Bloch Space

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Abstract: Necessary and sufficient conditions are given for a weighted composition operator $uC_\varphi f = uf \circ \varphi$ to be bounded or compact between the little logarithmic Bloch space β_L^0 and the α-Bloch space β_α.

MSC 2000: 47B38, 30H05, 30D05 .

Keywords: Weighted composition operators; Boundedness; Compactness; Weighted Bloch spaces.

1 Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$, and $H(D)$ denote the set of all analytic functions on D. Let φ be a holomorphic self-map of D and $u \in H(D)$. They induce the weighted composition operator uC_φ defined by:

$$uC_\varphi f = uf \circ \varphi, \quad \text{for } f \in H(D).$$

We can regard this operator as a generalization of the multiplication operator M_u denoted by $M_u f = uf$ for $f \in H(D)$, and the composition operator C_φ denoted by $C_\varphi f = f \circ \varphi$ for $f \in H(D)$. It is interesting to provide a function theoretic characterization when φ and u induce a bounded or compact weighted composition operator between different function spaces. The weighted composition operator was studied on the Bloch space and the little Bloch space in [5]. Some recent papers on weighted composition operators on Bloch type spaces can be found in [3, 9]. In [6], Ohno, Stroethoff and Zhao characterized φ and u for the weighted composition operator uC_φ to be bounded or compact between the α-Bloch spaces. Yoneda [10] studied the composition operator C_φ in the logarithmic Bloch space β_L and the little logarithmic Bloch space β_L^0. In [7] the author studied the multiplication operator M_u in β_L and β_L^0. In the paper we will characterize the weighted composition operator uC_φ to be bounded or compact between the α-Bloch spaces β_α and the little logarithmic Bloch space β_L^0.

For $f \in H(D)$, let

$$\|f\|_{\beta_\alpha} = \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in D\}, \quad 0 < \alpha < +\infty,$$

$$\|f\|_{\beta_L} = \sup\{(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| : z \in D\}.$$

As in [7, 11], the α-Bloch space β_α consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_\alpha} < +\infty$ and the little α-Bloch space β_α^0 consists of all $f \in H(D)$ satisfying $\lim\limits_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0$; the logarithmic Bloch space β_L consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_L} < +\infty$ and the little logarithmic Bloch space β_L^0 consists of all $f \in H(D)$ satisfying $\lim\limits_{|z|\to 1}(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| = 0$. It can easily proved that β_L is a Banach space under the norm

$$\|f\|_L = |f(0)| + \|f\|_{\beta_L}. \tag{1}$$

In [7], the author proved that β_L^0 is a closed subspace and coincides with the closure of polynomials. It is well known that with the norm $\|f\|_\alpha = |f(0)| + \|f\|_{\beta_\alpha}$, β_α is a Banach space and β_α^0 is a closed subspace of β_α. It is easily proved that for $0 < \alpha < 1$, $\beta_\alpha \subsetneqq \beta_L \subsetneqq \beta_1$. In this paper, C denotes the positive constant depending only on the index α, the C may differ at different places.

2 uC_φ from β_α to β_L^0

Lemma 2.1 (see [7]) *If $f \in \beta_L$, then $|f(z)| \le (2+\ln(\ln\frac{2}{1-|z|}))\|f\|_L$. Moreover, $|f(z)| \le 2\ln(\ln\frac{2}{1-|z|})\|f\|_L$, where $|z| \ge r_* = 1 - \frac{2}{e^{e^2}}$.*

Lemma 2.2 (see [7]) *If $f \in \beta_L^0$, then $\lim\limits_{|z|\to 1}\dfrac{|f(z)|}{\ln(\ln\frac{2}{1-|z|})} = 0$.*

Lemma 2.3 (see [8]) *Let $\alpha > 0$ and $f \in \beta_\alpha$. Then*

(1) $\|f_t\|_\alpha \le \|f\|_\alpha, 0 < t < 1$, where $f_t(z) = f(tz)$;

(2) $|f(z)| \le C\|f\|_\alpha$, where $\alpha < 1$;

(3) $|f(z)| \le C\ln(\dfrac{2}{1-|z|^2})\|f\|_\alpha$, where $\alpha = 1$;

(4) $|f(z)| \le \dfrac{C}{(\alpha-1)(1-|z|)^{\alpha-1}}\|f\|_\alpha$, where $\alpha > 1$.

Lemma 2.4 *Let $\alpha > 0$ and X be a Banach space. Then $uC_\varphi : \beta_\alpha^0 \to X$ is a weakly compact operator if and only if $uC_\varphi : \beta_\alpha^0 \to X$ is compact.*

The proof is similar to Lemma 4 in [2]. The details are omitted.

Lemma 2.5 (see [11]) *Let $t > 0$ and $f \in H(D)$. Then $\sup\limits_{z\in D}(1-|z|)^t|f(z)| < +\infty$ if and only if $\sup\limits_{z\in D}(1-|z|)^{t+1}|f'(z)| < +\infty$.*

Lemma 2.6 *Let $U \subset \beta_L^0$. Then U is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in U}(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| = 0.$$

The proof is similar to Lemma 1 in [4]. The details are omitted.

Theorem 2.1 *Let φ and u be analytic on D, φ a self-map of D. Let $\alpha > 0$. Then the following statements are equivalent:*

(1) $uC_\varphi : \beta_\alpha \to \beta_L^0$ is compact;
(2) $uC_\varphi : \beta_\alpha \to \beta_L^0$ is bounded;
(3) $uC_\varphi : \beta_\alpha^0 \to \beta_L^0$ is weakly compact;
(4) $uC_\varphi : \beta_\alpha^0 \to \beta_L^0$ is compact;
(5) (i) If $\alpha > 1$, $\lim_{|z|\to 1} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| = 0$ and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)| = 0$$

.

(ii) If $\alpha = 1$, $\lim_{|z|\to 1}(1-|z|^2)\ln(\frac{2}{1-|z|})\ln(\frac{2}{1-|\varphi(z)|^2})|u'(z)| = 0$ and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{1-|\varphi(z)|^2}|\varphi'(z)u(z)| = 0.$$

(iii) If $0 < \alpha < 1$, $u \in \beta_L^0$ and $\lim_{|z|\to 1} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)| = 0$.

Proof From Lemma 2.4, we see that (3) $\Longleftrightarrow$ (4). It is obvious that (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (4). Now we need to show that (2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5) $\Longrightarrow$ (1).

(2) $\Longrightarrow$ (3) From (2) we see that it holds $uC_\varphi(\beta_\alpha) \subset \beta_L^0$. Since $uC_\varphi : \beta_\alpha^0 \to \beta_L^0$ is bounded, by Basic Functional Analysis, we obtain that $(uC_\varphi)^{**} : (\beta_\alpha^0)^{**} \to (\beta_L^0)^{**}$ is bounded and $(uC_\varphi)^{**}(f) = uC_\varphi(f)$ for every $f \in \beta_\alpha^0$. Since β_α^0 is w^* dense in β_α, it follows that $(uC_\varphi)^{**}(f) = uC_\varphi(f)$ for every $f \in \beta_\alpha$.

Using Gantmacher's Theorem (see [1]), we obtain that $T : X \to Y$ is weakly compact if and only if $T^{**}(X^{**}) \subset Y$, where T^{**} is the second adjoint of T. Hence $uC_\varphi : \beta_\alpha^0 \to \beta_L^0$ is weakly compact if and only if $(uC_\varphi)^{**}((\beta_\alpha^0)^{**}) \subset \beta_L^0$. From the fact that $(uC_\varphi)^{**}(f) = uC_\varphi(f)$ for every $f \in \beta_\alpha$ and $(\beta_\alpha^0)^{**} \cong \beta_\alpha$ (see [11]), we obtain that $uC_\varphi : \beta_\alpha^0 \to \beta_L^0$ is weakly compact.

(4) $\Longrightarrow$ (5) Suppose that uC_φ is compact from β_α^0 to β_L^0. Then $u = uC_\varphi 1 \in \beta_L^0$. Also $u\varphi = uC_\varphi z \in \beta_L^0$, thus

$$\lim_{|z|\to 1}(1-|z|^2)\ln(\frac{2}{1-|z|})|u'(z)\varphi(z) + u(z)\varphi'(z)| = 0.$$

Since $|\varphi| \le 1$ and $u \in \beta_L^0$, we have

$$\lim_{|z|\to 1}(1-|z|^2)\ln(\frac{2}{1-|z|})|\varphi'(z)u(z)| = 0. \tag{2}$$

Next, by Lemma 2.6, we have

$$\lim_{|z|\to 1} \sup\{(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi f)'(z)| : f\in\beta_\alpha^0, \|f\|_\alpha \le M\} = 0 \qquad (3)$$

for some $M > 0$.

Suppose $\alpha > 1$. Fixing $w \in D$, we take the test function

$$f_w(z) = \frac{\alpha}{(1-\overline{\varphi(w)}z)^{\alpha-1}} - \frac{(\alpha-1)(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^\alpha}$$

for $z \in D$. It is clear that $f_w \in \beta_\alpha^0$ for every $w \in D$. By a direct calculation, we obtain that $\sup_{w\in D}\|f_w\|_\alpha \le \alpha + 3(\alpha-1)\alpha 2^\alpha$, $f_w'(\varphi(w)) = 0$, and $f_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\alpha-1}}$. Then

$$\lim_{|w|\to 1} \frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^{\alpha-1}}|u'(w)| = 0$$

by (3). Similarly, fix $w \in D$ and assume that $\varphi(w) \neq 0$. Consider the function g_w defined by

$$g_w(z) = \frac{1}{\overline{\varphi(w)}}\left(\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^\alpha} - \frac{1}{(1-\overline{\varphi(w)}z)^{\alpha-1}}\right)$$

for $z \in D$. It is clear that $g_w \in \beta_\alpha^0$. Since $g_w(\varphi(w)) = 0$, $g_w'(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^\alpha}$, and $\sup_{w\in D}\|g_w\|_\alpha \le 1 + \alpha 2^{\alpha+2}$, we have

$$\lim_{|w|\to 1} \frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^{\alpha}}|\varphi'(w)u(w)| = 0$$

for $\varphi(w) \neq 0$ by (3). However, if $\varphi(w) = 0$, by (2), we obtain that

$$\lim_{|w|\to 1}(1-|w|^2)\ln\left(\frac{2}{1-|w|}\right)|\varphi'(w)u(w)| = 0.$$

Suppose $\alpha = 1$. Fixing $w \in D$, we take the test function

$$h_w(z) = 2\ln\frac{2}{1-\overline{\varphi(w)}z} - \frac{1}{\ln\frac{2}{1-|\varphi(w)|^2}}\left(\ln\frac{2}{1-\overline{\varphi(w)}z}\right)^2$$

for $z \in D$. It is clear that $h_w \in \beta_1^0$ for every $w \in D$. By a direct calculation, we obtain that $h_w'(\varphi(w)) = 0$, $h_w(\varphi(w)) = \ln\frac{2}{1-|\varphi(w)|^2}$, and $\sup_{w\in D}\|h_w\|_1 \le 13 < +\infty$. It follows that

$$\lim_{|w|\to 1}(1-|w|^2)\ln\left(\frac{2}{1-|w|}\right)\ln\left(\frac{2}{1-|\varphi(w)|^2}\right)|u'(w)| = 0$$

by (3). The fact that another condition in (ii) is necessary for uC_φ to be compact from β_α^0 into β_L^0 is shown above.

Suppose $0 < \alpha < 1$. The proof is the same as that in the case $\alpha > 1$. The details are omitted.

(5) $\Longrightarrow$ (1) First, let $\alpha > 1$. Assume that $f \in \beta_\alpha$. By Lemma 2.5, we obtain that

$$(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi f)'(z)|$$

$$\leq (1-|z|^2)\ln(\frac{2}{1-|z|})|u'(z)f(\varphi(z))| + (1-|z|^2)\ln(\frac{2}{1-|z|})|u(z)||f'(\varphi(z))\varphi'(z)|$$

$$\leq C\|f\|_{\beta_\alpha}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| + \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)|\|f\|_{\beta_\alpha} \longrightarrow 0$$

as $|z| \to 1$. Then $uC_\varphi(f) \in \beta_L^0$, thus uC_φ is bounded from β_α into β_L^0. Thus we only prove that

$$\lim_{|z|\to 1}\sup\{(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi f)'(z)| : f \in \beta_\alpha, \|f\|_\alpha \leq 1\} = 0$$

by Lemma 2.6. However, it has just been proved above. Thus uC_φ is compact from β_α to β_L^0.

Next, let $\alpha = 1$. Similarly, assume that $f \in \beta_1$. By Lemma 2.3, we have

$$(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi f)'(z)|$$

$$\leq C\|f\|_\alpha(1-|z|^2)\ln(\frac{2}{1-|z|})\ln(\frac{2}{1-|\varphi(z)|^2})|u'(z)|$$

$$+\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{1-|\varphi(z)|^2}|\varphi'(z)u(z)|\|f\|_{\beta_\alpha} \longrightarrow 0$$

as $|z| \to 1$. Using the same method as in the case $\alpha > 1$, we also obtain that uC_φ is compact from β_1 to β_L^0.

Finally, let $0 < \alpha < 1$. Suppose that $f \in \beta_\alpha$. By Lemma 2.3, we have

$$(1-|z|^2)\ln\frac{2}{1-|z|}|(uC_\varphi f)'(z)|$$

$$\leq C\|f\|_\alpha(1-|z|^2)\ln\frac{2}{1-|z|}|u'(z)| + \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)|\|f\|_{\beta_\alpha} \longrightarrow 0$$

as $|z| \to 1$. This completes the proof of Theorem 2.1.

3 uC_φ from β_L^0 to β_α

Lemma 3.1 (see [7]) *Let* $f(z) = \dfrac{(1-|z|)\ln\frac{2}{1-|z|}}{|1-z|\ln\frac{4}{|1-z|}}$, $z \in D$. *Then* $|f(z)| < 2$.

Lemma 3.2 (see [7]) *Let* $0 \le t \le 1$, $g(z) = \dfrac{(1-|z|)\ln\frac{2}{1-|z|}}{(1-|tz|)\ln\frac{2}{1-|tz|}}$, $z \in D$. *Then* $|g(z)| < 2$.

Lemma 3.3 *Suppose* $f \in \beta_L$. *Let* $f_t(z) = f(tz)$, $0 < t < 1$. *Then* $\|f_t\|_L \le 4\|f\|_L$.

The result is easily proved by lemma 3.2.

Theorem 3.1 *Let* φ *and* u *be analytic on* D, φ *a self-map of* D. *Let* $\alpha > 0$. *Then the following statements are equivalent:*

(a) $uC_\varphi : \beta_L^0 \to \beta_\alpha$ *is bounded;*

(b) $uC_\varphi : \beta_L \to \beta_\alpha$ *is bounded;*

(c) $\sup\limits_{z\in D}(1-|z|^2)^\alpha|\ln(\ln\dfrac{2}{1-|\varphi(z)|})||u'(z)| < +\infty$ *and*

$$\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)| < +\infty$$

.

Proof $(a) \Longrightarrow (b)$ It is clear that for any $f \in \beta_L$, we have $f_t \in \beta_L^0$ for all $0 < t < 1$. Then, according to Lemma 3.3,

$$\|uC_\varphi(f_t)\|_\alpha \le \|uC_\varphi\|\|f_t\|_L \le 4\|uC_\varphi\|\|f\|_L < +\infty.$$

Hence $\|uC_\varphi(f)\|_\alpha \le 4\|uC_\varphi\|\|f\|_L < +\infty$, which shows that uC_φ is bounded from β_L to β_α.

$(b) \Longrightarrow (c)$ Suppose that uC_φ is bounded from β_L to β_α. Taking the test function $f(z) = 1$ and $f(z) = z$ respectively, we can easily obtain that

$$\sup_{z\in D}(1-|z|^2)^\alpha|\varphi'(z)u(z)| < +\infty. \tag{4}$$

Fix $w \in D$. Consider the function f_w defined by

$$f_w(z) = 2\ln(\ln\frac{4}{1-\overline{\varphi(w)}z}) - \frac{1}{\ln\ln\frac{4}{1-|\varphi(w)|^2}}(\ln\ln\frac{4}{1-\overline{\varphi(w)}z})^2 \tag{5}$$

for $z \in D$. According to Lemma 3.1 and 3.2 we get that $f_w \in \beta_L$ and $\|f_w\|_L \le 16$. Since $f_w'(\varphi(w)) = 0$ and $f_w(\varphi(w)) = \ln\ln\dfrac{4}{1-|\varphi(w)|^2}$, it follows that

$$\begin{aligned}(1-|w|^2)^\alpha \quad |u'(w)f_w(\varphi(w))| &= (1-|w|^2)^\alpha|(uC_\varphi f_w)'(w)| \\ &= \|uC_\varphi(f_w)\|_{\beta_\alpha} \le \|uC_\varphi(f_w)\|_\alpha \le \|uC_\varphi\|\|f_w\|_L \le 16\|uC_\varphi\| < +\infty.\end{aligned}$$

Then

$$\sup_{w\in D}(1-|w|^2)^\alpha|\ln(\ln\frac{2}{1-|\varphi(w)|})||u'(w)|\le 16\|uC_\varphi\|<+\infty. \tag{6}$$

Next, fix $w\in D$ with $w\neq 0$. Let

$$g_w(z)=\int_0^z(1-\frac{\overline{w}^2}{|w|^2}z^2)^{-1}(\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2})^{-1}\,dz. \tag{7}$$

By Lemma 3.1, we have

$$\sup_{z_1\in D}(1-|z_1|^2)(\ln\frac{2}{1-|z_1|^2})|1-z_1^2|^{-1}|\ln\frac{4}{1-z_1^2}|^{-1}<2<+\infty.$$

Applying $z_1=\frac{\overline{w}}{|w|}z$, we obtain that

$$\sup_{z\in D}(1-|z|^2)(\ln\frac{2}{1-|z|^2})|1-\frac{\overline{w}^2}{|w|^2}z^2||\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2}|^{-1}<2<+\infty.$$

Hence we have $g_w\in\beta_L$ and $\|g_w\|_L<4$. Then for $w\neq 0$ we obtain that

$$\|uC_\varphi(g_w)\|_{\beta_\alpha}\le\|uC_\varphi(g_w)\|_\alpha\le\|uC_\varphi\|\|g_w\|_L\le 4\|uC_\varphi\|<+\infty. \tag{8}$$

So for every $z\in D$ such that $\varphi(z)\neq 0$, let $w=\varphi(z)$ in (8), we have

$$\begin{aligned}|\varphi'(z)u(z)|\quad&\tfrac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{4}{1-|\varphi(z)|^2}}=(1-|z|^2)^\alpha|u(z)g_w'(\varphi(z))\varphi'(z)|\\&\le\|uC_\varphi(g_w)\|_{\beta_\alpha}+\sup_{z\in D}(1-|z|^2)^\alpha|u'(z)||g_w(\varphi(z))|\\&\le C+4\sup_{z\in D}(1-|z|^2)^\alpha(2+\ln(\ln\frac{2}{1-|\varphi(z)|}))|u'(z)|<+\infty,\end{aligned}$$

where we use Lemma 2.1 and (6).

For every $z\in D$ such that $\varphi(z)=0$, by (4), we have

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)|=\frac{1}{\ln 2}(1-|z|^2)^\alpha|u(z)\varphi'(z)|<+\infty.$$

$(c)\Longrightarrow(a)$ For $f\in\beta_L^0$, by Lemma 2.1, we have the following inequality:

$$\begin{aligned}\|uC_\varphi f\|_{\beta_\alpha}\quad&\le\sup_{z\in D}(1-|z|^2)^\alpha|u'(z)|(2+\ln\ln\frac{2}{1-|\varphi(z)|})\|f\|_L\\&+\sup_{z\in D}(1-|z|^2)^\alpha\frac{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|f'(\varphi(z))||\varphi'(z)u(z)|\\&\le C\|f\|_L+\|f\|_L\sup_{z\in D}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)|\end{aligned}$$

and

$$|u(0)f(\varphi(0))| \leq |u(0)|(2+\ln\ln\frac{2}{1-|\varphi(0)|})\|f\|_L.$$

Then uC_φ is bounded from β_L^0 to β_α. This completes the proof of Theorem 3.1.

Remark 1 *If the β_L with norm (1) is isometric to the second dual $(\beta_L^0)^{**}$, we can prove that $uC_\varphi : \beta_L^0 \to \beta_\alpha$ is compact if and only if $uC_\varphi : \beta_L^0 \to \beta_\alpha$ is bounded.*

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AVERAGED ITERATES FOR NON-EXPANSIVE NONSELF-MAPPINGS IN BANACH SPACES

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ABSTRACT. Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and K be a closed convex subset of E which is also a sunny non-expansive retract of E. Assume that $T : K \to E$ is a non-expansive mappings with a fixed point. The two averaged iterative sequences $\{x_n\}$ are given by

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n u + (1-\alpha_n)(PT)^j x_n), \quad \forall n \geq 0,$$

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n u + (1-\alpha_n)(TP)^j x_n), \quad \forall n \geq 0,$$

where P is sunny non-expansive retraction of E onto K and $\alpha_n \in (0,1)$ satisfying the conditions: $\lim\limits_{n\to\infty}\alpha_n = 0$ and $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$. We prove that $\{x_n\}$ converges strongly to some fixed point of T and, furthermore, as applications, we obtain the viscosity averaged approximation results for T.

Key Words and Phrases: Non-expansive nonself-mappings, averaged iterates; uniformly convex Banach space.

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1. INTRODUCTION

Let T be a mapping with domain $D(T)$ and range $R(T)$ in Banach space E. T is called *non-expansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D(T).$$

In 1967, Halpern [5] firstly introduced the following iteration scheme in Hilbert space: for a non-expansive self-mapping T and $u \neq 0, x_0 \in K$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geq 0.$$

He pointed out that the control conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$ are necessary for the convergence of the iteration scheme to a fixed point of T.

In last decades, many authors have studied the iterative algorithms for non-expansive nonself-mappings and obtained a series of good results. For example, for any given $u \in K$, define the following implicit iterative sequences $\{x_t\}$ as follows:

$$x_t = tu + (1-t)PTx_t, \quad \forall t \in (0,1),$$

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and

$$x_t = P(tu + (1-t)Tx_t), \quad \forall t \in (0,1),$$

where P is the projector (or non-expansive retraction) from E to K.

In 1995, Xu and Yin [23] showed that, as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point of T in Hilbert spaces. In 1997, Xu [20] proved and extended the results of Xu and Yin [23] from Hilbert spaces to uniformly smooth Banach spaces.

In 1998, Jung and Kim [6] obtained the same results in uniformly convex Banach spaces with the uniformly Gâteaux differentiable norm. In the same year, in the frame of reflexive Banach spaces with the uniformly Gâteaux differentiable norm, Takahashi and Kim [18] also gained the strong convergent results of the sequence $\{x_t\}$. But they all dealt with the implicit iteration of T, not the explicit iteration.

Recently, in a reflexive Banach space E with a weakly sequentially continuous duality mapping, Song and Chen [11] studied the following explicit iterative schemes for non-expansive nonself-mapping T defined by

$$x_{n+1} = P(\alpha_n f(x_n) + (1-\alpha_n)Tx_n), \quad \forall n \geq 0,$$

and showed that $\{x_n\}$ converges strongly to a fixed point p of T, which is also the unique solution of a variational inequality.

In 2002, Xu [21] also obtained the strong convergence of $\{x_n\}$ given by (1.1) for a non-expansive mapping in uniformly convex and uniformly smooth Banach spaces:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \quad \forall n \geq 0. \tag{1.1}$$

In 2004, Matsushita and Kuroiwa [7] investigated the following explicit averaging iterates of non-expansive nonself-mappings in Hilbert spaces and gained the strongly convergent outcomes of $\{x_n\}$ defined by (1.2) and (1.3), respectively,

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n u + (1-\alpha_n)(PT)^j x_n), \quad \forall n \geq 0, \tag{1.2}$$

and

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n u + (1-\alpha_n)(TP)^j x_n), \quad \forall n \geq 0. \tag{1.3}$$

In the above two results, the control conditions for the iterative schemes only need $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.

In this paper, our purpose is to extend the main results of Xu [21], Matsushita and Kuroiwa [7] to uniformly convex Banach spaces with the uniformly Gâteaux differentiable norm for nonexpansive non-self mappings, which also develop and complement the main corresponding results of [22, 20, 23, 11, 13, 14, 6, 18] and many others.

2. Preliminaries

Let E be a Banach space and J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall\, x \in E,$$

where E^* is the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

In the sequel, we denote the single-valued duality mapping by j and denote $F(T) = \{x \in D(T); x = Tx\}$ by the fixed point set of T, where $D(T)$ is the domain of T. We write $x_n \rightharpoonup x$ (respectively, $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence $\{x_n\}$ converges weakly (respectively, weak star) to x. As usual, $x_n \to x$ denotes the strong convergence. If $K \subset E$, $\overline{K}$ stands for the closure of K.

Recall that the norm of E is said to be *Gâteaux differentiable* (or E is said to be *smooth*) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.4}$$

exists for each x, y on the unit sphere $S(E) = \{x \in E : \|x\| = 1\}$ of E. Moreover, if for each y in $S(E)$ the limit defined by (1.4) is uniformly attained for x in $S(E)$, we say that the norm of E is *uniformly Gâteaux differentiable*. A Banach space E is said to *uniformly smooth* if the limit (1.4) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

It is well known that the normalized duality mapping J in a smooth Banach space is single-valued and norm topology to weak star topology continuous on any bounded sets of E [17, Theorems 4.3.1, 4.3.2, 4.3.3]. Also, see [8, 3].

A Banach space E is said to *strictly convex* if

$$\|x\| = \|y\| = 1, \ x \neq y \ \text{ implies } \ \frac{\|x + y\|}{2} < 1.$$

A Banach space E is said to *uniformly convex* if, for all $\varepsilon \in [0, 2]$, $\exists \delta_\varepsilon > 0$ such that

$$\|x\| = \|y\| = 1 \text{ with } \|x - y\| \geq \varepsilon \ \text{ implies } \frac{\|x + y\|}{2} < 1 - \delta_\varepsilon.$$

If C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is called a *retraction* from C to D if P is continuous with $F(P) = D$. A mapping $P : C \to D$ is called *sunny* if

$$P(Px + t(x - Px)) = Px, \quad \forall x \in C$$

whenever $Px + t(x - Px) \in C$ and $t > 0$. A subset D of Cis said to be a *sunny non-expansive retract* of C if there exists a sunny non-expansive retraction of C onto D. For more details, see [17, 20, 18, 4].

The following Lemma is well known [17]:

Lemma 2.1. *Let K be a nonempty convex subset of a smooth Banach space E, $\emptyset \neq D \subset K$, $J : E \to E^*$ the normalized duality mapping of E and $P : K \to D$ a retraction. Then the following are equivalent:*

(i) $\langle x - Px, J(y - Px) \rangle \leq 0$ *for all* $x \in K$ *and* $y \in D$.
(ii) P *is both sunny and non-expansive.*

Hence there exists at most a sunny non-expansive retraction P from K onto D.

In 1980, Reich [9] showed that, if E is uniformly smooth and $F(T)$ is the fixed point set of a non-expansive mapping T from K into itself, then there is the unique sunny non-expansive retraction from K onto $F(T)$.

Theorem R. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E and $T : K \to K$ a non-expansive mapping with a fixed point. Then $F(T)$ is a sunny non-expansive retract of K.*

In 1984, Takahashi and Ueda [19] obtained the same conclusion as Reich's in uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Recently, in a reflexive Banach spaces E with a weakly sequentially continuous duality mapping, Song and Chen [11] ($f(x) \equiv u \in K$) constructed the similar results to Reich's and Takahashi and Ueda's results.

Recently, Song et al. [10, 13, 14] also obtained the same outcomes in reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm.

Lemma 2.2 [16, Lemmas 3.1, 3.3] *Let E be a real smooth and strictly convex Banach space and K be a nonempty closed convex subset of E which is also a sunny nonexpansive retract of E. Assume that $T : K \to E$ is a nonexpansive mapping and P is a sunny nonexpansive retraction of E onto K, then $F(T) = F(PT)$.*

Using the similar technique of Song and Chen [10, Theorem 3.1] (also see [11, 12, 13]), we easily testify the following results (taking $f(x) \equiv u$ in [10, Theorem 3.1]).

Lemma 2.3. *Let E be a reflexive and strictly convex Banach space with the uniformly Gâteaux differentiable norm. Suppose that K is a nonempty closed convex subset of E and $T : K \to K$ is a non-expansive mapping satisfying $F(T) \neq \emptyset$. For any $t \in (0, 1)$, let $\{x_t\}$ be a sequence defined by the following equation:*

$$x_t = tu + (1 - t)Tx_t.$$

Then, as $t \to 0$, $\{x_t\}$ converges strongly to some fixed point $P_{F(T)}u$ of T, where $P_{F(T)}$ is a sunny non-expansive retraction from K to $F(T)$.

In the proof of our main theorems, we also need the following lemmas which can be found in [22, 21]:

Lemma 2.4. [21, Lemma 2.5] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\beta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$.

Then $\{a_n\}$ converges to zero as $n \to \infty$.

3. The Main Results

The following theorem was proved by Bruck in [1, 2]:

Theorem B. [2, Corollary 1.1] *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : K \to K$ a non-expansive*

mapping. Then, for any $x \in K$ *and the Cesàro means* $A_n x = \dfrac{1}{n}\sum_{j=0}^{n-1} T^j x$, *we have*

$$\lim_{n\to\infty} \sup_{x\in K} \|A_n x - T(A_n x)\| = 0.$$

Theorem 3.1. *Let* E *be a uniformly convex Banach space with the uniformly Gâteaux differentiable norm. Suppose that* K *is a nonempty closed convex subset of* E, *which is also a sunny non-expansive retract of* E, *and* $T : K \to E$ *is a non-expansive mapping with* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined by*

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n u + (1-\alpha_n)(TP)^j x_n), \quad \forall n \geq 0, \tag{3.1}$$

where P *is a sunny non-expansive retract from* E *to* K. *If* $\alpha_n \in (0,1)$ *satisfies the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, as $n \to \infty$, $\{x_n\}$ *converges strongly to some fixed point* $P_{F(T)}u$ *of* T, *where* $P_{F(T)}$ *is a sunny non-expansive retract from* K *to* $F(T)$.

Proof. For fixed $y \in F(T)$, we have $Py = y = Ty = PTy$ by Lemma 2.2 and the definition of the sunny non-expansive retraction. Moreover, it follows that

$$\begin{aligned}
&\|x_{n+1} - y\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} \|P(\alpha_n u + (1-\alpha_n)(TP)^j x_n) - PTy\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} (\alpha_n \|u - y\| + (1-\alpha_n)\|(TP)^j x_n - (TP)^j y\|) \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} (\alpha_n \|u - y\| + (1-\alpha_n)\|x_n - y\|) \\
&\leq \max\{\|u - y\|, \|x_n - y\|\} \\
&\quad \cdots \\
&\leq \max\{\|u - y\|, \|x_0 - y\|\}.
\end{aligned}$$

This implies the boundedness of $\{x_n\}$ and so are $\{Tx_n\}$ and $\{(TP)^j x_n\}$ for any fixed $j \geq 0$. If we set $A_n = \frac{1}{n+1}\sum_{j=0}^{n} (PT)^j$ and M is a constant such that $\|(TP)^j x_n - u\| \leq M$, then we have

$$\begin{aligned}
\|x_{n+1} - A_n x_n\| &\leq \frac{1}{n+1}\sum_{j=0}^{n} \|P(\alpha_n u + (1-\alpha_n)(TP)^j x_n) - (PT)^j x_n\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n} (\alpha_n \|u - (TP)^j x_n\|) \\
&\leq \alpha_n M \to 0 \quad (n \to \infty).
\end{aligned}$$

Therefore, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - A_n x_n\| = 0. \tag{3.2}$$

Take $v \in F(T)$ and define a subset D of K by

$$D = \{x \in K : \|x - v\| \le r\},$$

where $r = \max\{\|v - x_0\|, \|v - u\|, \|v - p\|\}$ and $p = P_{F(T)}u$. Then D is a nonempty closed bounded convex subset of K, $PT(D) \subset D$ and $\{x_n\} \subset D$. It follows from Theorem B for a nonexpansive self-mapping PT on D that

$$\lim_{n\to\infty} \|A_n x_n - PT(A_n x_n)\| \le \lim_{n\to\infty} \sup_{x\in D} \|A_n x - PT(A_n x)\| = 0. \tag{3.3}$$

On the other hand,

$$\begin{aligned}
&\|x_{n+1} - PTx_{n+1}\| \\
&\le \|x_{n+1} - A_n x_n\| + \|A_n x_n - PT(A_n x_n)\| + \|PT(A_n x_n) - PTx_{n+1}\| \\
&\le 2\|x_{n+1} - A_n x_n\| + \|PT(A_n x_n) - A_n x_n\|.
\end{aligned}$$

Combining (3.2) and (3.3), we have

$$\lim_{n\to\infty} \|x_{n+1} - PTx_{n+1}\| = 0. \tag{3.4}$$

By Lemmas 2.2 and 2.3, there exists a sunny non-expansive retraction from K to $F(T)$, say $P_{F(T)}$. In order to $x_n \to p = P_{F(T)}u$, that is, $\|x_n - p\| \to 0$, the application of Lemma 2.4 is desired. Since

$$\begin{aligned}
&\|x_{n+1} - P(\alpha_n u + (1-\alpha_n)p)\| \\
&\le \frac{1}{n+1}\sum_{j=0}^{n} \|P(\alpha_n u + (1-\alpha_n)(TP)^j x_n) - P(\alpha_n u + (1-\alpha_n)p)\| \\
&\le \frac{1}{n+1}\sum_{j=0}^{n} (1-\alpha_n)\|(TP)^j x_n - p\| \\
&\le (1-\alpha_n)\|x_n - p\|,
\end{aligned}$$

then we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \langle x_{n+1} - (\alpha_n u + (1-\alpha_n)p), J(x_{n+1} - p)\rangle + \alpha_n\langle u - p, J(x_{n+1} - p)\rangle \\
&\le \|x_{n+1} - P(\alpha_n u + (1-\alpha_n)p)\|\|x_{n+1} - p\| + \alpha_n\langle u - p, J(x_{n+1} - p)\rangle) \\
&\le (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| + \alpha_n\langle u - p, J(x_{n+1} - p)\rangle \\
&\le (1-\alpha_n)\frac{\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} + \alpha_n\langle u - p, J(x_{n+1} - p)\rangle.
\end{aligned}$$

Therefore, we have

$$\|x_{n+1} - p\|^2 \le (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle u - p, J(x_{n+1} - p)\rangle. \tag{3.5}$$

If we employ Lemma 2.4 into (3.5) and use the condition $\sum_{n=0}^{\infty} \alpha_n = \infty$, then it remains to prove that

$$\limsup_{n\to\infty} \langle f(p) - p, J(x_{n+1} - p)\rangle \le 0. \tag{3.6}$$

For this purpose, if we consider $z_t = tu+(1-t)PTz_t$, then it follows from Lemma 2.3 and Lemma 2.2 that $p = P_{F(T)}u = P_{F(PT)}u = \lim_{t\to 0} z_t$. Hence we have

$$\begin{aligned}
&\|z_t - x_{n+1}\|^2 \\
&= (1-t)\langle PTz_t - x_{n+1}, J(z_t - x_{n+1})\rangle + t\langle u - x_{n+1}, J(z_t - x_{n+1})\rangle \\
&= (1-t)(\langle PTz_t - PTx_{n+1}, J(z_t - x_{n+1})\rangle + \langle PTx_{n+1} - x_{n+1}, J(z_t - x_{n+1})\rangle) \\
&\quad + t\langle u - p, J(z_t - x_{n+1})\rangle + t\langle p - z_t, J(z_t - x_{n+1})\rangle + t\langle z_t - x_{n+1}, J(z_t - x_{n+1})\rangle \\
&\le \|x_{n+1} - z_t\|^2 + \|PTx_{n+1} - x_{n+1}\|M + t\langle u - p, J(z_t - x_{n+1})\rangle + \|z_t - p\|M
\end{aligned}$$

and so

$$\langle u - p, J(x_{n+1} - z_t)\rangle \le \frac{\|x_{n+1} - PTx_{n+1}\|}{t}M + M\|z_t - p\|, \tag{3.7}$$

where M is a constant such that $M \ge \|x_{n+1} - z_t\|$ by the boundedness of $\{x_n\}$ and $\{z_t\}$. Therefore, taking the upper limit as $n \to \infty$ firstly and then, as $t \to 0$ in (3.7), using (3.4) and $z_t \to p$, we get

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle u - p, J(x_{n+1} - z_t)\rangle \le 0. \tag{3.8}$$

On the other hand, since $\{x_{n+1} - z_t\}$ converges strongly to $(x_{n+1} - p)$ as $t \to 0$ and the duality mapping J is single-valued and the norm topology to the weak star topology is uniformly continuous in any bounded subset of E with the uniformly Gâteaux differentiable norm, it follows that, for all $n \ge 0$,

$$|\langle u - p, J(x_{n+1} - p) - J(x_{n+1} - z_t)\rangle| \to 0 \text{ uniformly } (t \to 0).$$

Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in (0, \delta)$ and $n \ge 0$,

$$\langle u - p, J(x_{n+1} - p)\rangle < \langle u - p, J(x_{n+1} - z_t)\rangle + \varepsilon.$$

Hence, noting (3.8), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - p, J(x_{n+1} - p)\rangle &\le \limsup_{t\to 0}\limsup_{n\to\infty}(\langle u - p, J(x_{n+1} - z_t)\rangle + \varepsilon) \\
&\le 0.
\end{aligned}$$

Namely, (3.6) is proved. Therefore, Lemma 2.4 is satisfied and so the theorem is proved. This completes the proof. □

Corollary 3.2. *Suppose that E, K, T, α_n, P are as Theorem 3.1. Let $\{x_n\}$ be a sequence defined by the following equation:*

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n u + (1-\alpha_n)(PT)^j x_n), \quad \forall n \ge 0. \tag{3.9}$$

Then, as $n \to \infty$, $\{x_n\}$ converges strongly to some fixed point $P_{F(T)}u$ of T, where $P_{F(T)}$ is a sunny non-expansive retraction of K onto $F(T)$.

Proof. Because P is a sunny nonexpansive retraction of E onto K, $PTx \in K$ for all $x \in K$ and $\|PTx - PTy\| \le \|x - y\|$ for all $x, y \in K$. Let $S = PT$. Then S is a non-expansive self-mapping on K and $Sx = PTx = PTPx = SPx$ since $Px = x$ for all $x \in K$.

On the other hand,

$$(PT)^j = (PT)(PPT)(PPT)^{j-2} = \cdots = (SP)^{j-1}(PT) = (SP)^j$$

since $P^2 = P$. Therefore, from (3.9), we have

$$\begin{aligned} x_{n+1} &= \frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n u + (1-\alpha_n)S^j x_n) \\ &= \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n u + (1-\alpha_n)(SP)^j x_n). \end{aligned}$$

Consequently, from Theorem 3.1, the conclusion is proved. This completes the proof. □

Remark 1. (1) Since every Hilbert space is a uniformly convex Banach space and the sunny non-expansive retraction in Hilbert space coincides with the metric projection, then Theorems 3.1 and 3.2 contain Theorems 2 and Theorem 1 of Matsushita and Kuroiwa [7] as a special case, respectively.

(2) The conclusions of Theorem 3.2 and 3,1 still hold if E is assumed to be a uniformly smooth Banach space instead of to have the uniformly Gâteaux differentiable norm since a uniformly smooth Banach space have the uniformly Gâteaux differentiable norm. In particular, if T is a non-expansive self-mapping on K and take $P = I$ (: the identity operator), then our result contains Theorem 3.2 in Xu [21].

4. Applications

As applications of Theorem 3.1, we present the following viscosity approximation results in virtue of Lemma 2.1 and the proof technique of Suzuki [15, Theorems 5, 6]:

Theorem 4.1. *Suppose that E, K, T, α_n, P are as Theorem 3.1. Let $\{x_n\}$ be a sequence defined by the following equation:*

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(x_n) + (1-\alpha_n)(TP)^j x_n), \quad \forall n \geq 0, \tag{4.1}$$

where f is a contractive self-mapping on K with the contractive coefficient $\beta \in (0,1)$. Then, as $n \to \infty$, $\{x_n\}$ converges strongly to p, which is an unique solution in $F(T)$ to the following variational inequality:

$$\langle f(p) - p, J(y-p)\rangle \leq 0, \quad \forall y \in F(T). \tag{4.2}$$

Proof. Theorem 3.1 of Song and Chen [10] guarantees that the variational inequality (4.2) has the unique solution p in $F(T)$. (also, see [22, 11, 12, 13]). Then it follows from Lemma 2.1 that $P_{F(T)}f(p) = p$. Define a sequence $\{y_n\}$ in K by $y_1 \in K$ and

$$y_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(p) + (1-\alpha_n)(TP)^j y_n), \quad \forall n \geq 0.$$

Then, by Theorem 3.1, $\{y_n\}$ converges strongly to $p = P_{F(T)}f(p) \in F(T)$.

Next, we testify $x_n \to p$ as $n \to \infty$. Indeed, it follows that

$$\begin{aligned}
&\|x_{n+1} - y_{n+1}\| \\
&\leq \frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n\|f(x_n) - f(p)\| + (1-\alpha_n)\|(TP)^j x_n - (TP)^j y_n\|) \\
&\leq \alpha_n\beta\|x_n - p\| + (1-\alpha_n)\|x_n - y_n\| \\
&\leq [1-(1-\beta)\alpha_n]\|x_n - y_n\| + \beta\alpha_n\|y_n - p\|.
\end{aligned}$$

Hence, from Lemma 2.4, we obtain $\|x_n - y_n\| \to 0$. Therefore, $\lim_{n\to\infty}\|x_n - p\| = 0$. This completes the proof. □

Corollary 4.2. *Suppose that E, K, T, α_n, P are as Theorem 3.1. Let $\{x_n\}$ be a sequence defined by the following equation:*

$$x_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}(\alpha_n f(x_n) + (1-\alpha_n)(PT)^j x_n), \quad \forall n \geq 0,$$

where f is a contractive self-mapping on K with the contractive coefficient $\beta \in (0,1)$. Then, as $n \to \infty$, $\{x_n\}$ converges strongly to p, which is the unique solution in $F(T)$ to the variational inequality (4.2).

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Error Analysis of the Semi-Implicit Four-Step Fractional Scheme and Adaptive Finite Element Method for the Unsteady Incompressible Navier-Stokes Equation*

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Abstract

In this paper we prove convergence of the velocity in the four-step time advanced fractional scheme for the unsteady incompressible Navier-Stokes equations. The techniques in proving the convergence for time steps are also be used in devicing the posteriori error estimator of the equations of the fractional scheme. The estimator yields upper bounds on the error which are global in space and time and lower bounds that are global in space and local in time. These functions satisfy two fundamental properties: (i) they are explicitly computable and thus their difference to the numerical solution is controlled a posteriori, and (ii) they lead to optimal order residuals as well as to appropriate pointwise representations of the error equation of the same form as the underlying evolution equation. The resulting estimators are shown to be of optimal order by deriving upper and lower bounds for them depending only on the discretization parameters and the data of our problem. We present an adaptive algorithm which can terminate in finite steps for a given tolerance. Numerical experiments validate the theory and yield refined meshes.

Key words Navier-Stokes equation, Crank-Nicolson, Fractional scheme, Posteriori error estimators, Adaptive mesh refinement, Data oscillation.

1 Introduction

The mathematical theories and computational methods of incompressible Navier-Stokes equations have always been important subjects of theoretical fluid dynamics. The basic error analysis of the nonstationary incompressible Navier-Stokes equations have been fully studied. Heywood and Rannacher[1]-[4] provided an error analysis for the Crank-Nicolson method of time discretization applied to spatially discrete Galerkin approximations of the nonstationary Navier-Stokes equations while Geveci [5]considered

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a linearized version of the implicit Euler scheme. However, most of the convergence analysis of the non-stationary incompressible Navier-Stokes equations concentrated on the standard schemes such as Euler or C-N scheme on linearized or nonlinearized version which does not split the time steps. Many useful schemes which are put forward in practice do not have the theoretical analysis. In this paper we introduce a scheme which is different from the normal Euler or C-N scheme and prove its stability and provide the error estimation.

In numerical computation of fluid dynamics a semi implicit scheme for the unsteady incompressible Navier-Stokes equations was developed in generalized coordinate systems [17]. The semi-implicit fractional-step Crank-Nicolson scheme is a special case of the fractional-θ-scheme which is called projection method [18]. We only need to solve a sequence of decoupled elliptic equations for the velocity and the pressure at each time step in projection methods. The main advantage of the semi-implicit fractional-step Crank-Nicolson scheme is its simplicity in prescribing the boundary conditions for the velocity and pressure formulation. The solution procedure is convenient for parallelizing. Liu C-H and Leung DYC developed the finite element solution for the unsteady Navier-Stokes equations using projection method and fractional-θ-scheme[19]. Liu and Leung [20] and Akal et al. [21] demonstrated the feasibility of overlapping domain decomposition for solving scalar transport equations and the incompressible Navier-Stokes equations, respectively. Though the fractional scheme was widely and successfully used in practice for its advantages, the convergence of the multiple steps for time discretization have not been proved. In the four-step time advancement scheme, one time step is splitted into four steps. There are two temporary variables which are denoted by $\widetilde{u}$ and u^* in this paper between two adjacent time steps. The pressure is updated by the temporary variables and the next time step velocity is obtained from the two temporary variables using the original and updated pressure. Therefore the way of obtaining the velocity and pressure in the fractional scheme is quite different from those in normal schemes. In this paper we consider the convergence of semi-implicit fractional-step Crank-Nicolson scheme. Since the updating process of the next time step velocity is second-order accurate, we use C-N scheme which is also second-order accurate to obtain the first temporary variable in order to complete the whole fractional scheme in second-order accurate. We first prove the stability of the fractional scheme which means that the numerical result is bounded by the time and the initial conditions. Then the stability of the velocity is used in the process of estimating the error between exact solution and the numerical results from the fractional scheme. The Uniform Gronwall Lemma plays an important rule in proving the convergence. At last we obtain error estimates of the velocity for the full discretization of the Navier-Stokes equations for sufficient small time increment.

Another object of this paper is to develop the fractional scheme by using adaptive method in some crucial steps of the scheme in order to decrease the computational scale in practice. Adaptive procedures for the numerical solution of partial differential equations started in the late 1970s and are now standard tools in science and engineering. Because of the success in practice, during the last 36 years the use of these adaptive methods became more and more widely spread. In 1984, Babuška and Vogelius [8] firstly showed the finite element solution of one-dimentional boundary value problems. Then Morin, Nochetto and Siebert [10] extended the convergence of adaptive FEM by using data oscillation in 2000. In 2004, Binev, Dahmen and DeVore [11] developed an adaptive finite element method which they showed to be of optimal computational complexity. In this paper the adaptive method is applied in the procedure of obtaining the first temporary velocity since first step of obtaining the $\widetilde{u}$ from original velocity by C-N scheme is the most complicate procedure among the four steps. The other three steps of updating the velocity and pressure do not need to use the adaptive method of mesh refinement. We present a

residual type error estimator and prove it bounds the true error by inequalities in both directions. The upper bound shows that the error estimator can be used as a reliable error indicator for the adaptive local refinement algorithm, while the lower bound suggests that an unnecessary amount of work can be avoid. Meanwhile we design the algorithm of computing the first step of the fractional scheme. In order to validate the algorithm we use the numerical computation results to illustrate procedure of the adaptive method and to show how the mesh is refined at the location where the error estimator is high. Since we have proved that the algorithm has the optimal rate of convergence, the adaptive algorithm provided in the paper is more effective than non-adaptive ones. Therefore we can decrease the degree and achieve better accuracy in parallelized computation in further studies.

In this paper, §2 shows the convergence of second-order finite element for an unsteady incompressible Navier-Stokes equations using the four-step time advancement scheme. Then in §3 we use the isotropic adaptive mesh through local refinement to solve the equation and by devicing the posteriori error estimator we present an adaptive algorithm which can terminate in finite steps for a given tolerance. The numerical computation result is presented in §4, from which we can validate the adaptive algorithm for the Navier-Stokes equations. §5 is the conclusion of this paper.

2 Error Estimation

We consider the Navier-Stokes equations on a finite time interval $[0, T]$. Let Ω be a open, connected and bounded subset of R^d, $d = 2, 3$ with a regular enough boundary $\partial\Omega$. Consider the following unsteady incompressible Navier-Stokes equations:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial}{x_j} \frac{\partial}{x_j} u_i, \qquad \text{for } i = 1, 2, 3. \tag{2.1}$$

and the continuity equations:

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.2}$$

where x_i are the Cartesian coordinates, u_i the velocity vectors, p the pressure, and $u = (u_i, u_j)$. Equations (2.1) and (2.2) are expressed in tensor notation in which indices i and j range over the dimension of the spatial domain n. The usual summation convention on repeated indices is employed.

The initial boundary conditions are:

$$u(x, t) = 0, \qquad x \in \partial\Omega,\ t > 0,$$

and

$$u(x, 0) = u_0(x), \qquad x \in \Omega,\ t = 0.$$

The coupled equations are decoupled by the fractional-step method [23]. A semi-implicit second-order accurate fractional-step method is used to decouple (2.1) and (2.2) that is the sequential solution to

the following equations [24]:

$$\frac{\widetilde{u}_i - u_i^n}{\tau} + \frac{\partial}{\partial x_j}\left(\frac{\widetilde{u}_i + u_i^n}{2}\right)\widehat{u}_j^n = -\frac{\partial p^n}{\partial x_i} + \frac{1}{Re}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{\widetilde{u}_i + u_i^n}{2}\right), \ \widetilde{u}_i|_{\partial\Omega} = 0 \tag{2.3}$$

$$\frac{u_i^* - \widetilde{u}_i}{\tau} = \frac{\partial p^n}{\partial x_i}, \ u_i^*|_{\partial\Omega} = 0 \tag{2.4}$$

$$\frac{\partial}{\partial x_i}\frac{\partial p^{n+1}}{\partial x_i} = \frac{1}{\tau}\frac{\partial u_i^*}{\partial x_i}, \ \frac{\partial p^{n+1}}{\partial n}|_{\partial\Omega} = 0 \tag{2.5}$$

$$\frac{u_i^{n+1} - u_i^*}{\tau} = -\frac{\partial p^{n+1}}{\partial x_i}, \ u_i^{n+1}|_{\partial\Omega} = 0. \tag{2.6}$$

It is a four-step time advancement scheme where τ is the time increment. Superscript represents variables at step n. $\widehat{u}_j^n$ is the average velocity of an element T. The convection and diffusion terms in (2.3) are integrated in time by second-order accurate Crank-Nicolson scheme. The intermediate velocities $\widetilde{u}_i$ and u_i^* do not necessarily satisfy (2.2) that are then used to update the pressure at time step $n+1$ by (2.2). Finally, the intermediate velocity u_i^* is corrected by 2.6 to obtain the divergence free velocity at the next time step u_i^{n+1}.

The weak FEM formulations of (2.3), (2.4), (2.5) and (2.6) are:

$$\int_\Omega v\left[\frac{\widetilde{u}_i - u_i^n}{\tau} + \sum_{j=1}^d \frac{\partial}{\partial x_j}\left(\frac{\widetilde{u}_i + u_i^n}{2}\right)\widehat{u}_j^n + \frac{\partial p^n}{\partial x_i}\right], \\ + \frac{1}{Re}\sum_{j=1}^d \frac{\partial v}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{\widetilde{u}_i + u_i^n}{2}\right)\mathrm{d}\Omega = \sum_{j=1}^d \oint_{\Gamma_N} v n_j \,\mathrm{d}\Gamma_N, \tag{2.7}$$

$$\int_\Omega v\left(\frac{u_i^* - \widetilde{u}_i}{\tau}\right)\mathrm{d}\Omega = \int_\Omega v\frac{\partial p^n}{\partial x_i}\,\mathrm{d}\Omega, \tag{2.8}$$

$$\tau\sum_{j=1}^d \int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial p^{n+1}}{\partial x_i}\,\mathrm{d}\Omega = \sum_{j=1}^d \int_\Omega \frac{\partial v}{\partial x_i}u_i^*\,\mathrm{d}\Omega - \sum_{j=1}^d \oint_{\Gamma_N} v u_i^{n+1} n_j \,\mathrm{d}\Gamma_N, \tag{2.9}$$

$$\int_\Omega v\left(\frac{u_i^{n+1} - u_i^*}{\tau}\right)\mathrm{d}\Omega = -\int_\Omega v\frac{\partial p^{n+1}}{\partial x_i}\,\mathrm{d}\Omega. \tag{2.10}$$

We introduce the standard Sobolev spaces $H^m(\Omega)$, $(m = 0, 1, \dots)$ whose norms are denoted by $||\cdot||_m$. The norm and inner product of L^2 are denoted by $||\cdot||$ and $(\cdot,\cdot)$. To account fot homogeneous Dirichlet boundary conditions we define $H_0^1(\Omega) = \{v \in H_0^1(\Omega) : v|_{\partial\Omega} = 0\}$. We introduce two spaces of incompressible vector fields:

$$H = \{v \in [L^2(\Omega)]^d : \nabla\cdot v = 0; v\cdot n|_{\partial\Omega} = 0\},$$
$$V = \{v \in [H_0^1(\Omega)]^d : \nabla\cdot v = 0\}$$

$d = 2$ *or* 3 for 2D or 3D case, respectively. Define the vector norm $||\cdot||$ such that

$$||u|| = (\sum_{i=1}^{n} ||u_i||^2)^{1/2}, \tag{2.11}$$

where $u = (u_1, u_2, \cdots, u_d) \in (H_0^1)^d$. We also use $(u, v) = \sum_{i=1}^{d}(u_i, v_i)$.

We define

$$\begin{aligned} a(u,v) &= (\nabla u, \nabla v) = \sum_{1\le i,j\le d} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}\Omega, \\ b(u,v,w) &= \sum_{1\le i,j\le d} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j\,\mathrm{d}\Omega. \end{aligned}$$

and

$$\begin{aligned} (Au, v) &= a(u, v) \\ (B(u, v), w) &= b(u, v, w) \end{aligned}$$

Within this framework, (2.1)-(2.2) can be expressed as the evolution equation in H: for $t > 0$

$$\frac{\partial u}{\partial t}(t) + Au(t) + B(u(t), u(t)) = 0,\ t \ge 0,\ u(0) = u_0, \tag{2.12}$$

From (2.4) and (2.5) we have

$$\Delta p^{n+1} = \frac{1}{\tau}\nabla u^{*n+1} = \frac{1}{\tau}\nabla(\widetilde{u}^{n+1} + \tau\nabla p^n) = \frac{1}{\tau}(\nabla\widetilde{u}^{n+1} + \tau\Delta p^n), \tag{2.13}$$

summing up (2.3) at time step $n + 1$, and (2.4), (2.6) at time step n, we derive

$$\frac{\widetilde{u}_i^{n+1} - \widetilde{u}_i^n}{\tau} + \frac{\partial}{\partial x_j}\left(\frac{\widetilde{u}_i^{n+1} + u_i^n}{2}\right)\widehat{u}_j^n = \frac{1}{Re}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{\widetilde{u}_i^{n+1} + u_i^n}{2}\right),$$

or

$$\frac{\widetilde{u}^{n+1} - \widetilde{u}^n}{\tau} + A(\frac{\widetilde{u}^{n+1} + u^n}{2}) + B(\widehat{u}^n, \frac{\widetilde{u}^{n+1} + u^n}{2}) = 0. \tag{2.14}$$

We start by introducing some inequalities for the operators defined above. These inequalities can be proved by using Hölder's inequality and Sobolev inequalities.

Lemma 2.1 *There exists a constant C_1 and C_2 depending only on the time t and the shape regularity constant such that*

$$\begin{aligned} |a(u,v)| &\le C_1||u||\cdot||v||, \\ |a(v,v)| &\ge C_2||v||^2, \\ b(u,v,v) &= 0, \end{aligned}$$

and

$$b(u,v,w) \leq \begin{cases} ||u||_1||v||_1||w||_1 & , \quad \forall u,v,w \in H_0^1(\Omega), \\ ||u||_2||v||||w||_1 & , \quad \forall u \in H^2(\Omega)\bigcap H_0^1(\Omega),\ v,w \in H_0^1(\Omega), \\ ||u||_2||v||_1||w|| & , \quad \forall u \in H^2(\Omega)\bigcap H_0^1(\Omega),\ v,w \in H_0^1(\Omega), \\ ||u||_1||v||_2||w|| & , \quad \forall v \in H^2(\Omega)\bigcap H_0^1(\Omega),\ u,w \in H_0^1(\Omega). \end{cases}$$

Here we recall a discrete version of the Uniform Gronwall Lemma[25] as Lemma 2.2 which will be useful in our discussion.

Lemma 2.2 *Let τ, B, and a_j, b_j, c_j, γ_j, for integers $j \geq 0$, be nonnegative numbers such that*

$$a_n + \tau \sum_{j=0}^{n} b_j \leq \tau \sum_{j=0}^{n} \gamma_j a_j + \tau \sum_{j=0}^{n} c_j + B. \qquad for\ n \geq 0.$$

Suppose that $\tau\gamma_j < 1$ for all j, then

$$a_n + \tau \sum_{j=0}^{n} b_j \leq e^{\tau \sum_{j=0}^{n} \frac{\gamma_j}{1-k\gamma_j}} (\tau \sum_{j=0}^{n} c_j + B). \qquad for\ n \geq 0.$$

Theorem 2.1 *If u^n is the FEM solution of Crank-Nicolson scheme (2.3) to (2.6), then there exists a constant C depending only on the initial date u_0 and time t such that*

$$||u^n||^2 \leq C(||u_0||, t).$$

Proof. From (2.8) and (2.10)

$$\int_\Omega v_i \left(\frac{u_i^{n+1} - \widetilde{u}_i}{\tau} \right) \mathrm{d}\Omega = \int_\Omega v_i (\frac{\partial p^n}{\partial x_i} - \frac{\partial p^{n+1}}{\partial x_i}) \, \mathrm{d}\Omega.$$

Use the summation over the dimensions of the spatial domain to obtain

$$(u^{n+1}, v) = (\widetilde{u}, v) + \tau \sum_{i=1}^{n} (v_i, \frac{\partial}{\partial x_i}(p^n - p^{n+1})).$$

Set $v = u^{n+1}$ and since $v \in V$

$$\begin{aligned} ||u^{n+1}||^2 &\leq ||u^{n+1}||||\widetilde{u}||, \\ ||u^{n+1}|| &\leq ||\widetilde{u}||, \end{aligned} \tag{2.15}$$

and the summation over the dimensions of the spatial domain on (2.7)

$$\left(\frac{\widetilde{u} - u^n}{\tau}, v \right) + a \left(\frac{\widetilde{u} + u^n}{2}, v \right) + b \left(\frac{\widetilde{u} + u^n}{2}, \widehat{u}^n, v \right) = (f, v), \qquad \forall v \in V.$$

Set $v = (\widetilde{u} + u^n)/2$ then $(f, v) = 0$ and $b\left((\widetilde{u} + u^n)/2, \widehat{u}^n, (\widetilde{u} + u^n)/2\right) = 0$,

$$\begin{aligned} \left(\frac{\widetilde{u} - u^n}{\tau}, \frac{\widetilde{u} + u^n}{2} \right) + a \left(\frac{\widetilde{u} + u^n}{2}, \frac{\widetilde{u} + u^n}{2} \right) &= 0, \qquad \forall v \in V, \\ ||\widetilde{u}||^2 - ||u^n||^2 + 2\tau C||\frac{\widetilde{u} + u^n}{2}||^2 &\leq 0. \end{aligned} \tag{2.16}$$

By recursion, (2.15) and (2.16) follows the conclusion

$$||u^n|| \leq ||u_0||.$$

The Theorem 2.1 is proved.

Theorem 2.2 *If u is the exact solution of (2.1) and (2.2), u^n is the solution of Crank-Nicolson scheme (2.3) to (2.6), then there exists a constant C depending on u and time t such that*

$$||u(t) - u^n|| \leq C\tau^2,$$

where time $t = n\tau$.

Proof. From (2.12) we have

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}B(u(s), u(s))\,\mathrm{d}s, \tag{2.17}$$

and from (2.4), (2.6), and (2.14) we obtain

$$u^n = E^n u_0 - \sum_{j=1}^{n} E^{n-j}(I + \frac{\tau}{2}A)^{-1}\tau B(u^{j-1}, \frac{u^{j-1} + u^j}{2}), \tag{2.18}$$

where $E = (I - \frac{\tau}{2}A)/(I + \frac{\tau}{2}A)$, I denoting the identity.
Then we have the equation about the error $e^n := u(t) - u^n$

$$\begin{aligned} e^n &= (e^{-tA}u_0 - E^n u_0) + (\sum_{j=1}^{n} E^{n-j}(I + \frac{\tau}{2}A)^{-1}\tau B(u^{j-1}, \frac{u^{j-1} + u^j}{2}) \\ &\quad - \int_0^t e^{-(t-s)A}B(u(s), u(s))\,\mathrm{d}s, \end{aligned} \tag{2.19}$$

For the first part of (2.19), we note that

$$\lim_{\tau \to 0} \frac{e^{-tA} - (\frac{I - \frac{\tau}{2}A}{I + \frac{\tau}{2}A})^n}{\tau^2} = \frac{1}{12}e^{-tA}tA^3,$$

thus for an sufficiently small τ

$$||e^{-tA}u_0 - E^n u_0|| \leq C(||u_0||)\tau^2.$$

The second part of (2.19)

$$\begin{aligned} &\int_0^t e^{-(t-s)A}B(u(s), u(s))\,\mathrm{d}s - \sum_{j=1}^{n} E^{n-j}(I + \frac{\tau}{2}A)^{-1}\tau B(u^{j-1}, \frac{u^{j-1} + u^j}{2}) \\ = \; &[\int_0^t e^{-(t-s)A}B(u(s), u(s))\,\mathrm{d}s - \sum_{j=1}^{n} E^{n-j}(I + \frac{\tau}{2}A)^{-1}\tau B(u(\tau(j-1)), \frac{u(\tau(j-1)) + u(\tau j)}{2})] \\ &+[\sum_{j=1}^{n} E^{n-j}(I + \frac{\tau}{2}A)^{-1}\tau B(u(\tau(j-1)), \frac{u(\tau(j-1)) + u(\tau j)}{2}) \\ &- \sum_{j=1}^{n} E^{n-j}(I + \frac{\tau}{2}A)^{-1}\tau B(u^{j-1}, \frac{u^{j-1} + u^j}{2})]. \end{aligned}$$

Since

$$\begin{aligned}
&\sum_{j=1}^{n} E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau[B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2})-B(u^{j-1},\frac{u^{j-1}+u^{j}}{2})]\\
=\ &\sum_{j=1}^{n} E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau B(u(\tau(j-1))-u^{j-1},\frac{u(\tau(j-1))+u(\tau j)}{2})\\
&+\sum_{j=1}^{n} E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau B(u^{j-1},\frac{u(\tau(j-1))+u(\tau j)}{2}-\frac{u^{j-1}+u^{j}}{2}).
\end{aligned}$$

We consider

$$\begin{aligned}
&\sum_{j=1}^{n} ||E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau B(u(\tau(j-1))-u^{j-1},\frac{u(\tau(j-1))+u(\tau j)}{2})||\\
\leq\ &\sum_{j=1}^{n}\frac{1}{2}||E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau||||u(\tau(j-1))-u^{j-1}||||u(\tau(j-1))+u(\tau j)||\\
\leq\ &\tau C(||u||,t)\sum_{j=1}^{n-1} C_j||u(\tau j)-u^{j}||.
\end{aligned}$$

Similarly, by virtue of Theorem 2.1

$$\begin{aligned}
&\sum_{j=1}^{n} ||E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau B(u^{j-1},\frac{u(\tau(j-1))+u(\tau j)}{2}-\frac{u^{j-1}+u^{j}}{2})||\\
\leq\ &\sum_{j=1}^{n}\frac{1}{2}||E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau||||u^{j-1}||\left[||u(\tau(j-1))-u^{j-1}||+||u(\tau j)-u^{j}||\right]\\
\leq\ &\tau C(||u||,t)\sum_{j=1}^{n-1} C_j||u(\tau j)-u^{j}||.
\end{aligned}$$

So we obtain

$$\begin{aligned}
&||\sum_{j=1}^{n} E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2})\\
&-\sum_{j=1}^{n} E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau B(u^{j-1},\frac{u^{j-1}+u^{j}}{2})||\\
\leq\ &\tau C(||u||,t)\sum_{j=1}^{n-1} C_j||u(\tau j)-u^{j}||.
\end{aligned}$$

Then we deal with

$$\begin{aligned}
&\int_0^t e^{-(t-s)A}B(u(s),u(s))\,\mathrm{d}s-\sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2})\\
=\;&\int_0^t e^{-(t-s)A}(B(u(s),u(s))-B(u(t),u(t)))\,\mathrm{d}s\\
&-\sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau[B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2})-B(u(t),u(t))]\\
&+[\int_0^t e^{-(t-s)A}\,\mathrm{d}s-\sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau]B(u(t),u(t)).
\end{aligned}$$

Since we have known that

$$\begin{aligned}
&\int_0^t e^{-(t-s)A}\,\mathrm{d}s-\sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau\\
=\;&(I-e^{-tA})A^{-1}-\frac{I-E^n}{I-E}\frac{\tau}{I+\frac{\tau}{2}A}\\
=\;&(I-e^{-tA})A^{-1}-(I-E^n)A^{-1}\\
=\;&(E^n-e^{-tA})A^{-1}.
\end{aligned}\tag{2.20}$$

It is easily verified that

$$\begin{aligned}
&||[\int_0^t e^{-(t-s)A}\,\mathrm{d}s-\sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau]B(u(t),u(t)||\\
=\;&||(E^n-e^{-tA})A^{-1}B(u(t),u(t)||\\
\leq\;&C\tau^2||u(t)||||\nabla u(t)||\\
\leq\;&C(||u||,t)\tau^2.
\end{aligned}$$

We write

$$\begin{aligned}
&\int_0^t e^{-(t-s)A}(B(u(s),u(s))-B(u(t),u(t)))\,\mathrm{d}s\\
&-\sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau[B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2})-B(u(t),u(t))]\\
=\;&\sum_{j=1}^{n}[e^{-(t-\tau(j-1/2))A}-E^{n-j}(I+\frac{\tau}{2}A)^{-1}]\tau[B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2})-B(u(t),u(t))]\\
&+\sum_{j=1}^{n}\int_{\tau(j-1)}^{\tau j}e^{-(t-\tau(j-1/2))A}[B(u(s),u(s))-B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2})]\,\mathrm{d}s\\
&+\sum_{j=1}^{n}\int_{\tau(j-1)}^{\tau j}[e^{-(t-s)A}-e^{-\tau(n-(j-1/2))A}](B(u(s),u(s))-B(u(t),u(t)))\,\mathrm{d}s.
\end{aligned}$$

Here since

$$\begin{aligned} & ||u(\tau j) - u(\tau(j-1)) - \tau\frac{\partial u}{\partial t}u(\tau(j-1/2))|| \\ = \;& \frac{1}{2}||\int_{\tau(j-1)}^{\tau j}(s-\tau(j-1))^2\frac{\partial^3 u}{\partial t^3}u(s)\,\mathrm{d}s + \int_{\tau(j-1)}^{\tau j}(s-\tau j)^2\frac{\partial^3 u}{\partial t^3}u(s)\,\mathrm{d}s|| \\ \leq \;& C\tau^2\int_{\tau(j-1)}^{\tau j}||\frac{\partial^3 u}{\partial t^3}u(s)||\,\mathrm{d}s \\ \leq \;& C(||u||,j)\tau^2. \end{aligned}$$

and as (2.20)

$$\begin{aligned} & \lim_{\tau\to 0}\sum_{j=1}^{n}[e^{-(t-\tau(j-1/2))A} - E^{n-j}(I+\frac{\tau}{2}A)^{-1}]\tau \\ = \;& \lim_{\tau\to 0}\sum_{j=1}^{n}[e^{-(t-\tau(j-1/2))A}\tau - \sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau \\ = \;& \int_0^t e^{-(t-s)A}\,\mathrm{d}s - (I-E^n)A^{-1} \\ = \;& (E^n - e^{-tA})A^{-1}. \end{aligned}$$

Noting that $||B(u(t_1),u(t_2)) - B(u(t),u(t))|| \leq C(||u||,t) \qquad \forall\, t_1,t_2 \in [0,t]$, then

$$\begin{aligned} & ||\int_0^t e^{-(t-s)A}(B(u(s),u(s)) - B(u(t),u(t)))\,\mathrm{d}s \\ & -\sum_{j=1}^{n}E^{n-j}(I+\frac{\tau}{2}A)^{-1}\tau[B(u(\tau(j-1)),\frac{u(\tau(j-1))+u(\tau j)}{2}) - B(u(t),u(t))]|| \\ \leq \;& C(||u||,t)\tau^2. \end{aligned}$$

Now we have obtained that for sufficient small τ

$$||u(t) - u^n|| \leq C(||u||,t)\tau^2 + C(||u||,t)\sum_{j=1}^{n-1}C_j||u(\tau j) - u^j||\tau.$$

Using the Lemma 2.2, the theorem is established.

$$||u(t) - u^n|| \leq C(||u||,t)\tau^2.$$

The Theorem 2.2 is proved.

Theorem 2.3 *We obtain error estimates for the full discretization of the Navier-Stokes equations. For a constant C there holds at time $t = n\tau$*

$$||u(t) - u_h^n|| \leq C(||u||,t,\Omega)(h^2+\tau^2). \tag{2.21}$$

Proof. If (2.7), (2.8), (2.9) and (2.10) are solved step by step using second-order finite element. Let $V_h \in V$ be the finite dimensional subspaces, $u_h^n \in V_h$ be the solutions of FEM. There exists a constant C such that

$$||u^n - u_h^n|| \leq Ch^2||u^n||.$$

and thanks to Theorem 2.1, we have

$$||u(t) - u_h^n|| \leq ||u(t) - u^n|| + ||u^n - u_h^n|| \leq C(h^2 + \tau^2).$$

The Theorem 2.3 is proved.

Since the four-step scheme is complex and error estimation of pressure is difficult to derive, we will put forward the approximation result about pressure in further studies to complete the convergence of the scheme.

3 Convergence of Adaptive Finite Element Method

Now we shall show the adaptive finite element method for the equation (2.7). Firstly we present a residual type error estimator η and prove it bounds the true error by inequalities in both directions. The upper bound shows that η can be used as a reliable error indicator for the adaptive local refinement algorithm, while the lower bound suggests that an unnecessary amount of work can be avoid.

For the purposes of the error analysis it is convenient to introduce the fact that for $t \in (t_{j-1}, t_j], j = 1, 2, \ldots, n$

$$U(x,t) = \frac{t - t_{j-1}}{\tau} u_h^j(x) + \frac{t_j - t}{\tau} u_h^{j-1}(x) \tag{3.1}$$

Then we can define the inner residual

Define the initial error

$$\eta_0 = ||u(0) - U(0)||_V = ||u_0 - u_h^0||_{0,\Omega},$$

the jump residual

$$\eta_1 = (\int_0^t \tau \sum_{l \bigcap \partial\Omega = \emptyset} \int_{K_l} ||h^{1/2}[\nabla U]_l||_V^2 \, \mathrm{d}x \, \mathrm{d}s)^{1/2},$$

where $[U]_l$ is the jump of U on the edge $l = \overline{K}_l^1 \bigcap \overline{K}_l^2$,
the time residual

$$\eta_2 = (\sum_j \tau ||\nabla(u_h^j - u_h^{j-1})||^2)^{1/2}.$$

Theorem 3.1 *Let u and u_h^n be the exact solution and the full discretization solution of 2.1 and 2.2, respectively, and the U is defined as 3.1. For all time $t \in [0, T]$, there exists a constant $C_1 > 0$ depending only on the shape regularity constant such that*

$$||(u - U)(t)||_{0,\Omega}^2 + \int_0^t ||(u - U)(s)||_{0,\Omega}^2 \, ds \leq C(\eta_0^2 + \eta_1^2 + \eta_2^2). \tag{3.2}$$

Proof. From the proof of Theorem 2.2 we arrive at

$$\begin{aligned} & ||(u-U)(t)||_{0,\Omega}^2 + \int_0^t ||(u-U)(s)||_{0,\Omega}^2 \,\mathrm{d}s \\ \leq\ & (||(u-U)(0)|| + \int_0^t ||\frac{\partial}{\partial t}(u-U)(s)|| \,\mathrm{d}s)^2 + \int_0^t ||(u-U)(s)||_{0,\Omega}^2 \,\mathrm{d}s \\ \leq\ & C(||(u-U)(0)||^2 + \int_0^t ||\frac{\partial}{\partial t}(u-U)(s)||^2 \,\mathrm{d}s + \int_0^t ||(u-U)(s)||_{0,\Omega}^2 \,\mathrm{d}s) \end{aligned}$$

for simplicity we set $t \in [t_{n-1}, t_n]$, where $t_j = \tau j, j = 1, 2, \ldots, n$, then we derive

$$\begin{aligned} & ||(u-U)(t)||_{0,\Omega}^2 + \int_0^t ||(u-U)(s)||_{0,\Omega}^2 \,\mathrm{d}s \\ \leq\ & C(||(u-U)(0)||^2 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} ||\frac{\partial}{\partial t}(u-U)(s)||^2 + ||(u-U)(s)||^2 \,\mathrm{d}s). \end{aligned}$$

We note that at time $s = t_j, U(s) = u_h^j$ and substitute (2.17) and (2.18) into the equation above

$$\begin{aligned} & ||(u-U)(t)||_{0,\Omega}^2 + \int_0^t ||(u-U)(s)||_{0,\Omega}^2 \,\mathrm{d}s \\ \leq\ & C(||(u-U)(0)||^2 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} ||Au(s) - AU(s)||^2 \\ & + ||B(u(s),u(s)) - B(U(s),U(s))||^2 + ||(u-U)(s)||^2 \,\mathrm{d}s) \end{aligned}$$

From the proof of Theorem 2.2 we know that for time step $[t_{j-1}, t_j]$

$$\begin{aligned} & \int_{t_{j-1}}^{t_j} ||Au(s) - AU(s)||^2 \,\mathrm{d}s \\ \leq\ & \int_{t_{j-1}}^{t_j} (||Au(s) - AU(t_{j-1})|| + ||AU(s) - AU(t_{j-1})||)^2 \,\mathrm{d}s \\ \leq\ & C \int_{t_{j-1}}^{t_j} (C_1 ||u(s) - U(s)||^2 + \sum_{l \bigcap \partial\Omega = \emptyset} \int_{K_l} ||h^{1/2} [\nabla U]_l|| \,\mathrm{d}x) \,\mathrm{d}s \\ =\ & C \int_{t_{j-1}}^{t_j} ||u(s) - U(s)||^2 \,\mathrm{d}s + \tau \sum_{l \bigcap \partial\Omega = \emptyset} \int_{K_l} ||h^{1/2} [\nabla U]_l|| \,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned}
&\int_{t_{j-1}}^{t_j} ||B(u(s),u(s)) - B(U(s),U(s))||^2 \,\mathrm{d}s \\
\leq\ & \int_{t_{j-1}}^{t_j} (||B(u(s),u(s)-U(s))|| + ||B(u(s)-U(s),U(s))||)^2 \,\mathrm{d}s \\
\leq\ & C\int_{t_{j-1}}^{t_j} (||\nabla u(s)||^2 + ||\nabla U(s)||^2 + ||\nabla(u(s)-U(s))||^2) \,\mathrm{d}s \\
\leq\ & C(\tau(||U_{t_j} - U(t_{j-1})||^2 + \int_{t_{j-1}}^{t_j} ||u(s)-U(s)||^2 \,\mathrm{d}s + \tau \sum_{l\bigcap\partial\Omega=\emptyset} \int_{K_l} ||h^{1/2}[\nabla U]_l|| \,\mathrm{d}x
\end{aligned}$$

Hence adding up the time step $[t_{j-1}, t_j]$ and recalling the definition of $\eta_i, i = 0, 1, 2$, we have that

$$||(u-U)(t)||^2 + \int_0^t ||(u-U)(s)||^2 \,\mathrm{d}s \leq C(\eta_0^2 + \eta_1^2 + \eta_2^2)$$

The proof of Theorem 3.1 is complete.

Theorem 3.2 *Let u, u_h^n and U be defined as same as in Theorem 3.1. For all time $t \in [0,T]$, there exists a constant $C_2 > 0$ depending only on the shape regularity constant such that*

$$\eta_1^2 - C_2 osc_\Omega(u_h) \leq C_2(||(u-U)(t)||^2 + \int_0^t ||(u-U)(s)||^2 \, ds), \tag{3.3}$$

with

$$osc_\Omega(u_h)^2 \ := \ \int_0^t \sum_K \int_K \left[||h^2(U - \overline{U})||^2 + ||h^2(\frac{\partial u_h}{\partial t} - \overline{\frac{\partial u_h}{\partial t}})||^2 + ||h^2 \nabla U||^2 \right] dx \, ds.$$

where $\overline{v}|_K = \int_K v/|K| \, dx$

Proof: We recall the definition of η_1, set $l = K_l^1 \bigcap K_l^2$, then

$$||[\nabla U]_l|| \ = \ ((\nabla U)_{K_l^1} - (\nabla U)_{K_l^1})n$$

where n is the unit normal vector on loutwards K_l^1. Since U is the linear interpolation of $u_h^j, j = 0, 1, \ldots, n$we arrive at

$$\begin{aligned}
&\sum_{l\bigcap\partial\Omega=\emptyset} \int_{K_l} ||h^{1/2}[\nabla U]_l||_V^2 \,\mathrm{d}x \\
\leq\ & C \sum_{l\bigcap\partial\Omega=\emptyset} (|K_l^1| + |K_l^2|) \int_{K=\max(K_l^1,K_l^2)} h||\nabla U - \nabla\overline{U}|_K||^2 \,\mathrm{d}x \\
\leq\ & C\sum_K \int_K h(||U - \overline{U}|_K||^2 + ||\nabla U||^2)
\end{aligned}$$

Integrate the equation above from 0 to t, then

$$\eta_1^2 \leq C_2(osc_\Omega(u_h) + (||(u-U)(t)||^2 + \int_0^t ||(u-U)(s)||^2 \, \mathrm{d}s)),$$

The proof of Theorem 3.2 is complete..

With the same argument, we have the two sided bounds in nested finite element spaces. Let T_H be a shape regular tetrahedralation of Ω and T_h is a refinement of T_H such that $V_H \subset V_h$. Let u_h and u_H be the finite element approximation of u in V_h and V_H. Now we present a localized version of the upper bound for the difference of solutions.

The interior node property used in the above results is important since without the interior nodes V^h in not fine enough to capture the difference of the successive solutions. If the refinement does not produce interior nodes, the error will not change. In this sense, interior node is a necessary condition for the error condition.

Now we are going to prove that there exists a local refinement algorithm which produces a sequence of meshes $\{T_k\}$ such that the corresponding finite element solutions $\{u_k\}$ satisfy

$$||u - u_k|| \leq C\delta^k, \qquad \text{for some } \delta \in (0,1). \tag{3.4}$$

We are now in the positon to present the algorithms:

Algorithm 3.1

$[u_H, \eta_H] =$ **POSTERIORI**(T_H)

1. Solve (2.7) on T_H to get the solution u_H .
2. Compute the residual type error estimator $\eta(u_H) := (\eta_0 + \eta_1 + \eta_2)^{1/2}$

END POSTERIORI

Let $\theta, \tilde{\theta} \in (0,1)$ be two fixed numbers and T_H is the current mesh. Denote $\Omega_{\overline{M_H}} := \bigcup_{T \in \overline{M}} \Omega_T$. Our main subroutine is the refinement in one loop:

Algorithm 3.2

$T_h =$ **REFINE**$(T_H, \eta_H, \theta, \tilde{\theta})$

1. **Marking Strategy**
 (a) Mark the minimal edge set M_H such that
 $$\eta_{M_H}(u_H) \geq \theta\eta(u_H). \tag{3.5}$$
 (b) Enlarge M_H to be $\overline{M_H}$ with minimal extension such that
 $$osc_{\Omega_{\overline{M_H}}}(u_H) \geq \tilde{\theta} osc_{T_H}(u_H). \tag{3.6}$$
2. **Refinement**
 (a) Refine each element $T \in \Omega_{\overline{M_H}}$ into elements such that it will satisfy the interior node property for T and ∂T.
 (b) Complete the hanging point to be a conforming partition T_h

END REFINE

Before we proof (3.4), we need the lemma:

Lemma 3.1 *Let $T_h = \boldsymbol{REFINE}(T_H, \eta_H, \theta, \tilde{\theta})$. There exists a number $\beta \in (0,1)$ depending only on θ, the shape regularity of T_H such that*

$$||u - u_h|| \leq \beta ||u - u_H|| + osc_{T_H}(u_H).$$

Proof. Using the upper bound inequality in the refinement strategy and the discrete version of lower bound inequality we find

$$\begin{aligned} \theta^2 ||u - u_H||^2 &\leq C_1^2 \eta_{M_H}(u_H)^2 \\ &\leq 3C_1^2/C_2^2((||u_h - u_H||)^2 + osc_{T_H}^2(u_H)). \end{aligned}$$

Then

$$||u - u_h||^2 \leq (1 - \frac{\theta^2 C_2^2}{3C_1^2})||u - u_H||^2 + osc_{T_H}(u_H)^2.$$

We define $\beta^2 := 1 - \frac{\theta^2 C_2^2}{3C_1^2}$, such that

$$\begin{aligned} ||u - u_h||^2 &\leq \beta^2 (||u - u_H||)^2 + osc_{T_H}(u_H)^2 \\ &\leq \beta (||u - u_H|| + osc_{T_H}(u_H))^2. \end{aligned}$$

The proof of Lemma 3.1 is completed.

Theorem 3.3 *Let u_k be the solution obtained in the k-th loop in the algorithm AFEM, then there exists a constant $\delta \in (0,1)$ depending only on θ, $\tilde{\theta}$ and the shape regularity of T_0 such that*

$$||u - u_k|| \leq C\delta^k$$

and thus the algorithm AFEM will terminate in finite steps.

Proof. Obviously, we have the inequations for the data oscillation on the tow mashes T_H and T_h generated in the algorithm AFEM.

$$osc_{K_h}(u_h) \leq \alpha osc_{K_H}(u_H).$$

By Lemma 3.1 and the inequation above there exist α, $\beta \in (0,1)$ such that

$$||u - u_{k+1}|| \leq \beta(||u - u_k||) + osc_{T_k}(u_k),$$

and

$$osc_{T_k}(u_k) \leq \alpha^k osc_{T_0}(u_0).$$

If we let $e_k = ||u - u_k||$, we then get

$$e_{k+1} \leq \beta e_k + \alpha^k osc_{T_0}(u_0),$$

which by recursion implies

$$e_{k+1} \leq \beta^{k+1} e_0 + osc_{T_0}(u_0) \sum_{j=0}^{k} \beta^j \alpha^{k-j}.$$

We choose δ such that $1 \geq \delta \geq \max(\alpha, \beta)$ to obtain

$$\begin{aligned} \sum_{j=0}^{k} \beta^j \alpha k - j &\leq \sum_{j=0}^{k} \max(\alpha, \beta)^j \delta^{k-j} \\ &\leq \delta^k \sum_{j=0}^{k} (\frac{\max(\alpha, \beta)}{\delta})^j \\ &\leq \delta^{k+1} \frac{1}{\delta - \max(\alpha, \beta)}. \end{aligned}$$

Then the assertion follows immediately

$$e_{k+1} \leq C\delta^{k+1},$$

with

$$C = e_0 + \frac{osc_{T_0}(u_0)}{\delta - \max(\alpha, \beta)}.$$

The proof of Theorem 3.3 is completed.

From Theorem 3.3, the adaptive algorithm AFEM will stop in finite steps for a given tolerance.

4 Numerical Results

In this section, we take the numerical solution of the N-S equation at different Reynolds number as examples to test the four-step time advancement scheme and show the mesh refinement to illustrate the adaptive method.

The case is the backward facing step. The step height h is set to half the height of the expanded channel. The inflow and outflow boundaries are located at $x = 0$ and $x = 20h$, respectively. The bottom and top of the channel are fixed by stationary walls. We set

$$\Omega = \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$$

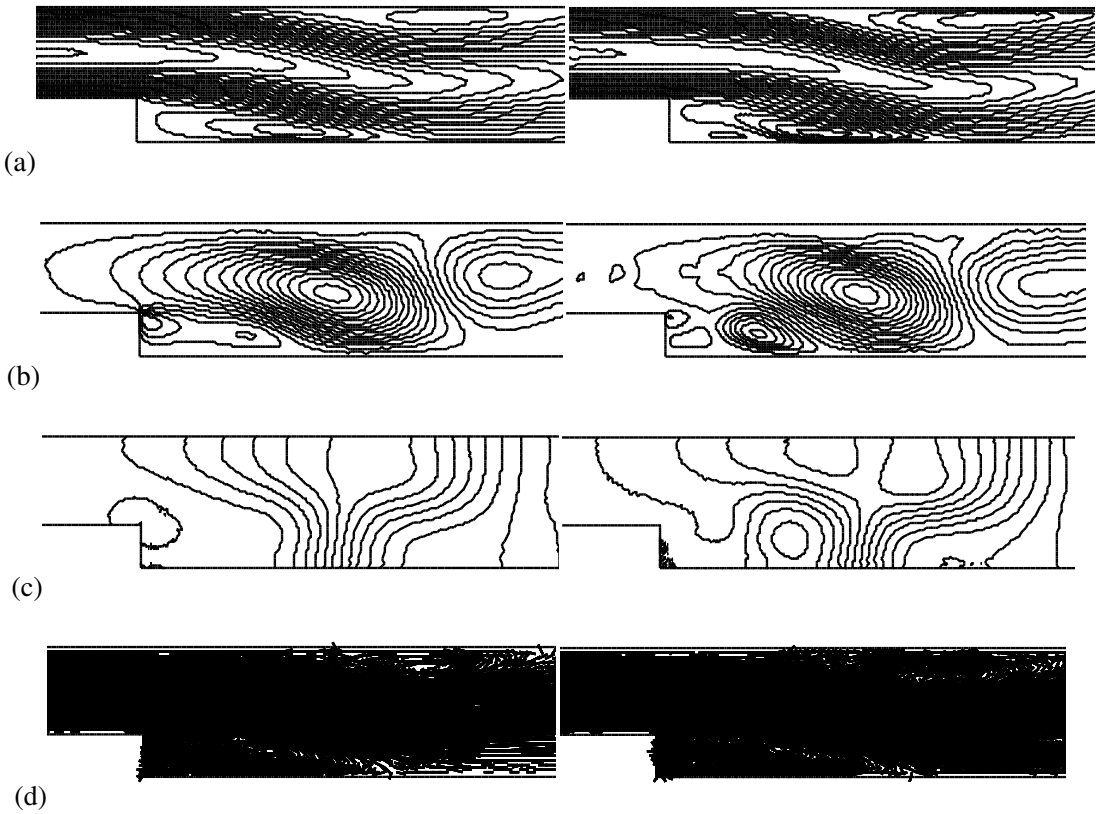
where

$$\begin{aligned} \Gamma_0 &= \{(x, y) : x = 0, y = t, t \in [0, 1]\} \\ \Gamma_1 &= \{(x, y) : x = 2t, y = 0, t \in [0, 1]\} \\ \Gamma_2 &= \{(x, y) : x = 2, y = -t/2, t \in [0, 1]\} \\ \Gamma_3 &= \{(x, y) : x = 2 + 20t, y = -0.5, t \in [0, 1]\} \\ \Gamma_4 &= \{(x, y) : x = 20, y = -0.5 + 1.5t, t \in [0, 1]\} \\ \Gamma_5 &= \{(x, y) : x = 20t, y = 1, t \in [0, 1]\} \end{aligned}$$

with initial condition $u_{i0} = 4y^2(1-y)$ $(0 < x < 2, 0 < y < 1)$, $u_{j0} = 0$, $p_0 = 0$ and boundary condition

$$\begin{aligned} u_i &= 4y^2(1-y), \text{ on } \Gamma_0 \\ u_i &= 0, \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_5 \\ u_j &= 0, \text{ on } \Gamma_0 \cup \Gamma_1 \cup \Gamma_3 \cup \Gamma_5 \\ p &= 0, \text{ on } \Gamma_4 \end{aligned}$$

And we set $dt = 0.2$, N(The number of nodes) ≈ 1000



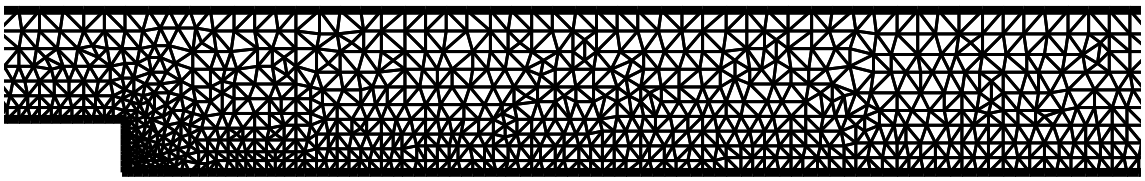
(a)

(b)

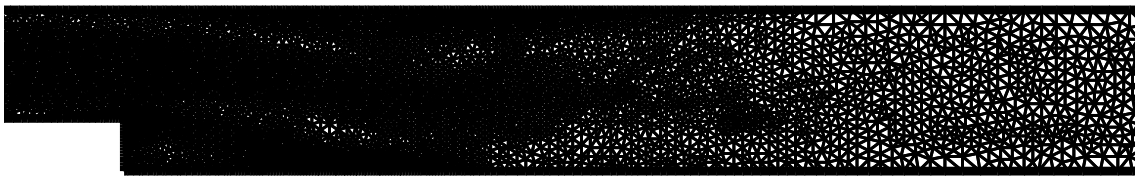
(c)

(d)

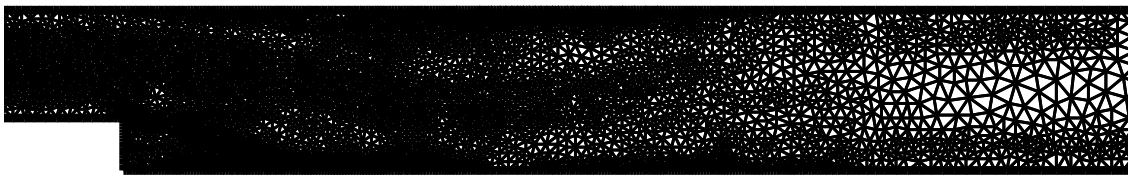
The pictures show the speed contours(a), spanwise contours(b), pressure contours(c),and stream(d) at Reynolds number $Re = 400$(left) and $Re = 1500$(right). From (a) and (c) we can see that there are two secondary vortices. The dividing streamlines of the recirculation at the bottom are indicated by the zero streamfunction contours and the recirculation regions are indicated by negative streamfunctions. At the Reynolds number $Re = 400$ increased a recirculation region is observed at the top of the expanded channel and its size increased at $Re = 1500$.



(e1)



(e2)



(e3)

(e1) is the original while (e2) and (e3) are the refined mesh at $Re = 400$ and $Re = 2000$, respectively. (e1) is nearly uniform, (e2) and (e3) are refined at the left bottom and top center of the expanded channel where near the locations of the two secondary recirculations.

5 Conclusion

The convergence of the four-step time advancement scheme for the unsteady incompressible Navier-Stokes equations is proved. The governing equations are decoupled by the fractional method. The spatial domain is solved by the second-order Galerkin FEM and the temporal domain is integrated by the Crank-Nicolson scheme. We complete the proof of stability and error estimation of the fractional scheme in second-order accurate. The posteriori error estimator of adaptive method is devised to satisfy the both upper and lower bounds. And we device an adaptive algorithm using the error estimator as an indicator. The numerical result is validated with the cases: flow over a backward facing step. The calculated results also show the step of mesh refinement.

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A SQP Algorithm Based on a Smoothing Lower Order Penalty Function for Inequality Constrained Optimization

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Abstract

In this paper, based on a smoothing approximation of a lower order penalty function and Facchinei's method of dealing with the inconsistency of subproblems in SQP methods, we propose a SQP algorithm for nonlinear inequality constraints optimization problems. The presented algorithm incorporates automatic adjustment rules for the choice of parameters. The algorithm is proved to be globally convergent and superlinearly convergent under some mild conditions without the strict complementarity.

Key words. Inequality constrained optimization, quadratic programming, global convergence, convergence rate.

MR(2000)Subject Classification: 65K05,90C30

1. Introduction

In this paper, we consider the following nonlinear inequality constrained optimization:

$$(P)\qquad \begin{array}{ll}\min & f(x)\\ \text{s.t.} & g_j(x)\le 0,\quad j\in I=\{1,2,...,m\},\end{array} \tag{1.1}$$

where the functions $f_0,\quad f_j(j\in I):R^n\to$R are all twice continuously differentiable.

It is well known that sequential quadratic programming (SQP) algorithms are widely acknowledged to be among the most successful algorithms for solving (P)(See[1], [7]–[11], [13], [18]-[21]). A good survey of SQP algorithms by Boggs and Toll can be found in [6].

At each iteration of a SQP algorithm, a search direction d^k is calculated by solving the following QP subproblem:

$$(\text{QP})\qquad \begin{array}{ll}\min\limits_{d} & \nabla f(x)^Td+\frac{1}{2}d^TB^kd\\ \text{s.t.} & g_j(x)+\nabla g_j(x)^Td\le 0,\quad j\in I,\end{array} \tag{1.2}$$

where B^k is symmetric positive definite. The iteration then has the form

$$x^{k+1}=x^k+\alpha^kd^k,$$

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where the stepsize α^k is chosen to yield a sufficient decrease of a suitable merit function.

A serious shortcoming in the conventional SQP method for problem (P) is the possible inconsistency of the constrains in (1.2). To overcome this difficulty, various method have been proposed. Generally speaking, there are two ways to overcome it, one is to modify subproblem (1.2) to ensure that this subproblem is always feasible, e.g. see [3, 8, 19]. The other one is Facchinei's method. Its basic idea is: By using the differentiable exact penalty function developed by Lucidi [4] as merit function, if the subproblem (1.2) is consistent and its solution is acceptable, then the solution is used as the search direction; otherwise, a first order direction, which is an approximation of the gradient of the merit function, is used. The algorithm is proved to be global and superlinear convergence under some mild conditions without the strict complementarity.

Among many SQP algorithms for problem (P), the classical l_1 penalty function

$$F(x,\mu) = f(x) + \mu \sum_{i=1}^{m} {g_i}^+(x),$$

where ${g_i}^+(x) = \max_{i\in I}\{0, g_i(x)\}$, has been often used as a merit function. Since lower order penalty function have shown some promising in establishing optimality conditions, and in particular, they require weaker conditions than the l_1 penalty function for the existence of exactness, e.g. see ([15, 16, 17]). Recently, based on a smoothing approximation of the following lower order penalty function

$$F^s(x,\mu) = f(x) + \mu \sum_{i=1}^{m} ({g_i}^+(x))^s, \tag{1.3}$$

where $s \in (0,1)$, and Facchinei's method of dealing with the inconsistency of subproblems in SQP methods, K.W.Meng, S.J.Li and X.Q.Yang [22] have presented a robust SQP algorithm for solving a nonlinear constrained optimization problem. This algorithm incorporates automatic adjustment rules for the choice of parameters. Under a new regularity condition at infeasible points, the algorithm is proved to be globally convergent. However, no superlinearly convergent result is presented.

In this paper, based on another smoothing approximation of a lower order penalty function and Facchinei's method of dealing with the inconsistency of subproblems in SQP methods, we propose a SQP algorithm for nonlinear inequality constraints optimization problems. The presented algorithm incorporates automatic adjustment rules for the choice of parameters. The algorithm is proved to be globally convergent and superlinearly convergent under some mild conditions without the strict complementarity.

The remainder of this paper is organized as follows. The proposed algorithm is stated in Section 2. In Section 3 and Section 4, under some mild assumptions, we show that this algorithm is globally convergent and locally superlinear convergent, respectively. In section 5, some preliminary numerical results are reported. Finally, we give concluding remarks about the proposed algorithm.

For the rest of this section, we give a list of notation to be employed in this paper. We define the following index sets:

$$I_P(x) = \{i \in I : g_i(x) > 0\},$$

$$I_N(x) = \{i \in I : g_i(x) < 0\},$$

$$I_0(x) = \{i \in I : g_i(x) = 0\}.$$

2. Description of algorithm

In this section, following the similar way as in [5], we firstly give a smoothing approximation for (1.3) as follows:

$$F^s(x, \mu, \varepsilon) = f(x) + \mu \sum_{i=1}^{m} ((g_i{}^+(x))^{\alpha s} + \varepsilon^{\alpha})^{\frac{1}{\alpha}}, \tag{2.1}$$

where $\varepsilon > 0, \quad \alpha > \frac{2}{s}$.

It is not difficult to verify that $F^s(x, \mu, \varepsilon)$ is twice continuously differentiable for any x, with gradient

$$\nabla F^s(x, \mu, \varepsilon) = \nabla f(x) + \mu \sum_{i=1}^{m} \frac{s(g_i{}^+(x))^{(s\alpha-1)}}{((g_i{}^+(x))^{\alpha s} + \varepsilon^{\alpha})^{1-\frac{1}{\alpha}}} \nabla g_i(x). \tag{2.2}$$

In the sequel, using the smoothing lower order penalty function $F^s(x, \mu, \varepsilon)$ as the merit function, we propose a SQP algorithm for (P). If the QP subproblem is feasible, we use its solution as a search direction. Otherwise, we use the negative gradient direction of the merit function, i.e., $-\nabla F^s(x, \mu, \varepsilon)$ as a search direction.

We state now the algorithm for solving problem (P).

Algorithm 2.1

Data: $x^0 \in R^n, \quad B^0 > 0, \quad \varepsilon^{-1}, \quad \mu^{-1}, \quad T_\mu > 1, \quad T_\varepsilon \in (0, 1), \ \theta \in (0, 1), \quad \sigma \in (0, \frac{1}{2})$.

Step 0: Set $k = 0$.

Step 1: Let $\varepsilon^k = T_\varepsilon \varepsilon^{k-1}$ and $\mu^k = \mu^{k-1}$.

Step 2: Compute $(d^k, \ \lambda^k)$, the KKT pair for QP subproblem. If the QP subproblem is infeasible, go to Step 5. Otherwise, if $d^k = 0$, stop.

Step 3: If $\nabla F^s(x^k, \mu^k, \varepsilon^k)^T d^k \leq -\frac{1}{2}(d^k)^T B_k d^k$, go to Step 9.

Step 4: Set $\mu^k = T_\mu \mu^k$, go to Step 3.

Step 5: If $\sum_{i \in I_P(x^k)} \frac{s(g_i{}^+(x^k))^{(s\alpha-1)}}{((g_i{}^+(x^k))^{\alpha s} + \varepsilon^{\alpha})^{1-\frac{1}{\alpha}}} \nabla g_i(x^k) \neq 0$, go to Step 7.

Step 6: Set $\varepsilon^k = T_\varepsilon \varepsilon^k$, go to Step 5.

Step 7: If $-\|\nabla F^s(x^k, \mu^k, \varepsilon^k)\|^2 \leq -H(x^k)$, where $H(x^k) = \max_{i \in I}\{g_i(x^k), \ 0\}$, set $d^k = -\nabla F^s(x^k, \mu^k, \varepsilon^k)$, go to Step 9.

Step 8: Set $\mu^k = T_\mu \mu^k$, go to Step 7.

Step 9: Let α^k be the largest one of the sequence $\{1, \theta, \theta^2, \cdots\}$ satisfying the following condition:

$$F^s(x^k + \alpha^k d^k, \mu^k, \varepsilon^k) \leq F^s(x^k, \mu^k, \varepsilon^k) + \sigma \alpha^k \nabla F^s(x^k, \mu^k, \varepsilon^k)^T d^k, \tag{2.3}$$

Step 10: If the QP subproblem is feasible, generate B^{k+1} using the damped BFGS formula proposed by Powell ([11]). Otherwise, set $B^{k+1} = B^k$.

Step 11: Set $k = k + 1$, go to Step 1.

To prove that Algorithm 3.1 is well defined and the convergence of the above algorithm, we need the following assumptions.

Assumption 1: The sequences $\{x^k\}$ and $\{d^k\}$ generated by Algorithm 3.1 are bounded.

Assumption 2: $\{B^k\}$ are positive definite and there exist two positive constants β_1 and β_2 such that

$$\beta_1\|d\|^2 \leq d^T B^k d \leq \beta_2\|d\|^2.$$

Assumption 3: (Regularity Condition at Feasible Points) At each feasible point x of problem (P), the gradients of the active constraints are linearly independent.

Since a general constrained problem can be viewed as a combination of two problems: (i) the feasible problem, i.e., the problem of finding a feasible point; (ii) the problem of finding a local minimum point of the objective function over the feasible set. The former problem is a hard one, since it is essentially global; in fact, we have to find a global minimum of the generally nonconvex function expressing the violation of the constraints. Therefore, in order to be able to easily solve this problem, we have to resort to some suitable condition. In our approach, similar to [13], this condition is expressed as follows.

Assumption 4: (Regularity Condition at infeasible Points) At each infeasible point x of problem (P), there exists $\widetilde{\varepsilon} > 0$ satisfying the following condition

$$\sum_{i\in I_P(x)} \frac{s(g_i{}^+(x))^{(s\alpha-1)}}{((g_i{}^+(x))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}}\nabla g_i(x) \neq 0, \quad \forall \varepsilon \in (0,\widetilde{\varepsilon}]. \tag{2.4}$$

If subproblem (1.2) is feasible and B^k is positive definite, then it admits a unique solution d^k, and d^k is such a solution if and only if there exists a multiplier vector λ^k satisfying the following KKT conditions:

$$\begin{aligned}
&\nabla f(x^k) + \sum_{i=1}^{m} \lambda_i^k \nabla g_i(x^k) + B_k d^k = 0\\
&\lambda_i^k(\nabla g_i(x^k)^T d^k + g_i(x^k)) = 0, \quad \forall i \in I\\
&\lambda_i^k \geq 0, \quad \forall i \in I\\
&\nabla g_i(x^k)^T d^k + g_i(x^k) \leq 0, \quad \forall i \in I.
\end{aligned} \tag{2.5}$$

The following result can be obtained immediately.

Lemma 2.1 *For any positive definite matrix B^k, the pair (x^*, λ^*) is a KKT pair for problem (P) if and only if $(d^k, \lambda^k) = (0, \lambda^*)$ is a KKT pair for subproblem (1.1).*

Taking into account Assumption 3, we can obtain the following result.

Lemma 2.2 *(See [13]) Let x^k be a feasible point, and let the matrix B^k be positive definite. Then, $d(x, B)$ exists in a neighborhood of (x^k, B^k) and is continuous at (x^k, B^k), where $d(x, B)$ indicates the solution of the subproblem QP(x,B).*

Now we show that Algorithm 2.1 is well defined.

Lemma 2.3 *Algorithm 2.1 does not cycle between Step 3 and Step 4 infinitely.*

Proof. When

$$\mu^k \geq \max_{g_i(x^k)>0}\{\frac{\lambda_i^k((g_i(x))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}}{s(g_i(x))^{(s\alpha-1)}}\}, \tag{2.6}$$

it follows from (2.5) that we have

$$\begin{aligned}
\nabla F^s(x^k,\mu,\varepsilon)^T d^k &= \nabla f(x^k)^T d^k + u^k \sum_{i=1}^{m} \frac{s({g_i}^+(x^k))^{(s\alpha-1)}}{(({g_i}^+(x^k))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}} \nabla g_i(x^k)^T d^k \\
&= \nabla f(x^k)^T d^k + u^k \sum_{g_i(x^k)>0} \frac{s(g_i(x^k))^{(s\alpha-1)}}{((g_i(x^k))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}} \nabla g_i(x^k)^T d^k \\
&= -(d^k)^T B^k d^k - \sum_{i=1}^{m} \lambda_i^k \nabla g_i(x^k)^T d^k + u^k \sum_{g_i(x^k)>0} \frac{s(g_i(x^k))^{(s\alpha-1)}}{((g_i(x^k))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}} \nabla g_i(x^k)^T d^k \\
&\le -(d^k)^T B^k d^k - \sum_{g_i(x^k)>0} (\mu^k \frac{s(g_i(x^k))^{(s\alpha-1)}}{((g_i(x^k))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}} - \lambda_i^k) g_i(x^k) \\
&\le -(d^k)^T B^k d^k \\
&\le -\tfrac{1}{2}(d^k)^T B^k d^k.
\end{aligned}$$

The proof is completed.
By Assumption 4, we have

Lemma 2.4 *Algorithm 2.1 does not cycle between Step 5 and Step 6 infinitely.*

Lemma 2.5 *Algorithm 2.1 does not cycle between Step 7 and Step 8 infinitely.*

Proof. From the proof of Lemma 2.3, we have

$$\begin{aligned}
-\|\nabla F^s(x^k,\mu,\varepsilon^k)\|^2 &= -\|\nabla f(x^k)\|^2 - \mu^2 \|\frac{s(g_i(x^k))^{(s\alpha-1)}}{((g_i(x^k))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}} \nabla g_i(x^k)\|^2 \\
&- 2\mu \sum_{g_i(x^k)>0} \frac{s(g_i(x^k))^{(s\alpha-1)}}{((g_i(x^k))^{\alpha s}+\varepsilon^\alpha)^{1-\frac{1}{\alpha}}} \nabla g_i(x^k)^T \nabla f(x^k).
\end{aligned}$$

By Assumption 4 and lemma 2.4, we have that

$$\lim_{\mu\to+\infty} -\|\nabla F^s(x^k,\mu,\varepsilon^k)\|^2 = -\infty.$$

The proof is completed.

Lemma 2.6 *Algorithm 2.1 is well defined at Step 9.*

Proof. Suppose by contradiction that the condition (2.4) is not satisfied when Algorithm 2.1 reaches Step 9. Then there exist some $x^k, d^k, \mu^k, \varepsilon^k$ and $\alpha^n \to 0+$ such that the following inequality holds.

$$F^s(x^k+\alpha^n d^k,\mu^k,\varepsilon^k) > F^s(x^k,\mu^k,\varepsilon^k) + \sigma\alpha^n \nabla F^s(x^k,\mu^k,\varepsilon^k)^T d^k$$

Then we have

$$\frac{1}{\alpha^n}[F^s(x^k+\alpha^n d^k,\mu^k,\varepsilon^k) - F^s(x^k,\mu^k,\varepsilon^k)] - \sigma\nabla F^s(x^k,\mu^k,\varepsilon^k)^T d^k > 0$$

Since $\alpha^n \to 0+$, we obtain $(1-\sigma)\nabla F^s(x^k,\mu^k,\varepsilon^k)^T d^k \ge 0$. Noting that $\sigma \in (0,\frac{1}{2})$, we have $\nabla F^s(x^k,\mu^k,\varepsilon^k)^T d^k \ge 0$.

 On the other hand, if Algorithm 2.1 goes to Step 9 from Step 3, we have

$$\nabla F^s(x^k, \mu^k, \varepsilon^k)^T d^k \leq -\frac{1}{2}(d^k)^T B^k d^k < 0,$$

and if Algorithm 2.1 goes to Step 9 from Step 7, we also have

$$\nabla F^s(x^k, \mu^k, \varepsilon^k)^T d^k = -\|\nabla F^s(x^k, \mu^k, \varepsilon^k)\|^2 \leq -H(x^k) < 0,$$

either of which causes a contradiction. The proof is completed.

3. Global Convergence

In this section, we analyze the global convergence of the proposed algorithm. For this, we show that the penalty parameter μ keeps as a constant when k is sufficiently large.

Lemma 3.1 *There exists an iterate index k_0 satisfying that*

$$\mu^k = \mu^{k_0} = \overline{\mu}, \quad \forall k \geq k_0.$$

Proof. We know that the penalty parameter μ only increase at Step 4 and Step 8. It follows from Lemma 2.3 that Algorithm 2.1 will not go to Step 4 from Step 3 after a finite number of iterations, that is, the value of the penalty parameter μ is not changed between Step 3 and Step 4 after a finite number of iterations. Thus, it is sufficient to prove that Algorithm 2.1 will not go to Step 8 from Step 7 after a finite number of iterations.

Suppose by contradiction that Algorithm 2.1 goes to Step 8 from Step 7 infinitely. Then there exist subsequences $\{x^r\},\ \{\mu^r\},\ \{\varepsilon^r\},\ \{B^r\}$ generated by Algorithm 2.1, satisfying that

$$x^r \to x^*,\ \mu^r \to +\infty,\ \varepsilon^r \to 0,\ B^r \to B^*$$

and

$$-\|\nabla F^s(x^r, \mu^r, \varepsilon^r)\|^2 > -H(x^r). \tag{3.1}$$

For x^*, we have possible cases:

Case(i). x^* is infeasible. From the proof of Lemma 2.5, we have

$$\lim_{r \to +\infty} -\|\nabla F^s(x^r, \mu^r, \varepsilon^r)\|^2 = -\infty.$$

Since

$$\lim_{r \to +\infty} -H(x^r) = -H(x^*).$$

Step 8 is satisfied eventually. Thus, we have a contradiction to (3.1).

Case(ii). x^* is feasible. From lemma 3.2, $d(x^r, B^r)$ is well defined and continuous in a neighborhood of (x^*, B^*). Thus, when r is sufficiently large, subproblem $QP(x^r, B^r)$ is always feasible, i.e., Step7 and Step 8 cannot be reached. This contradicts that Algorithm 2.1 goes to Step 8 from Step 7 infinitely. This completes the proof.

In the sequel, we can prove that after a finite number of iterations, the solution d^k of subproblem $QP(x^k, B^k)$ always exists and can be selected as a search direction of the algorithm.

Lemma 3.2 *After a finite number of iterations, we have $d^k := d(x^k, B^k)$, i.e., Steps 5, 6, 7 and 8 are not reached.*

Proof. From Lemma 4.1. without loss of generality, we can assume that $\mu^k = \overline{\mu}$, $\forall k$. Suppose by contradiction that there exists a subsequence $\{x^r\}$ of $\{x^k\}$ satisfying that Algorithm 2.1 goes to Step 5 from Step 2 at x^r. It follows from Assumption 1 that x^r is a bounded sequence. So we can assume that $x^r \to x^*$. As ε^k is a monotonically decreasing sequence, by the construction of Algorithm 2.1, we have

$$\begin{aligned} & F^s(x^0,\overline{\mu},\varepsilon^0) \geq F^s(x^0,\overline{\mu},\varepsilon^1) \geq F^s(x^1,\overline{\mu},\varepsilon^1) \geq \cdots \geq F^s(x^{k-1},\overline{\mu},\varepsilon^{k-1}) \\ \geq \quad & F^s(x^{k-1},\overline{\mu},\varepsilon^k) \geq F^s(x^k,\overline{\mu},\varepsilon^k) \geq \cdots . \end{aligned} \tag{3.2}$$

Therefore, $\{F^s(x^k,\overline{\mu},\varepsilon^k)\}$ is monotonically decreasing. It follows from Assumption 1 that $\{F^s(x^k,\overline{\mu},\varepsilon^k)\}$ is a bounded sequence. Thus it is convergent and

$$F^s(x^k,\overline{\mu},\varepsilon^k) - F^s(x^{k+1},\overline{\mu},\varepsilon^{k+1}) \to 0. \tag{3.3}$$

By (3.2) and (3.3), we have

$$F^s(x^k,\overline{\mu},\varepsilon^k) - F^s(x^{k+1},\overline{\mu},\varepsilon^k) \to 0.$$

It follows from Step 9 and $d^r = -\|\nabla F^k(x^r,\overline{\mu},\varepsilon^r)\|$ that

$$\alpha^r \|\nabla F^s(x^r,\overline{\mu},\varepsilon^r)\|^2 \to 0.$$

Now we consider two cases.

Case(i). $\|\nabla F^s(x^r,\overline{\mu},\varepsilon^r)\|^2 \to 0$. From the test of Step 7, we have

$$\lim_{r\to+\infty} H(x^r) \to 0.$$

Taking into account the definition of $H(\cdot)$, we know that x^* is feasible. Following Case(ii) in Lemma 3.1, we conclude that Algorithm 2.1 will not go to Step 5 from Step 2, which is a contradiction.

Case(ii) $\alpha^r \to 0$. By assumption, Algorithm 2.1 goes to Step 5 from Step 2 infinitely. It follows from $d^r = -\nabla F^s(x^r,\overline{\mu},\varepsilon^r)$, Lemma 3.1 and the construction of Step 9 that $\frac{\alpha^r}{\theta}$ does not satisfy (2.4). Thus, we get

$$F^s(x^k - \tfrac{\alpha^k}{\theta}\nabla F^s(x^k,\overline{\mu},\varepsilon^k),\overline{\mu},\varepsilon^k) > F^s(x^k,\overline{\mu},\varepsilon^k) - \sigma\tfrac{\alpha^k}{\theta}\|\nabla F^s(x^k,\overline{\mu},\varepsilon^k)\|^2. \tag{3.4}$$

By the continuous differentiability of $F^k(x,\mu,\varepsilon)$, we have

$$F^s(x^r - \tfrac{\alpha^r}{\theta}\nabla F^s(x^r,\overline{\mu},\varepsilon^r),\overline{\mu},\varepsilon^r) = F^s(x^r,\overline{\mu},\varepsilon^r) - \tfrac{\alpha^r}{\theta}\|\nabla F^s(x^r,\overline{\mu},\varepsilon^r)\|^2 + o(\alpha^r). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$-(1-\sigma)\tfrac{1}{\theta}\|\nabla F^s(x^r,\overline{\mu},\varepsilon^r)\|^2 + \tfrac{o(\alpha^r)}{\alpha^r} > 0. \tag{3.6}$$

From (3.6), $\theta > 0$ and $\sigma \in (0,\frac{1}{2})$, we have

$$\|\nabla F^s(x^r,\overline{\mu},\varepsilon^r)\|^2 \to 0, \ \textit{when } \alpha^r \to 0.$$

By a similar proof of case(i), we also have that Steps 5, 6, 7 and 8 are not reached. Thus, we get a contradiction. The proof is completed.

Now, we are read to prove the global convergence property of Algorithm 2.1.

Theorem 3.1 *Every limit point of the sequence $\{x^k\}$ generated by Algorithm 2.1 is a KKT point of problem (P).*

Proof. From Lemma 4.1 and Lemma 4.2, without loss of generality, we can assume that $\mu^k = \overline{\mu},\ d^k = d(x^k, B^k),\ \forall k$. First we prove that

$$\lim_{k\to+\infty} \|d^k\| = 0. \tag{3.7}$$

As $\{\varepsilon^k\}$ is a monotonically decreasing sequence, by the construction of Algorithm 2.1, we have

$$\begin{array}{ll} & F^s(x^0,\overline{\mu},\varepsilon^0) \geq F^s(x^0,\overline{\mu},\varepsilon^1) \geq F^s(x^1,\overline{\mu},\varepsilon^1) \geq \cdots \geq F^s(x^{k-1},\overline{\mu},\varepsilon^{k-1}) \\ \geq & F^s(x^{k-1},\overline{\mu},\varepsilon^k) \geq F^s(x^k,\overline{\mu},\varepsilon^k) \geq \cdots. \end{array} \tag{3.8}$$

Therefore, $\{F^s(x^k,\overline{\mu},\varepsilon^k)\}$ is monotonically decreasing. It follows from Assumption 1 that $\{F^s(x^k,\overline{\mu},\varepsilon^k)\}$ is a bounded sequence. Thus it is convergent and

$$F^s(x^k,\overline{\mu},\varepsilon^k) - F^s(x^{k+1},\overline{\mu},\varepsilon^{k+1}) \to 0. \tag{3.9}$$

By (3.8) and (3.9), we have

$$F^s(x^k,\overline{\mu},\varepsilon^k) - F^s(x^{k+1},\overline{\mu},\varepsilon^k) \to 0.$$

Thus, from Step 9, we have

$$\alpha^k \nabla F^s(x^k,\overline{\mu},\varepsilon^k)^T d^k \to 0. \tag{3.10}$$

Suppose by contradiction that there exists a subsequence relabelled $\{x^k\}$ again, satisfying that

$$\|d^k\| \geq \delta > 0,\ \forall k. \tag{3.11}$$

Then, by Step 3, (3.11) and (3.12) and Assumption 2, we deduce that

$$\lim_{k\to\infty} \alpha^k = 0. \tag{3.12}$$

From Assumption 1 and (3.11), we can assume without loss of generality that there exist $\overline{x}$ and $\overline{d} \neq 0$ such that $x^k \to \overline{x}$ and $d^k \to \overline{d}$. Then, by Step 3 and Assumption 2, we have

$$\lim_{k\to\infty} \nabla F^s(x^k,\overline{\mu},\varepsilon^k)^T d^k = \nabla F^s(\overline{x},\overline{\mu},0)^T \overline{d} < 0. \tag{3.13}$$

Now, by (3.12) and Step 9, we can write

$$F^s(x^k + \tfrac{\alpha^k}{\theta} d^k, \mu^k, \varepsilon^k) > F^s(x^k,\mu^k,\varepsilon^k) + \sigma \tfrac{\alpha^k}{\theta} \nabla F^s(x^k,\mu^k,\varepsilon^k). \tag{3.14}$$

By the mean theorem, we get from (3.14)

$$\nabla F^s(x^k + \nu \tfrac{\alpha^k}{\theta} d^k, \mu^k, \varepsilon^k) > \sigma \nabla F^s(x^k,\mu^k,\varepsilon^k), \tag{3.15}$$

where $\nu \in (0,1)$. Taking limits of (3.15) for $k \to \infty$, by (3.12) we obtain

$$(1-\sigma)\nabla F^s(\overline{x},\overline{\mu},0)^T \overline{d} \geq 0,$$

As $\sigma < 1$, by (3.13), we get a contradiction. Hence, (3.7) is proved. By (1.2), (3.7) implies that $\overline{x}$ is feasible, so that, (3.7), Proposition 2.1 and Proposition 2.2 imply the assertion.

4. Rate of convergence

In this section, we will analyze the convergent rate of the proposed algorithm. The main result of the analysis will be that, if a unit stepsize ensures superlinear convergence, then the (2.4) is eventually satisfied by $\alpha^k = 1$ so that the Maratos effect does not occur. To this end, we need the following Assumption. Let x^* be an accumulation point of the sequence $\{x^k\}$ generated by Algorithm 2.1 and λ^* be related multiplier. Then, it follows from Theorem 3.1 that (x^*, λ^*) is a pair for problem (P).

Assumption 5: Strong second-order sufficient condition holds at (x^*, λ^*), i.e., the Hessian $\nabla^2 L(x^*, \lambda^*)$ is positive definite on the space $\{u | \nabla g_i(x^*)^T u = 0,\ \forall i \in I_+(x^*)\}$, where $\nabla^2 L(x^*, \lambda^*) = \nabla^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 g_i(x^*)$ and $I_+(x^*) = \{i \in I_0(x^*) : \lambda_i^* > 0\}$.

Lemma 4.1 *Under the stated assumptions, the whole sequence $\{x^k\}$ convergence to x^*.*

Proof. Assumption 3 and 5 mean that x^* is an isolated accumulation point of $\{x^k\}$. The assertion then follows from [2] and (3.7).

To prove the main result of this section, we first recall the definitions of semismooth function and SC^1-function.

Definition 4.1 *(See [14])Let $F : R^n \to R^m$ be locally Lipschitz at $x \in R^n$. We say that F is semismooth at x if*

$$\lim_{H \in \partial F(x+t\nu'),\ \nu' \to \nu, t \downarrow 0} H\nu'$$

exists for any $\nu \in R^n$.

Definition 4.2 *(See [13])A function $H : R^n \to R$ is said to SC^1-function on an open set X if*
(a) h is continuously differentiable on X;
(b) ∇h is semismooth on X;

Since $F^s(x, \mu, \varepsilon)$ is twice continuously differentiable for any x, $F^s(x, \mu, \varepsilon)$ is SC^1-function. Then applying Theorem 3.2 in [12] and similar to the proof of Theorem 5.2 in [13], we deduce the next theorem.

Theorem 4.1 *Suppose that the stated assumptions hold. If*

$$\lim_{k \to \infty} \frac{\|x^k + d^k - x^*\|}{\|x^k - x^*\|} = 0,$$

then for all sufficiently large k, the unit stepsize is accepted by the Algorithm 2.1, that is, $x^{k+1} = x^k + d^k$.

5. Concluding remarks

By applying another smoothing approximation of a lower order penalty function and Facchinei's method of dealing with the inconsistency of subproblems in SQP methods, we propose a SQP

algorithm for nonlinear inequality constraints optimization problems. The presented algorithm incorporates automatic adjustment rules for the choice of parameters. The algorithm is proved to be globally convergent and superlinearly convergent under some mild conditions without the strict complementarity.

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On a new application of quasi power increasing sequences

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Abstract

In the present paper, a general theorem on $\mid \bar{N}, p_n; \delta \mid_k$ summability factors of infinite series has been proved under more weaker conditions. We have also obtained some results dealing with $\mid \bar{N}, p_n \mid_k$, $\mid C, 1 \mid_k$ and $\mid C, 1; \delta \mid_k$ summability factors.

1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad and \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{3}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [6], [9])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} \frac{\mid t_n^\alpha \mid^k}{n} < \infty. \tag{4}$$

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and it is said to be summable $\mid C,\alpha;\delta\mid_k$, $k\geq 1$ and $\delta\geq 0$, if (see [7])

$$\sum_{n=1}^{\infty} n^{\delta k-1}\mid t_n^{\alpha}\mid^k<\infty. \tag{5}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v\to\infty \quad as \quad n\to\infty, \quad (P_{-i}=p_{-i}=0, i\geq 1). \tag{6}$$

The sequence-to-sequence transformation

$$\sigma_n=\frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{7}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\mid \bar{N},p_n\mid_k$, $k\geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}\mid \Delta\sigma_{n-1}\mid^k<\infty, \tag{8}$$

and it is said to be summable $\mid \bar{N},p_n;\delta\mid_k$, $k\geq 1$ and $\delta\geq 0$, if (see [4])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1}\mid \Delta\sigma_{n-1}\mid^k<\infty, \tag{9}$$

where

$$\Delta\sigma_{n-1}=-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v, \quad n\geq 1. \tag{10}$$

In the special case $p_n=1$ for all values of n (resp. $\delta=0$) $\mid \bar{N},p_n;\delta\mid_k$ summability is the same as $\mid C,1;\delta\mid_k$ (resp. $\mid \bar{N},p_n\mid_k$) summability. Also, if we take $\delta=0$ and $k=1$, then we get $\mid \bar{N},p_n\mid$ summability. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n\leq b_n\leq Bc_n$ (see ([1]).We denote by $\mathcal{BV}_{\mathcal{O}}$ the expression $\mathcal{BV}\cap\mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}}$ and $\mathcal{BV}$ are the set of all null sequences and the set of all sequences with bounded variation, respectively.Concerning the $\mid \bar{N},p_n\mid_k$ summability factors, Bor [5] has recently proved the following theorem.

Theorem A. Let (X_n) be an almost increasing sequence and there be sequences (β_n) and (λ_n) such that

$$| \Delta\lambda_n | \leq \beta_n, \tag{11}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{13}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{14}$$

If

$$\sum_{v=1}^{n} \frac{| t_v |^k}{v} = O(X_n) \quad as \quad n \to \infty \tag{15}$$

, where (t_n) is the n-th (C,1) mean of the sequence (na_n), and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{16}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{17}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $| \bar{N}, p_n |_k, k \geq 1$.

2. The main result. The aim of this paper is to generalize Theorem A under more weaker conditions for $| \bar{N}, p_n; \delta |_k$ summability. Therefore we need the concept of quasi β-power increasing sequence. A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^{\beta}\gamma_n \geq m^{\beta}\gamma_m \tag{18}$$

holds for all $n \geq m \geq 1$). It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$.

Now we shall prove the following theorem.

Theorem. Let $(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}$ and (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$ and the sequences (β_n) and (λ_n) are such that conditions (11)-(17) of Theorem A are satisfied with the condition (15) replaced by:

$$\sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{| t_v |^k}{v} = O(X_n) \quad as \quad n \to \infty. \tag{19}$$

If

$$\sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}=O((\frac{P_v}{p_v})^{\delta k}\frac{1}{P_v}) \quad as \quad m\to\infty, \tag{20}$$

then the series $\sum_{n=1}^{\infty}a_n\frac{P_n\lambda_n}{np_n}$ is summable $\mid \bar{N},p_n;\delta\mid_k$, $k\geq 1$ and $0\leq\delta<1/k$.

It should be noted that if we take $\delta=0$ and (X_n) as an almost increasing sequence, then we get Theorem A. In this case the condition $(\lambda_n)\in\mathcal{BV}_{\mathcal{O}}$ is not needed and condition (20) reduces to

$$\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}=\sum_{n=v+1}^{m+1}\left(\frac{1}{P_{n-1}}-\frac{1}{P_n}\right)=O\left(\frac{1}{P_v}\right) \quad as \quad m\to\infty, \tag{21}$$

which always holds.

We require the following lemmas for the proof of our theorem.

Lemma 1 ([10]). Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold :

$$nX_n\beta_n=O(1), \tag{22}$$

$$\sum_{n=1}^{\infty}\beta_nX_n<\infty. \tag{23}$$

Lemma 2 ([3]). If the conditions (16) and (17) are satisfied, then $\Delta(P_n/p_nn^2)=O(1/n^2)$.

3. Proof of the Theorem. Let (T_n) be the sequence of $(\bar{N},p_n)$ mean of the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$. Then, by definition, we have

$$T_n=\frac{1}{P_n}\sum_{v=1}^{n}p_v\sum_{r=1}^{v}\frac{a_rP_r\lambda_r}{rp_r}=\frac{1}{P_n}\sum_{v=1}^{n}(P_n-P_{v-1})\frac{a_vP_v\lambda_v}{vp_v}. \tag{24}$$

Then, for $n\geq 1$

$$\begin{aligned}T_n-T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}P_va_v\lambda_v}{vp_v}\\ &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}P_va_v\lambda_v}{v^2p_v}.\end{aligned}$$

Using Abel's transformation, we get

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \Delta \left(\frac{P_{v-1} P_v \lambda_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{\lambda_n}{n^2} \sum_{v=1}^{n} v a_v \\
&= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{P_v}{p_v} (v+1) t_v p_v \frac{\lambda_v}{v^2} \\
&+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v (v+1) \frac{t_v}{v^2 p_v} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} (v+1) t_v \Delta (P_v / v^2 p_v) \\
&+ \lambda_n t_n (n+1)/n^2 \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say.
\end{aligned}$$

To complete the proof of the Theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{25}$$

Now, applying Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{\delta k + k - 1} \mid T_{n,1} \mid^k &= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \mid t_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k - 1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v} \right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \\
&\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k p_v \mid t_v \mid^k \mid \lambda_v \mid^k \frac{1}{v^k} \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k - 1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^k \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid p_v \mid t_v \mid^k \frac{1}{v^k} \frac{1}{P_v} (\frac{P_v}{p_v})^{\delta k} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v} \right)^{k-1} \mid \lambda_v \mid\mid t_v \mid^k \frac{1}{v^k} (\frac{P_v}{p_v})^{\delta k} \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} v^{k-1} \frac{1}{v^k} \mid \lambda_v \mid\mid t_v \mid^k \\
&= O(1) \sum_{v=1}^{m} \mid \lambda_v \mid (\frac{P_v}{p_v})^{\delta k} \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta \mid \lambda_v \mid \sum_{r=1}^{v} (\frac{P_r}{p_r})^{\delta k} \frac{\mid t_r \mid^k}{r}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad O(1) \mid \lambda_m \mid \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \frac{\mid t_v \mid^k}{v} \\
&= \quad O(1) \sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) \mid \lambda_m \mid X_m \\
&= \quad O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) \mid \lambda_m \mid X_m = O(1)
\end{aligned}
$$

as $m \to \infty$, by (11), (14), (19), and (23). Now ,using the fact that $(P_v/v) = O(p_v)$ by (16), we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k \ (\beta_v)^k \mid t_v \mid^k p_v \\
&\times \quad \left\{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \ (\beta_v)^k \mid t_v \mid^k p_v \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \left(\frac{P_v}{p_v}\right)^{k-1} \ (v\beta_v)^{k-1} v\beta_v \frac{1}{v^k} \mid t_v \mid^k \\
&= = O(1) \sum_{v=1}^{m} v\beta_v (\frac{P_v}{p_v})^{\delta k} \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} (\frac{P_r}{p_r})^{\delta k} \frac{\mid t_r \mid^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} \frac{\mid t_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by (11), (13), (16), (19), (20), (22) and (23). Now, since $\Delta(\frac{P_v}{v^2 p_v}) = O(\frac{1}{v^2})$ by Lemma 2, we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid T_{n,3} \mid^k &= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{\sum_{v=1}^{n-1} P_v \mid \lambda_{v+1} \mid\mid t_v \mid \frac{1}{v} \frac{v+1}{v}\right\}^k \\
&= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{\sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \mid \lambda_{v+1} \mid \frac{1}{v} \mid t_v \mid\right\}^k \\
&= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k p_v \frac{1}{v^k} \mid \lambda_{v+1} \mid^k \mid t_v \mid^k
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1} \\
&= \quad O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k} p_v \frac{1}{v^k} \mid \lambda_{v+1}\mid^{k-1} \mid \lambda_{v+1}\mid\mid t_v\mid^{k} \sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}} \\
&= \quad O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} \frac{1}{v^k} \mid \lambda_{v+1}\mid\mid t_v\mid^{k} (\frac{P_v}{p_v})^{\delta k} \\
&= \quad O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k} v^{k-1}\frac{1}{v^k} \mid \lambda_{v+1}\mid\mid t_v\mid^{k} \\
&= \quad O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k} \mid \lambda_{v+1}\mid \frac{\mid t_v\mid^{k}}{v} \\
&= \quad O(1)\sum_{v=1}^{m-1}\Delta \mid \lambda_{v+1}\mid \sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k}\frac{\mid t_r\mid^{k}}{r} + O(1)\mid \lambda_{m+1}\mid \sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v\mid^{k}}{v} \\
&= \quad O(1)\sum_{v=1}^{m-1}\mid \Delta\lambda_{v+1}\mid X_v + O(1)\mid \lambda_{m+1}\mid X_m \\
&= \quad O(1)\sum_{v=1}^{m-1}\mid \Delta\lambda_{v+1}\mid X_{v+1} + O(1)\mid \lambda_{m+1}\mid X_{m+1} \\
&= \quad O(1)\sum_{v=2}^{m}\mid \Delta\lambda_{v}\mid X_{v} + O(1)\mid \lambda_{m+1}\mid X_{m+1} \\
&= \quad O(1)\sum_{v=1}^{m}\beta_v X_v + O(1)\mid \lambda_{m+1}\mid X_{m+1} = O(1)
\end{aligned}
$$

as $m \to \infty$, by (11), (14), (16), (19), and (20). Finally, as in $T_{n,3}$, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid T_{n,4}\mid^{k} &= O(1)\sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{n+1}{n}\right)^{k}\frac{1}{n^k}\mid \lambda_n\mid^{k}\mid t_n\mid^{k} \\
&= O(1)\sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k} n^{k-1}\frac{1}{n^k}\mid \lambda_n\mid^{k-1}\mid \lambda_n\mid\mid t_n\mid^{k} \\
&= O(1)\sum_{n=1}^{m}\mid \lambda_n\mid (\frac{P_n}{p_n})^{\delta k}\frac{\mid t_n\mid^{k}}{n} = O(1) \quad as \quad m\to\infty.
\end{aligned}
$$

Therefore, we get that

$$
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\mid T_{n,r}\mid^{k} = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.
$$

This completes the proof of the Theorem. If we take $\delta = 0$, then we get a result for $\mid \bar{N}, p_n\mid_k$ summability factors. Also if we take $p_n = 1$ for all values of n, then we obtain

a new result dealing with $\mid C,1;\delta \mid_k$ summability factors. Furthermore , if we take $p_n = 1$ for all values of n and $\delta = 0$, then we get another new result concerning the $\mid C,1 \mid_k$ summability factors.

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On a simple criteria of convexity of order α for meromorphic functions

Adriana Cătaş

Abstract

The aim of the paper is to provide sufficient conditions for meromorphic functions defined in the punctured disc, $\dot{U} = U \setminus \{0\}$, to be convex functions of order α. The present work is based on some results involving differential subordinations.

Key words: Convex functions, differential subordination, meromorphic functions.

AMS Subject Classification: 30C45.

1 Introduction and preliminaries

For integer $n \geq 0$, denote by Σ_n the class of meromorphic functions, defined in the punctured disc

$$\dot{U} = \{z \in C : \ 0 < |z| < 1\} = U \setminus \{0\}$$

which are of the form

$$f(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + \dots \tag{1.1}$$

and let $\Sigma = \Sigma_0$.

A function $f \in \Sigma$ is said to be starlike if it is univalent and the complement of $f(\dot{U})$ is starlike with respect to the origin. Denote by Σ^* the class of such functions. A function $f \in \Sigma_n$ is said to be in the class $\Lambda(\alpha)$ of meromorphic convex functions of order α in $\dot{U}$ if and only if

$$\text{Re } \left[-\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \alpha, \quad z \in \dot{U},\ 0 \le \alpha < 1. \tag{1.2}$$

The following definitions and lemma will be used in the next section.

Let $\mathcal{H}(U)$ denote the space of analytic functions in U. For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}_n = \{f \in \mathcal{H}(U):\ f(z) = a_n z^n + a_{n+1}z^{n+1} + \dots\} \tag{1.3}$$

and

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U):\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots\}. \tag{1.4}$$

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 1.1 [2] *Let m be a positive integer and let α be real, with $0 \le \alpha < m$. Let $q \in \mathcal{H}(U)$, with $q(0) = 0$, $q'(0) \ne 0$ and*

$$\text{Re } \left(1+\frac{zq''(z)}{q'(z)}\right) > \frac{\alpha}{m}. \tag{1.5}$$

Define the function h as

$$h(z) = mzq'(z) - \alpha q(z). \tag{1.6}$$

If $p \in \mathcal{H}_m$ and

$$zp'(z) - \alpha p(z) \prec h(z), \tag{1.7}$$

then $p(z) \prec q(z)$ and this result is sharp.

2 Main results

Theorem 2.1 *If $f \in \Sigma_n$, $n \in \mathbb{N}^*$, with $-1 \leq \alpha < n$ satisfies the condition*

$$|(1-\alpha)z^2 f'(z) + z^3 f''(z) - \alpha - 1| < M, \tag{2.1}$$

then

$$|z^2 f'(z) + 1| < \frac{M}{n-\alpha} \tag{2.2}$$

and this result is sharp.

Proof. If we let

$$p(z) = z^2 f'(z) + 1$$

then $p \in \mathcal{H}_{n+1}$ and (2.1) can be rewritten as

$$|zp'(z) - (\alpha+1)p(z)| < M \tag{2.3}$$

or

$$zp'(z) - (\alpha+1)p(z) \prec Mz. \tag{2.4}$$

If we take

$$q(z) = \frac{Mz}{n-\alpha}, \quad q \in \mathcal{H}(U)$$

with $q(0) = 0$, $q'(0) \neq 0$ and

$$\text{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \frac{\alpha+1}{n+1}$$

then from (1.6), $h(z) = Mz$ and the result follows from Lemma 1.1, that is $p(z) \prec q(z)$

$$|z^2 f'(z) + 1| < \frac{M}{n-\alpha}.$$

□

By applying our previous result we can obtain a simple criterion for the convexity of order α of a meromorphic function.

Theorem 2.2 *Let $n \in \mathbb{N}^*$, let $\alpha \in [0,1)$ and let*

$$M_n(\alpha) = \frac{(1-\alpha)^2(n-\alpha)}{2\alpha(1-\alpha) + \sqrt{(\alpha+1)^2 + (n-\alpha)^2}}. \tag{2.5}$$

If $f \in \Sigma_n$ satisfies the condition

$$|(1-\alpha)z^2 f'(z) + z^3 f''(z) - \alpha - 1| < M_n(\alpha), \quad z \in U,$$

then $f \in \Lambda(\alpha)$.

Proof. Let

$$0 < M \leq M_n(\alpha), \tag{2.6}$$

where $M_n(\alpha)$ is given by (2.5), and suppose that $f \in \Sigma_n$ satisfies

$$|(1-\alpha)z^2 f'(z) + z^3 f''(z) - \alpha - 1| < M, \quad z \in U. \tag{2.7}$$

If we set $P(z) = z^2 f'(z) + 1$, then by Theorem 2.1 we obtain

$$|P(z)| < \frac{M}{n-\alpha} \equiv R, \quad z \in U. \tag{2.8}$$

Hence if we let

$$p(z) = -\left(\alpha + 1 + \frac{zf''(z)}{f'(z)}\right) \tag{2.9}$$

then $p(z) \in \mathcal{H}[1-\alpha, n+1]$ and (2.7) can be written in the form

$$|p(z)(-P(z)+1) - 2\alpha P(z) + \alpha - 1| < M. \tag{2.10}$$

We claim that this inequality implies Re $p(z) > 0$, $z \in U$. If this is false, then there exists a point $z_0 \in U$, such that $p(z_0) = i\rho$, where ρ is real. We will show that at such a point the negation of condition (2.10) holds, that is

$$|i\rho(1 - P(z_0)) + 2\alpha P(z_0) + \alpha - 1| \geq M, \tag{2.11}$$

for all real ρ.

If we let $P_0 = P(z_0)$, we have

$$|i\rho(P_0-1)+2\alpha P_0+1-\alpha|^2 = \rho^2|P_0-1|^2+|2\alpha P_0+1-\alpha|^2-2\rho(\alpha+1)\text{Im } P_0.$$

Hence inequality (2.11) is equivalent to

$$\rho^2|P_0-1|^2+|2\alpha P_0+1-\alpha|^2-2\rho(\alpha+1)\text{Im } P_0 - M^2 \geq 0. \tag{2.12}$$

The above inequality holds if and only if

$$(\alpha+1)^2(\text{Im } P_0)^2 \leq |P_0-1|^2(|2\alpha P_0+1-\alpha|^2-M^2). \tag{2.13}$$

Since from (2.8) we have

$$|2\alpha P_0+1-\alpha| > 1-\alpha-2\alpha R \tag{2.14}$$

and

$$\frac{(1+\alpha)^2(\text{Im } P_0)^2}{|P_0-1|^2} < (\alpha+1)^2R^2 \tag{2.15}$$

by using (2.14) and (2.15) we deduce that inequality (2.13) holds if

$$\frac{(\alpha+1)^2(\text{Im } P_0)^2}{|P_0-1|^2} < (\alpha+1)^2R^2 \leq (1-\alpha-2\alpha R)^2 - R^2(n-\alpha)^2 \tag{2.16}$$

which yields

$$R^2[(\alpha+1)^2-4\alpha^2+(n-\alpha)^2]+4\alpha(1-\alpha)R-(1-\alpha)^2 \leq 0. \tag{2.17}$$

Because of the definition of M given in (2.6) this forces inequalities (2.17), (2.13) and (2.11) to hold. Thus we have a contradiction of (2.10). Therefore, Re $p(z) > 0$, that is

$$\text{Re } \left[-\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \alpha$$

and we obtain the desired result. □

We were applied the same techniques used in [3].

Corollary 2.1 *If $f \in \Sigma_n$, $n \in \mathbb{N}^*$ satisfies the condition*

$$|z^2 f'(z) + z^3 f''(z) - 1| < \frac{n}{\sqrt{n^2+1}} \tag{2.18}$$

then f is a meromorphic convex function.

Theorem 2.2 can be written in the following equivalent form, that is useful for the other results.

Theorem 2.3 *Let $f \in \Sigma_n$, $n \in \mathbb{N}^*$, have the form*

$$f(z) = \frac{1}{z} + g(z), \quad 0 < |z| < 1 \tag{2.19}$$

where $g \in \mathcal{H}_n$. If $\alpha \in [0,1)$ and

$$|(1-\alpha)z^2 g'(z) + z^3 g''(z)| < \frac{(1-\alpha)^2(n-\alpha)}{2\alpha(1-\alpha) + \sqrt{(\alpha+1)^2 + (n-\alpha)^2}} \tag{2.20}$$

then $f \in \Lambda(\alpha)$.

Example 2.1 For the Theorem 2.3 we consider the following function

$$f(z) = \frac{1}{z} + \lambda(1 - \cos z). \tag{2.21}$$

In this case $g \in \mathcal{H}_2$ and for $\alpha = \frac{1}{2}$ we get

$$\left| \frac{1}{2} z^2 g'(z) + z^3 g''(z) \right| < |\lambda| \frac{3e^2+1}{8e}.$$

Hence, by Theorem 2.3 if

$$|\lambda| < \frac{6e}{(3e^2+1)(1+3\sqrt{2})} = 0.1342\ldots$$

then the function

$$f(z) = \frac{1}{z} + \frac{1}{8}(1 - \cos z)$$

is a meromorphic convex function of order 1/2 that is, $f \in \Lambda(\frac{1}{2})$.

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The Symbol Series Expression and Hölder Exponent Estimates of Fractal Interpolation Function

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Abstract

Now fractal interpolation functions (FIFs) are mainly described in terms of concrete interpolation region and its partition form. Many theory and applications only deal with low dimensional interpolation. In this paper, we consider a high dimensional fractal interpolation problem and describe a generalized definition of FIFs based on multi-scale partition and refinable set. This definition enables us to investigate the expression with symbol series of FIFs under more general setting. By applying the expression with series of FIFs, we discuss Lipschitz continuity of FIFs and give the estimates of *Hö*lder exponent of FIFs. The obtained results can be easily applied to concrete fractal interpolation problems.

keyword fractal interpolation function, multiscale partition, refinable set, Lipschitz continuity, *Hölder* exponent.

1 Introduction

Fractal interpolation methods introduced by Barnsley [1] are a relatively new techniques for data imitation which has generated much interest in both theoretical and applied mathematics over the past decade. FIFs are attractors of iterated function system (IFS) of a special form, the main difference from traditional interpolation functions consists in the definition of a function relation assuming a self-similarity on small scales[1,2,3]. For the applications of FIFs theory, the properties of FIFs are usually considered, such as the expression with series in terms of a suitable function system, smoothness and estimates of *Hölder* exponent, etc.

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Since FIFs are continuous, in general nowhere differentiable and self-similar, their analysis can not be done satisfactorily by restricting to classical analytic tools. This leads to the interest in the properties of FIFs growing considerably. Sha[4] gave a series representation of self-affine FIFs through a special function system and studied the *Hölder* properties of FIFs. Sha and Chen[5] expanded equidistant FIFs on [0, 1] by using Haar-wavelet function system and obtained their global *Hölder* property, in which the number of interpolation points is N = 2p + 1, p being a definite positive integer. Hardin, Kessler and Massopust[6] showed that certain classes of fractal interpolation functions generate a multi-resolution analysis of $L^2(R)$. Subsequently, construction methods of orthogonal wavelets based on fractal functions were provided[7,8], in which the Lipschitz continuity and the estimates of *Hölder* exponent were discussed. Lévy-Véhel, Daoudi and Lutton[9] considered the problem of speech modelling with the aid of FIFs, the idea of matching *Hölder* singularities of the FIF to those of the original series was employed. Sebastian, Navascues and Valdizan[10] proposed an explicit formulae to compute the fractal dimension of experimental recordings on the brain cortex results by means of fractal interpolation, and discussed two and three-dimensional brain mapping representations. See [11-16] for more relative references.

The main purpose of this paper is to describe a high dimensional fractal interpolation problem, and to give its expression with symbol series and estimates of *Hölder* exponent by using two relatively new nations: refinable set and partition. These results, presented under general setting, provide a simple method for analyzing properties of fractal interpolation functions. We proceed as follows: In Section 2, we describe a high dimensional interpolation problem based on multiscale partitions and refinable set. Then in Section 3, we give the definition and the expression with symbol series of FIFs belonging to rather general class. In Section 4, the Lipschitz continuity and the estimates of *Hölder* exponent are discussed. Finally, in Section 5 we apply our results to concrete fractal interpolation functions.

2 Multiscale partition and description of a high dimensional interpolation problem

In this section, we will describe the method we use to generate a multiscale partition of an invariant set $\Omega \in R^d (d \in N)$, and give the description of a high dimensional interpolation problem.

We start with a positive integer $\mu \geq 1$ and a family $W := \{w_e : e \in Z_\mu\}$ of contractive mappings on R^d, where $Z_\mu = \{0, 1, ..., \mu - 1\}$. There exists a unique compact subset Ω of R^d such that

$$\Omega = W(\Omega) := \bigcup_{e \in Z_\mu} w_e(\Omega). \tag{2.1}$$

This set Ω is called the invariant set associated with the family of mappings W(see [17]). Generally, it has a complex fractal structure. For example, there are choices of W for which Ω is the Cantor subset of [0,1], the Sierpinski gasket contained in an equilateral triangle or the twin dragons form wavelet analysis. In this paper, we are interested in the cases when Ω is usually convex polygonal region which has a simple structure, for example, the cube and simplex in R^d. With these cases in mind, we make the following additional restriction on the family of mapping W

(a) $\forall e \in Z_e$, the mapping w_e has a continuous inverse on Ω.

(b) The region Ω has non-empty interior and

$$meas(w_e(\Omega) \cap w_{e'}(\Omega)) = 0,\ e \neq e',\ \forall e,\ e' \in Z_\mu, \tag{2.2}$$

where meas denotes Lebesgue measure on R^d. Let

$$\Omega_1 = \{\Omega_{1,e} : \Omega_{1,e} = w_e(\Omega),\ e \in Z_\mu\}, \tag{2.3}$$

from equation (2.1) and conditions (a) and (b), Ω_1 forms a partition of region Ω.

Now we use W to obtain a more general partition of Ω in the following way. Given any $e = (e_0, ..., e_{n-1}) \in Z_\mu^n = Z_\mu \times Z_\mu \times \cdots Z_\mu$, n times , we define the mappings

$$w_e = w_{e_0} \circ \circ w_{e_{n-1}}, \tag{2.4}$$

let

$$\Omega_n = \{\Omega_{n,e} : \Omega_{n,e} = w_e(\Omega),\ e \in Z_\mu^n\},\ n \in N, \tag{2.5}$$

then $\{\Omega_n\}$ forms a multiscale partition of Ω for any $n \in N$. For a more detailed presentation of multiscale partition we refer to Chen et.al. [18].

To describe high dimensional interpolation problem, we introduce the definition of refinable set.

Definition 2.1 A subset V_0 of Ω is said to be refinable relative to the mapping W if $V_0 \subset W(V_0)$.

Let V_0 be a nonempty refinable subset of Ω relative to the contractive mapping W, we have([19])

(1) Let $W^k := \{w_{e_i} : e_i \in Z_\mu^k\}$, then $W^k(V_0)$ is also refinable set relative to the contractive mapping W for $k \in N$.

(2) Let

$$V_i = W(V_{i-1}),\ i \in N, \tag{2.6}$$

then

$$V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots, \tag{2.7}$$

and

$$\Omega = \overline{\bigcup_{i \in N_0} V_i}, \tag{2.8}$$

where N_0 denotes the set of nonnegative integer.

In traditional interpolation method, interpolation conditions are given based on partition knots. Now we give the definition of interpolation knots under multiscale partition.

Definition 2.2 Let V_0 be the vertex set of convex polygonal region Ω. If V_0 is refinable set relative to the mapping W, then $V_k = W^k(V_0)$ is said to be partition knots of Ω_k, where $k \in N$.

We note that Definition 2.2 is the natural generalization of the description of interpolation knots in traditional interpolation methods. To illustrate this point, we give an example.

Let $\Omega \subset R^2$ be the triangle with vertices at

$$A_0 = (0,\, 0),\ A_1 = (1,\, 0),\ A_2 = (0,\, 1).$$

Considering the contractive mapping $W = \{w_e : e \in Z_4\}$, where w_e is defined as follows

$$w_e(x) = \frac{1}{2}(y_e + (-1)^{\tau(e)}x),\ e \in Z_4,\ x \in R^2, \tag{2.9}$$

where $\tau(e) = 0, y_e = A_e, e \in Z_3,\ \tau(3) = 1, y_3 = (1,1)$. The invariant subset of R^2 relative to the mapping W is the triangle Ω, $V_0 = \{A_0,\, A_1,\, A_2\}$ is a refinable set, and $V_1 = W(V_0) = \{A_i : i = 0, 1, \cdots, 5\}$ form knots of the partition Ω_1, where $A_3 = (1/2, 0), A_4 = (1/2, 1/2), A_5 = (0, 1/2)$.

Now we give the description of a high dimension interpolation problem.

For some positive integer k, let V_k be partition knots satisfying Definition 2.2. Given function values $\{Y_{P_i} : P_i \in V_k\}$, we want to seek a continuous function f(x) such that

$$f(P_i) = Y_{P_i},\ P_i \in V_k. \tag{2.10}$$

Differing from traditional interpolation methods, FIF is defined as the fixed point of an iterated function system(IFS), one can adjust shape and dimension of FIFs by changing attractive factors. For simplicity, we will consider interpolation problem corresponding to the partition Ω_1.

3 Fractal interpolation function and its expression with symbol series

In the rest of this paper, we always assume that polygonal interpolation region Ω is the invariant set associated with the family of mappings W, conditions (a) and (b) are satisfied. Let B(Ω) denote the Banach space of bounded real-valued functions on Ω with $\infty-$norm, $\beta = \otimes_{j=0}^{\mu-1} B(\Omega)$, and $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_{\mu-1}) \in \beta$. Let $v_i : \Omega \times R \to R$ be defined as follows

$$v_i(x,\, y) = \lambda_i(x) + s_i y,\ i \in Z_\mu, \tag{3.1}$$

it will always be assumed that $s = \max\{|s_i|\} < 1$. Define $T : B(\Omega) \to B(\Omega)$ by

$$(Tf)(x) = v_i\left(w_i^{-1}(x),\ f(w_i^{-1}(x))\right),\ x \in \Omega_{1,i},\ i \in Z_\mu, \tag{3.2}$$

then T is a contractive mapping on $B(\Omega)$, and there exists a unique attractive fixed point $f_\lambda(x) \in B(\Omega)$. In general, G=graph $f_\lambda(x)$ is typically a fractal set in $R^{(d+1)}$ made up of images of itself.

In the event that $f_\lambda(x)$ should be well-defined on the intersection of $\Omega_{1,i}$ and $\Omega_{1,j}$, in other words, one should impose additional 'join-up' conditions on these faces such that $f_\lambda(x)$ is continuous. This is an interesting work in the literature of FIFs, as we can find numerous papers dealing with this subject. For example, in [20] a concise fractal iteration form based on the triangle partitions was proposed, which generates continuous three-dimensional fractal interpolation function. See [1-2,4, 12, 21-22] for more references.

If fixed point $f_\lambda(x)$ of (3.2) is continuous, and satisfies interpolation condition (2.10), then $f_\lambda(x)$ is called a fractal interpolation function.

For simplicity, in this paper we always assume that $f_\lambda(x)$ is continuous, then $f_\lambda(x)$ satisfies

$$f_\lambda(x) = \lambda_i\left(w_i^{-1}(x)\right) + s_i f_\lambda\left(w_i^{-1}(x)\right),\ x \in \Omega_{1,i},\ i \in Z_\mu. \tag{3.3}$$

Now we consider the expression with symbol series of FIFs. Introduce a symbol sequence $\{\sigma_i : \sigma_i \in Z_\mu\}_{i=1}^{\infty}$. For any positive integer n, let

$$\sigma(n) = (\sigma_1, \cdots, \sigma_n) \in Z_\mu^n, \tag{3.4}$$

$$w_{\sigma(n)} = w_{\sigma_1} \circ \circ w_{\sigma_n}, \tag{3.5}$$

then multiscale partition Ω_n can be written as

$$\Omega_n := \{\Omega_{\sigma(n)} : \Omega_{\sigma(n)} = w_{\sigma(n)}(\Omega),\ \sigma(n) \in Z_\mu^n\},\ n \in N. \tag{3.6}$$

Proposition 3.1 $\forall x \in \Omega$, $n \in N$, there exists a symbol sequence $\sigma(n) = \{\sigma_1, \cdots, \sigma_n\}$ relative to oint x, such that

$$f_\lambda(x) = \sum_{k=1}^{n} S_{\sigma(k-1)}\lambda_{\sigma_k}\left(w_{\sigma(k)}^{-1}(x)\right) + S_{\sigma(n)} f_\lambda\left(w_{\sigma(n)}^{-1}(x)\right), \tag{3.7}$$

where

$$S_{\sigma(0)} = 1,\ S_{\sigma(n)} = \prod_{i=1}^{n} s_{\sigma_i}.$$

Proof. Suppose $x \in \Omega_{1,i}$. Let $\sigma_1 = i$, it follows from (3.3) that

$$f_\lambda(x) = S_{\sigma(0)}\lambda_{\sigma_1}\left(w_{\sigma(1)}^{-1}(x)\right) + S_{\sigma(1)} f_\lambda\left(w_{\sigma(1)}^{-1}(x)\right),$$

which shows that equation (3.7) holds for n=1. Assume that

$$f_\lambda(x) = \sum_{k=1}^{l} S_{\sigma(k-1)}\lambda_{\sigma_k}\left(w_{\sigma(k)}^{-1}(x)\right) + S_{\sigma(l)}f_\lambda\left(w_{\sigma(l)}^{-1}(x)\right). \tag{3.8}$$

According to the mathematical induction, we need only to prove that (3.7) holds for $n = l+1$. We decompose the last term of equation (3.8) by utilizing equation (3.3). Suppose $w_{\sigma(l)}^{-1}(x) \in \Omega_{1,j}$, and let $\sigma_{l+1} = j$, we have

$$\begin{aligned} f_\lambda(x) &= \sum_{k=1}^{l} S_{\sigma(k-1)}\lambda_{\sigma_k}\left(w_{\sigma(k)}^{-1}(x)\right) + S_{\sigma(l)}\lambda_{\sigma_{l+1}}\left(w_{\sigma(l+1)}^{-1}(x)\right) \\ &+ S_{\sigma(l)}s_{\sigma_{n+1}}f_\lambda\left(w_{\sigma(l+1)}^{-1}(x)\right) \\ &= \sum_{k=1}^{l+1} S_{\sigma(k-1)}\lambda_{\sigma_k}\left(w_{\sigma(k)}^{-1}(x)\right) + S_{\sigma(l+1)}f_\lambda\left(w_{\sigma(l+1)}^{-1}(x)\right), \end{aligned}$$

which shows that (3.7) holds for $n = l + 1$. Thus, we gain the conclusion of the Proposition 3.1.

We note that (3.7) is determined uniquely by real x when $x \notin \partial\Omega_n$. Otherwise, it follows from the continuity of $f_\lambda(x)$ that function values of $f_\lambda(x)$ corresponding to different expressions are equal.

Let $n \to \infty$, we find from (3.7) that

Corollary 3.1 $\forall x \in \Omega$, there exists a symbol sequence $\{\sigma_1, \cdots, \sigma_n, \cdots\}$ relative to real x, such that

$$f_\lambda(x) = \sum_{k=1}^{\infty} S_{\sigma(k-1)}\lambda_{\sigma_k}\left(w_{\sigma(k)}^{-1}(x)\right). \tag{3.9}$$

The obtained symbol series expressions of FIFs have different forms corresponding to different real x, which is very inconvenient for applications. To this end, we give the following theorem

Theorem 3.1 For any $x \in \Omega_{\sigma(n)}$, if x is interior node of $\Omega_{\sigma(n)}$, then there exists a unique sequence $\sigma(n)$, such that

$$f_\lambda(x) = \sum_{k=1}^{n} S_{\sigma(k-1)}\lambda_{\sigma_k}\left(w_{\sigma(k)}^{-1}(x)\right) + S_{\sigma(n)}f_\lambda\left(w_{\sigma(n)}^{-1}(x)\right), \tag{3.10}$$

Proof. According to Proposition 3.1, there exists a sequence $\sigma(n) = (\sigma_1, \cdots, \sigma_n)$, such that equation (3.7) holds. Let $w_{\sigma(n)}^{-1}(x) = \overline{x}$, then

$$x = w_{\sigma(n)}(\overline{x}) \in \Omega_{\sigma(n)},$$

and the above expression is determined uniquely by real x. Let $y \in \Omega_{\sigma(n)}$ be another interior node, similarly , there exists a sequence $\widetilde{\sigma}(n)$ such that

$$f_\lambda(y) = \sum_{k=1}^{n} S_{\widetilde{\sigma}(k-1)} \lambda_{\widetilde{\sigma_k}} \left(w_{\widetilde{\sigma}(k)}^{-1}(x) \right) + S_{\widetilde{\sigma}(n)} f_\lambda \left(w_{\widetilde{\sigma}(n)}^{-1}(x) \right).$$

Let $w_{\widetilde{\sigma}(n)}^{-1}(y) = \overline{y}$, then

$$y = w_{\sigma(n)}(\overline{y}) \in \Omega_{\widetilde{\sigma}(n)}.$$

If $\widetilde{\sigma}(n) \neq \sigma(n)$, we have $\Omega_{\widetilde{\sigma}(n)} \neq \Omega_{\sigma(n)}$, it is not consistent with that y is interior node of $\Omega_{\sigma(n)}$, such we gain $\widetilde{\sigma}(n) = \sigma(n)$.

4 The estimates of Hölder exponent of fractal interpolation functions

For the sake of simplicity, it will be assumed that $D(\Omega) = 1$, where D(A) represents the diameter of some set A, i.e.,

$$D(A) = \sup\{|x - y| : x, y \in A\},$$

with $|\cdot|$ denotes the Euclidean norm on the space R^d. For any d-dimensional vector $P = (p_i) \in R^d$ and $d \times d$ matrix $A = (a_{ij}) \in R^{d \times d}$, define norm $|\cdot|$ and $\|\cdot\|$ by

$$|P| = \left(\sum_{i=1}^{N} |p_i|^2 \right)^{\frac{1}{2}}, \ \|A\| = \sup_{|P|=1} |AP|.$$

In the section, we assume that W is a family of contractive affine mappings with the following form

$$w_i(x) = A_i x + B_i, \ x \in R^d, \ i \in Z_\mu,$$

where A_i is $d \times d$ matrix, B_i is d-dimensional vector. For given symbol sequence $\sigma(n) = (\sigma_1, \sigma_2, \cdots, \sigma_n)(\sigma_i \in Z_\mu)$, $A_{\sigma(n)}$ is defined by

$$A_{\sigma(n)} = A_{\sigma_1} A_{\sigma_2} \cdots A_{\sigma_n}.$$

Here we note that $\Omega_{\sigma(n)} \in \Omega_n$ is also convex polygonal region for any $n \in N$, with the assumption that W is a family of contractive affine mappings. Now we present one proposition, the conclusion of the proposition is well known.

Proposition 4.1 The sufficient and necessary condition for map $w_i(x)$ being contractive is

$$\|A_i\| < 1.$$

In this section, we also assume that $\lambda_i(x) \in C^1(\Omega)$ for $i \in Z_\mu$. Let

$$G(\lambda_i(x)) = \left(\frac{\partial \lambda_i}{\partial x_1}, \cdots, \frac{\partial \lambda_i}{\partial x_d}\right),$$

then there exists a positive constant M such that

$$\max_{i \in Z_\mu}\{\sup_{x \in \Omega} |G(\lambda_i(x))|\} \leq M. \tag{4.1}$$

Now we discuss Lipschitz continuity and the estimates of *Hölder* exponent of FIFs. At first, we recall relevant basic concepts.

A function f defined on Ω is said to be Lipscitz continuous if for some constant C, there holds the inequality

$$|f(x) - f(y)| \leq c|x - y|, \forall x, y \in \Omega.$$

The smallest possible constant in the above inequality is called the Lipscitz constant of f. More generally, a function f is said to be *Hölder* continuous with exponent $\alpha \in (0, 1]$ if for some constant C,

$$|f(x) - f(y)| \leq C|x - y|^\alpha, \forall x, y \in \Omega.$$

The Hölder exponent of $f \in C(\Omega)$ at x is

$$h_x = \inf_{\varepsilon \to 0} \left\{ \frac{\log |f(x) - f(y)|}{\log |x - y|} : y \in B(x, \varepsilon) \right\},$$

and $h = \inf\{h_x : x \in \Omega\}$ is called *Hölder* exponent of f.

Theorem 4.1 Let $h > 0$ be a positive number, $\rho_i = |s_i| \|A_i^{-1}\|^h$. If $\rho = \max\{\rho_i : i \in Z_\mu\} < 1$, and

$$|\lambda_i(x) - \lambda_i(y)| \leq C_0|x - y|^h, x, y \in \Omega, i \in Z_\mu,$$

where C_0 is a positive constant. Then there exists constant $C > 0$, such that

$$|f_\lambda(x) - f_\lambda(y)| \leq C|x - y|^h, \forall x, y \in \Omega.$$

Proof. Let

$$F(x, y) = \begin{cases} \dfrac{f_\lambda(x) - f_\lambda(y)}{|x - y|^h}, & x, y \in \Omega, x \neq y, \\ 0, & x, y \in \Omega, x = y. \end{cases}$$

It is sufficient only to prove that $F(x, y)$ is bounded when $x - y \to 0$.

Form (3.3), we have that for $x - y \neq 0$

$$F(x, y) = \widetilde{\rho}_i(x, y) F(w_i^{-1}(x), w_i^{-1}(y)) + \psi_i(x, y),$$

where

$$\widetilde{\rho}_i(x,y)=\frac{s_i|w_i^{-1}(x-y)|^h}{|x-y|^h},$$

$$\psi_i(x,\,y)=\begin{cases}\dfrac{\lambda_i(w_i^{-1}(x)-\lambda_i(w_i^{-1}(y))}{|x-y|^h}, x\neq y,\\ 0,\, x=y\end{cases}$$

It follows conditions of theorem that $|\widetilde{\rho}_i(x,y)|\leq|s_i|\|A_i^{-1}\|^h\leq\rho<1$ and $|\psi_i(x,y)|\leq C_0\|A_i^{-1}\|^h$.

We construct iterated function system $\{R^{2d+1}:W_i,\,i\in Z_\mu\}$, where

$$W_i\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}A_ix+B_i\\A_iy+B_i\\\widetilde{\rho}_i(x,y)z+\psi_i(x,y)\end{pmatrix}.$$

It is easy to see that it is a hyperbolic IFS, and its attractor is compact set in R^{2d+1}. Since that $\{(x,y,F(x,y)):(x,y)\in\Omega\times\Omega\}$ is decided by the attractor of IFS and the attractor is a compact set, we have that $\lim\limits_{x-y\to0}F(x,y)$ is bounded. Thus conclusion of Theorem 4.1 holds.

Now we give two corollaries. Let $h=1$, we find from Theorem 4.1 that

Corollary 4.1 Given $|s_i|<1(i\in Z_\mu)$, let $r=\max\{|s_i|\|A_i^{-1}\|\}$. If $r<1$, then $f_\lambda(x)$ is Lipschitz continuous, i.e., there exists a constant $C>0$ such that

$$|f_\lambda(x)-f_\lambda(y)|\leq C|x-y|,\,\forall x,y\in\Omega. \tag{4.2}$$

Corollary 4.2 Given $|s_i|<1(i\in Z_\mu)$, let $r=\max\{|s_i|\|A_i^{-1}\|\}$, $\overline{\|A\|}=\max\{\|A_i\|\}$, and

$$\underline{h}=1+\frac{\log r}{\log\overline{\|A\|}}. \tag{4.3}$$

If $1<r<\overline{\|A\|}^{-1}$, then for any given $h\in(0,\underline{h})$, there exists constant $C>0$ such that

$$|f_\lambda(x)-f_\lambda(y)|\leq C|x-y|^h,\,\forall x,\,y\in\Omega, \tag{4.4}$$

Proof. Let $\rho=\max\{|s_i|\|A_i^{-1}\|^h\},\underline{\rho}=\max\{|s_i|\|A_i^{-1}\|^{\underline{h}}\}$. Then $\rho<\underline{\rho}$, and we have that

$$\underline{\rho}=\max\{|s_i|\|A_i^{-1}\|\cdot\|A_i^{-1}\|^{\underline{h}-1}\}\leq r\left(\max\{\|A_i^{-1}\|\}\right)^{\underline{h}-1},$$

and

$$\max\{\|A_i^{-1}\|\}\overline{\|A\|}\geq\max\{\|A_i\|^{-1}\}\overline{\|A\|}=\frac{\overline{\|A\|}}{\min\{\|A_i\|\}}\geq1.$$

By induction, we find that

$$\log\underline{\rho}\leq\frac{\log r}{\log\overline{\|A\|}}\log\left(\max\{\|A_i^{-1}\|\}\overline{\|A\|}\right)\leq0.$$

Thus, $\rho < \underline{\rho} \leq 1$, The inequality (4.4) is immediate from Theorem 4.1.

Theorem 4.2 Given $|s_i| < 1 (i \in Z_\mu)$, let $r = \max\{|s_i| \|A_i^{-1}\|\}$, $\overline{\|A\|} = \max\{\|A_i\|\}$. $r = |s_p| \|A_p^{-1}\|$. If $f_\lambda(x)$ is not a plane in $\Omega \subset R^d$, λ_p is linear polynomial, then when $1 < r < \overline{\|A\|}^{-1}$, the *Hölder* exponent h of $f_\lambda(x)$ satisfies: $\underline{h} \leq h \leq \overline{h}$, where

$$\underline{h} = 1 + \frac{\log r}{\log \overline{\|A\|}}, \ \overline{h} = \frac{\log |s_p|}{\log \|A_p\|}. \tag{4.5}$$

Proof. The inequality $\underline{h} \leq h$ can be derived from Corollary 4.2, here we consider the estimate of upper bound $\overline{h}$. Let

$$\sigma(n) = (p, p, \cdots, p) \in Z_\mu^n,$$

given two nodes $\overline{x}, \overline{y} \in \Omega$, we define nodes x^*, $y^* \in \Omega_{\sigma(n)}$ as follows

$$x^* = w_{\sigma(n)}(\overline{x}), \ y^* = w_{\sigma(n)}(\overline{y}).$$

For any $x, y \in \Omega_{\sigma(n)}$, We find from (3.10) that

$$\begin{aligned} f_\lambda(x) - f_\lambda(y) = & \sum_{k=1}^{n} S_{\sigma(k-1)} \left[\lambda_{\sigma_k} \left(w_{\sigma(k)}^{-1}(x) \right) - \lambda_{\sigma_k} \left(w_{\sigma(k)}^{-1}(y) \right) \right] \\ & + S_{\sigma(n)} \left[f_\lambda \left(w_{\sigma(n)}^{-1}(x) \right) - f_\lambda \left(w_{\sigma(n)}^{-1}(y) \right) \right]. \end{aligned} \tag{4.6}$$

It follows from differential mean-value theorem that

$$\begin{aligned} |f_\lambda(x^*) - f_\lambda(y^*)| &= \left| \sum_{k=1}^{n} s_p^{k-1} G(\lambda_p) A_p^{-k} A_p^n (\overline{x} - \overline{y}) + s_p^n \left[f_\lambda(\overline{x}) - f_\lambda(\overline{y}) \right] \right| \\ &= |s_p|^n |f_\lambda(\overline{x}) - f_\lambda(\overline{y})| \left| \frac{G(\lambda_p) \sum_{k=1}^{n} s_p^{k-n} A_p^{n-k} (\overline{x} - \overline{y})}{s_p \left[f_\lambda(\overline{x}) - f_\lambda(\overline{y}) \right]} + 1 \right| \\ &= |s_p|^n | f_\lambda(\overline{x}) - f_\lambda(\overline{y})| \left| \frac{g_1(\overline{x} - \overline{y}) + g_2(\overline{x} - \overline{y})}{s_p \left[f_\lambda(\overline{x}) - f_\lambda(\overline{y}) \right]} + 1 \right|, \end{aligned} \tag{4.7}$$

where

$$g_1(x) = G(\lambda_p)(I - s_p^{-1} A_p)^{-1} x,$$

$$g_2(x) = -G(\lambda_p)(I - s_p^{-1} A_p)^{-1} (s_p^{-1} A_p)^n x.$$

Since $g_1(x)$ is a plane, $f_\lambda(x)$ is fractal surface, there exist distinct points $\overline{x}$ and $\overline{y} \in \Omega$ such that $f_\lambda(\overline{x}) \neq f_\lambda(\overline{y})$ and $g_1(\overline{x} - \overline{y}) / s_p [f_\lambda(\overline{x}) - f_\lambda(\overline{y})] > 0$, we let

$$\alpha = \frac{g_1(\overline{x} - \overline{y})}{s_p \left[f_\lambda(\overline{x}) - f_\lambda(\overline{y}) \right]}.$$

The inequality $\|s_p^{-1}A_p\| < 1$ implies that

$$(s_p^{-1}A_p)^n \to 0(n \to \infty),$$

namely

$$g_2(x) \to 0(n \to \infty),\ \forall x \in \Omega.$$

We choose n large enough such that

$$\varepsilon = \left| \frac{g_2(\overline{x} - \overline{y})}{s_p\left[f_\lambda(\overline{x}) - f_\lambda(\overline{y})\right]} \right| < \alpha + 1.$$

It follows from (4.7) that

$$|f_\lambda(x^*) - f_\lambda(y^*)| \geq C_1|s_p|^n, \tag{4.8}$$

where

$$C_1 = [f_\lambda(\overline{x}) - f_\lambda(\overline{y})](\alpha + 1 - \varepsilon).$$

Noting that $\|A_p\|^{\overline{h}} = |s_p|$(see (4.5)), we gain

$$|x^* - y^*| = |A_p^n(\overline{x} - \overline{y})| \leq \|A_p\|^n|\overline{x} - \overline{y}|,$$

this gives

$$|x^* - y^*|^{\overline{h}} \leq |A_p|^{\overline{h}n}||\overline{x} - \overline{y}|^{\overline{h}} = |s_p|^n|\overline{x} - \overline{y}|^{\overline{h}}. \tag{4.9}$$

Submit (4.9) into (4.8), then

$$|f(x^*) - f(y^*)| \geq C|x^* - y^*|^{\overline{h}},$$

where $C = C_1|\overline{x} - \overline{y}|^{-\overline{h}}$. Since $|x^* - y^*| < 1$, we have that

$$\lim_{\varepsilon \to 0}\left\{ \frac{\log|f_\lambda(x^*) - f_\lambda(y^*)|}{\log|x^* - y^*|} : y^* \in B(x^*, \varepsilon) \right\} \leq \overline{h},$$

it follows from the definition of *Hölder* exponent that $h \leq \overline{h}$.

In Theorem 4.2, we restrict λ_p to linear functions of variable x. Next we want to seek more general results. To this end, we give some marks.

For any $\lambda(x) \in C^1(\Omega)$, and d-dimensional vector $\overrightarrow{v} = (v_1, v_2, \cdots, v_d)^T$, let

$$GD(\lambda(x), \overrightarrow{v}) = v_1\frac{\partial\lambda}{\partial x_1} + v_2\frac{\partial\lambda}{\partial x_2} + \cdots + v_d\frac{\partial\lambda}{\partial x_d}.$$

We denote by $L(x_0, \overrightarrow{v})$ the straight line which has direction vector $\overrightarrow{v} \neq 0$ and passes through point $x_0 \in \Omega$, namely

$$L(x_0, \overrightarrow{v}) : x = x_0 + t\overrightarrow{v}, t \in R.$$

Denote by $\overline{L(x_0, \overrightarrow{v})}$ the line segment which is expressed as follows

$$\overline{L(x_0, \overrightarrow{v})} = \{x \in \Omega | x = x_0 + t\overrightarrow{v}, t \in R\}.$$

Theorem 4.3 Given $|s_i| < 1 (i \in Z_\mu)$, let $r = \max\{|s_i|\|A_i^{-1}\|\}$, $\overline{\|A\|} = \max\{\|A_i\|\}$, $r = |s_p|\|A_p^{-1}\|$. Denote fixed point of contractive map w_p by x_p, real eigenvalue of matrix A_p by γ and the eigenvector corresponding to γ by $\overrightarrow{P}$. For any $n \in N$, we let

$$Q(n, x) = \sum_{k=1}^{n} GD(\lambda_p(\xi_k), \overrightarrow{P})\rho^{n-k}, \tag{4.10}$$

where $\xi_k = w_p^{-k}(x)$, $x \in \overline{L(x_p, \overrightarrow{P})} \cap w_p^n(\Omega)$ and $\rho = s_p^{-1}\gamma$.

If there is choice of γ and $\overrightarrow{P}$, such that

(i) $1 < r < \overline{\|A\|}^{-1}$, $0 < |\rho| < 1$.

(ii) $f_\lambda(x)$ is not a line in $\overline{L(x_p, \overrightarrow{P})} \subset \Omega$.

(iii) For any given $x \in \overline{L(x_p, \overrightarrow{P})} \cap w_p^n(\Omega)$, there exists a positive integer N_0 such that $Q(n, x) \geq 0 (or \quad Q(n, x) \leq 0)$ for every $n > N_0$.

Then the *Hölder* exponent h of $f_\lambda(x)$ satisfies: $\underline{h} \leq h \leq \overline{h}$, where

$$\underline{h} = 1 + \frac{\log r}{\log \overline{\|A\|}}, \ \overline{h} = \frac{\log |s_p|}{\log \|A_p\|}. \tag{4.11}$$

Proof. According the proving procession of Theorem 4.2, we need only to prove that there exist two nodes x^* and y^* such that (4.8) holds.

Let $\sigma(n) = (p, p, \cdots, p) \in Z_\mu^n$, $\overline{x}, \overline{y} \in \overline{L(x_p, \overrightarrow{P})}$ satisfy $\overline{x} \neq \overline{y}$. Then nodes $\overline{x}$ and $\overline{y}$ can be written as

$$\overline{x} = x_p + t_x\overrightarrow{P}, \ \overline{y} = x_p + t_y\overrightarrow{P}, \ t_x \neq t_y,$$

we have that

$$A_p^k(\overline{x} - \overline{y}) = (t_x - t_y)\gamma^k\overrightarrow{P}, \ \forall k \in N. \tag{4.12}$$

Define nodes x^* and y^* as follows

$$x^* = w_{\sigma(n)}(\overline{x}), \ y^* = w_{\sigma(n)}(\overline{y}),$$

by using differential mean-value theorem, we find from (4.6) and (4.12) that

$$\begin{aligned} |f_\lambda(x^*) - f_\lambda(y^*)| &= \left|\sum_{k=1}^{n} s_p^{k-1} G(\lambda_p(\xi_k)) A_p^{n-k}(\overline{x} - \overline{y}) + s_p^n \left[f_\lambda(\overline{x}) - f_\lambda(\overline{y})\right]\right| \\ &= \left|\sum_{k=1}^{n} s_p^{k-1} G(\lambda_p(\xi_k))\gamma^{n-k}(t_x - t_y)\overrightarrow{P} + s_p^n \left[f_\lambda(\overline{x}) - f_\lambda(\overline{y})\right]\right| \\ &= \left|\sum_{k=1}^{n} s_p^{k-1} GD(\lambda_p(\xi_k), \overrightarrow{P})\gamma^{n-k}(t_x - t_y) + s_p^n \left[f_\lambda(\overline{x}) - f_\lambda(\overline{y})\right]\right|, \end{aligned}$$

where $\xi_k = w_p^{-k}(\xi)$ and $\xi \in \overline{L(x_p, \overrightarrow{P})} \cap w_p^n(\Omega)$. If $f_\lambda(\overline{x}) \neq f_\lambda(\overline{y})$, then

$$|f_\lambda(x^*) - f_\lambda(y^*)| = |s_p|^n |f_\lambda(\overline{x}) - f_\lambda(\overline{y})| \cdot \left| \frac{Q(n, \xi)(t_x - t_y)}{s_p [f_\lambda(\overline{x}) - f_\lambda(\overline{y}))]} + 1 \right|. \quad (4.13)$$

We chose n large enough such that $Q(n, \xi) = a|Q(n, \xi)|$, where a=1 or -1(see condition (iii)). It follows from condition (ii) that there exist distinct points t_x and t_y, such that $f_\lambda(\overline{x}) \neq f_\lambda(\overline{y})$, and

$$\frac{a(t_x - t_y)}{s_p[f_\lambda(\overline{x}) - f_\lambda(\overline{y})]} > 0.$$

We find from (4.13) that

$$\begin{aligned} |f_\lambda(x^*) - f_\lambda(y^*)| &= |s_p|^n |f_\lambda(\overline{x}) - f_\lambda(\overline{y})| \left| \frac{|Q(n, \xi)| a(t_x - t_y)}{s_p [f_\lambda(\overline{x}) - f_\lambda(\overline{y})]} + 1 \right| \\ &\geq |s_p|^n |f_\lambda(\overline{x}) - f_\lambda(\overline{y})|, \end{aligned}$$

and the above inequality gives that

$$|f_\lambda(x^*) - f_\lambda(y^*)| \geq C_1 |s_p|^n, \quad (4.14)$$

where

$$C_1 = |f_\lambda(\overline{x}) - f_\lambda(\overline{y})|$$

Thus inequality (4.8)(i.e. (4.14)) holds, and the proof is completed.

Obviously, it may be somewhat difficult to check condition (iii), here we give some simple remarks. if λ_p is a linear polynomial, or $GD(\lambda_p, \overrightarrow{P}) \neq 0$ for any $x \in \overline{L(x_p, \overrightarrow{P})}$ and $\rho > 0$, then condition (iii) holds. In general, for given x, $Q(n, x)$ can be viewed as a serie, if $\lim_{n \to \infty} Q(n, x) = q_0 \neq 0$, then condition (iii) holds.

5 Examples of the Hölder exponent estimates

The results presented in section 4 are only relative to vertical contractive factor s_i and the family of contractive mappings $W := \{w_i : i \in Z_\mu\}$ satisfied $W(\Omega) = \Omega$. In applications, these results can be applied immediately to concrete fractal interpolation functions, such as FIFs based on interval, triangle and quadrangle area, etc. Here we give two examples.

5.1 One-dimensional example

Let $I = [0, 1]$, given a partition of I

$$\triangle : x_0 = 0 < x_1 < x_2 < \cdots < x_N = 1. \quad (5.1)$$

The family of contractive affine mappings $W = \{w_i : i \in Z_N\}$ is defined as follows

$$w_i(x) = x_i + |I_i|x,\ i \in Z_N,$$

where $|I_i| = x_{i+1} - x_i$ corresponding to the length of interval $I_i = [x_i,\ x_{i+1}]$. $f_\lambda(x)$ denotes affine fractal interpolation function defined on I, and satisfies suitable conditions. Now we consider the Hölder exponent of $f_\lambda(x)$. Corresponding to Theorem 4.2, $A_i = |I_i|$, we have the following theorem

Theorem 5.1 Given $|s_i| < 1 (i \in Z_N)$, let $r = \max\{|s_i||I_i|^{-1}\}$, $\overline{I} = \max\{|I_i|\}$, $r = |s_p||I_p|^{-1}$. If $f_\lambda(x)$ is not line, and $r > 1$, then the *Hölder* exponent h of $f_\lambda(x)$ satisfies: $\underline{h} \le h \le \overline{h}$, where

$$\underline{h} = 1 + \frac{\log r}{\log \overline{I}},\ \overline{h} = \frac{\log |s_p|}{\log I_p} \tag{5.2}$$

Let $\overline{I} = \max\{|I_i|\}$, $\underline{I} = \min\{|I_i|\}$. If $I_p = \overline{I}$, then $s = \max\{|s_i|\} = |s_p|$, $\underline{h} = \overline{h}$. We find from Theorem 5.1 that

Corollary5.1 Under suppositions of Theorem 5.1, if $I_p = \overline{I}$, then the *Hölder* exponent of $f_\lambda(x)$ is

$$h = \frac{\log |s|}{\log \overline{I}}. \tag{5.3}$$

Corollary 5.2 Under suppositions of Theorem 5.1, Let $\overline{I} = \beta \underline{I}$, if $I_p = \underline{I}$, then

$$\underline{h} = \frac{\log(|s_p|\beta)}{\log \overline{I}},\ \overline{h} = \frac{\log |s_p|}{\log \underline{I}}. \tag{5.4}$$

Especially, if $\beta = 1$, $\triangle$ is a uniform partition, it is easily derived that

$$I_p = \frac{1}{N},\ s = |s_p|,$$

from Corollary 5.1, we have $h = -\log s / \log N$, and this result can be found in [8].

5.2 Two-dimensional example

Let $\Omega = \{(x,\ y) : 0 \le x \le 1,\ 0 \le y \le 1\}$, given a partition of Ω

$$0 = x_0 < x_1 < \cdots < x_M = 1$$
$$0 = y_0 < y_1 < \cdots < x_N = 1$$

Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, then the family of contractive affine mappings $W = \{w_{ij} : i = 1, 2, \cdots, M,\ j = 1, 2, \cdots, N\}$ can be defined by

$$w_{ij} = \begin{pmatrix} \Delta x_i & 0 \\ 0 & \Delta y_j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_{i-1} \\ y_{j-1} \end{pmatrix} \equiv A_{ij} \begin{pmatrix} x \\ y \end{pmatrix} + B_{ij}.$$

Consider the following iterated function system

$$(Tf)(x,y) = \lambda_{i,j}(w_{ij}^{-1}(x,y)) + s_{ij}f(w_{ij}^{-1}(x,y)), x, y \in \Omega_{ij},$$

where $\Omega_{ij} = w_{ij}(\Omega)$, s_{ij} denotes vertical contractive factor, and

$$\lambda_{ij} = a_{ij}x + b_{ij}y + c_{ij}xy + d_{ij}.$$

With some suitable conditions, the existence and uniqueness of fractal interpolation function $f_\lambda(x,y)$ was proved in [23]. Now we consider the Hölder exponent of $f_\lambda(x,y)$.

Let γ_x, γ_y and $\overrightarrow{P}_x$, $\overrightarrow{P}_y$ denote eigenvalues and eigenvectors of A_{ij} respectively, then

$$\gamma_x = \Delta x_i,\ \overrightarrow{P}_x = (1,\ 0),$$
$$\gamma_y = \Delta y_j,\ \overrightarrow{P}_y = (0,\ 1).$$

Corresponding to Theorem 4.3, we have

Theorem 5.2 Given $|s_{ij}| < 1$, let $r = \max\{|s_{ij}|\|A_{ij}\|^{-1}\}$, $\overline{\|A\|} = \max\{\|A_{ij}\|\}$, $r = |s_{pq}|\|A_{pq}^{-1}\|$, $\rho = s_{pq}^{-1}\gamma_x$, and (x_{pq}, y_{pq}) denote the fixed point of w_{pq}. if $1 < r < \overline{\|A\|}^{-1}$, $|\rho| < 1$, and $f_\lambda(x,y)$ is not a line in $L\left((x_{pq}, y_{pq}),\ \overrightarrow{P}_x\right)$, then the *Hölder* exponent h of $f_\lambda(x,y)$ satisfies: $\underline{h} \le h \le \overline{h}$, where

$$\underline{h} = 1 + \frac{\log r}{\log \overline{\|A\|}},\ \overline{h} = \frac{\log |s_{pq}|}{\log \|A_{pq}\|}.$$

Proof. If condition (iii) of Theorem 4.3 holds, Theorem 5.2 is immediate from Theorem 4.3.

In fact, for any given positive number n and $(x,\ y) \in L\left((x_{pq}, y_{pq}),\ \overrightarrow{P}_x\right) \cap w_{pq}^n(\Omega)$(noting that $y = y_{pq}$), by induction we find that

$$Q(n,x,y) = (a_{pq} + c_{pq}y_{pq})\sum_{k=1}^{n}\rho^{n-k},$$

it is easy to see that condition (iii) of Theorem 4.3 holds.

In theorem 5.2, if replace $\rho = s_{pq}^{-1}\gamma_x$ by $\rho = s_{pq}^{-1}\gamma_y$, we can obtain the same conclusion.

6 Conclusions

In this paper, we borrow some notions such as multiscale partition and refinable set to describe a high dimensional fractal interpolation problem. This enables us to investigate the properties of FIFs under more general setting. With the aids of the expression with symbol series of FIFs, we discuss Lipschitz continuity, and give the estimates of H*ölder* exponent. Compared with the existing results, ours conclusions are provided in more general meaning, which can be more easily applied to concrete fractal interpolation problems.

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On Some Generalized Difference Sequence Spaces of Invariant Means Defined by a Sequence of Orlicz Functions

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Abstract

The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of a sequence of Orlicz functions, invariant mean and lacunary convergence. We also examine some topological properties of these sequence spaces and establish some elemantary connections between them. These are generalizations of those defined and studied by Savaş and Rhoades [22] and Bataineh and Azar [2] and some others before.

Key words and phrases: Lacunary sequence, invariant mean, Orlicz function, difference sequence.

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1 Introduction

Let l_∞, c and c_o denote the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, normed by $\|x\|=\sup_k|x_k|$, respectively. A sequence $x=(x_k)\in l_\infty$ is said to be almost convergent if all of its Banach limits coincide (see, Banach [1]). Let f denote the space of all almost convergent sequences. Lorentz [11] proved that $f=\{x=(x_k)\in l_\infty:\ \lim_{k\to\infty}t_{km}(x)$ exists, uniformly in $m\}$, where $t_{km}(x)=\frac{x_m+x_{m+1}+....x_{m+k}}{k+1}$. The space $[f]$ of strongly almost convergent sequences was introduced by Maddox [13] and also independently by Freedmann et al. [6] as follows: $[f]=\{x=(x_k)\in l_\infty:\ \lim_{k\to\infty}|t_{km}(x-Le)|\ =0$, uniformly in m, for some $L\}$, where $e=(1,1,...)$. Schaefer [23] defined the $\sigma-$*convergence* as follows: Let σ be a mapping of the set positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be invariant mean or $\sigma-$*mean* if and only if (i) $\phi(x)\geq 0$ when the sequence $x=(x_k)$ has $x_k\geq 0$ for all $k\in N$, (ii) $\phi(e)=1$,(iii) $\phi\left(x_{\sigma(k)}\right)=\phi(x)$ for all $x=$

$(x_k) \in l_\infty$. In the case σ is the translation mapping $k \to k+1$, a $\sigma-mean$ is often called a Banach limit and V_σ , the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences. If $x = (x_k)$, write $Tx = T(x_k) = \left(x_{\sigma(k)}\right)$. It can be shown that $V_\sigma = \{x = (x_k) \in l_\infty : \lim_{k\to\infty} t_{km}(x) = l$, uniformly in m, $l = \sigma - \lim x\}$, where $t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \ldots . x_{\sigma^k(m)}}{k+1}$. Here $\sigma^k(m)$ denotes the kth iterate of the mapping σ at m . A $\sigma-mean$ extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits; that is to say, if and only if for all $n \geq 0, j \geq 0, \sigma^j(n) \neq n$ (see, Mursaleen [15]). A sequence $x = (x_k)$ is said to be strongly $\sigma-convergent$ if there exists a number L such that $(|x_k - l|) \in V_\sigma$ with the limit zero (see, Mursaleen [16]). We write as the set of all strongly $\sigma-convergent$ sequences. If $(|x_k - l|) \in V_\sigma$, we write $V_\sigma - \lim x_k = L$. Taking $\sigma(n) = n+1$, we obtain $[V_\sigma] = [f]$ so that strong $\sigma-convergence$ generalizes the concept of strong almost convergence. Note that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$.Using the concept of invariant means, the following sequence spaces have been recently introduced and examined by Mursaleen et al. [17] is a generalization of the results of Das and Sahoo [4].

$$w_\sigma = \left\{ x = (x_k) : \lim_s s^{-1} \sum_{k=1}^{s} t_{km}(x - Le) = 0, \text{ for some } L, \text{ uniformly in } m \right\},$$

$$[w]_\sigma = \left\{ x = (x_k) : \lim_s s^{-1} \sum_{k=1}^{s} |t_{km}(x - Le)| = 0, \text{ uniformly in } m \right\},$$

$$[w_\sigma] = \left\{ x = (x_k) : \lim_s s^{-1} \sum_{k=1}^{s} t_{km}(|x - Le|) = 0, \text{ uniformly in } m \right\}.$$

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, ...$, where $k_o = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r . The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [6] as follows

$$N_\theta = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

The concept of lacunary strong $\sigma-corvergence$ was introduced by Savaş [21] which is a generalization of the idea of lacunary strong almost convergence due to Das and Mishra [3]. If $\left[V_\sigma^\theta\right]$ denotes the set of all lacunary strongly $\sigma-convergent$ sequences then Savaş [21] defined

$$\left[V_\sigma^\theta\right] = \left\{ x = (x_k) : \lim_r h_r^{-1} \sum_{k \in I_r} \left| x_{\sigma^k(n)} - L \right| = 0, \text{ for some } L, \text{ uniformly in } n \right\}.$$

Recall [7, 10] that an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$

for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$ then this function is called modulus function, defined and discussed by Ruckle [20] , Maddox [14] and many others. Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct the sequence space $l_M = \left\{x = (x_k) : \sum_k M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$. The space l_M with the norm $\|x\| = \inf\left\{\rho > 0 : \sum_k M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$ becomes a Banach space which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. A generalization of Orlicz sequence space is due to Woo [24]. Let $\Omega = (M_k)$ be a sequence of Orlicz functions. Define the sequence space $l(\Omega)$ by $l(\Omega) = \left\{x = (x_k) : \sum_k M_k\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}$ and equip this space with the norm $\|x\| = \inf\left\{\rho > 0 : \sum_k M_k\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$. The space $l(\Omega)$ is a Banach space and is called a modular sequence space. The space $l(\Omega)$ also generalizes the concept of modulared sequence space introduced earlier by Nakano [18], who considered the space $l(\Omega)$ when $M_k(x) = x^{\alpha_k}$, where $1 \leq \alpha_k < \infty$ for $k \geq 1$.

An Orlicz function M is said to satisfy the Δ_2-*condition* for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ $(u \geq 0)$. It is easy to see that always $K > 2$. The Δ_2-*condition* is equivalent to the satisfaction of the inequality $M(Lu) \leq KLM(u)$ for every value of u and for $L > 1$ (see Krasnoselskii and Rutickii [9]).

Parashar and Choudhary [19] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalizes the well-known Orlicz sequence space l_M and strong summable sequence spaces $[C,1,p]$,$[C,1,p]_o$ and $[C,1,p]_\infty$. It may be noted that the spaces of strongly summable sequences were discussed by Maddox [12].

The difference sequence space $X(\Delta)$ was introduced by Kizmaz [8] as follows: $X(\Delta) = \{x = (x_k) : \ (\Delta x_k) \in X\}$, for $X = l_\infty, c$ and c_o, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$. Later, these difference sequence spaces were generalized by Et and Çolak [5] as follows: Let $n \in N$ be fixed, then $X(\Delta^n) = \{x = (x_k) : \ (\Delta^n x_k) \in X\}$, for $X = l_\infty$, c and c_o, where $\Delta^n x_k = \Delta^{n-1}x_k - \Delta^{n-1}x_{k+1}$ and $\Delta^0 x_k = x_k$ for all $k \in N$. The generalized difference has the following binomial representation: $\Delta^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+i}$ for each $k \in N$.

The main object of this paper is to define and study the sequence spaces $\left[w^\theta, \Omega, u, \Delta^n, p\right]_\sigma$, $\left[w^\theta, \Omega, u, \Delta^n, p\right]_\sigma^0$ and $\left[w^\theta, \Omega, u, \Delta^n, p\right]_\sigma^\infty$ which are defined by combining the concept of a sequence of Orlicz functions, invariant mean and lacunary convergence. We examine some linearity and inclusion relations of these sequence spaces. These are generalizations of those defined and studied by Savaş and Rhoades [22], Bataineh and Azar [2] and some others before.

Let $\Omega = (M_k)$ be any sequence of Orlicz functions. Now, if $u = (u_k)$ is any sequence such that $u_k \neq 0$ $(k = 1, 2, ...)$ and for any sequence $x = (x_k)$,the generalized difference sequence $(\Delta^n x_k)$ is given by $\Delta^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+i}$,

then we define the following sequence spaces:

$$\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}=\left\{x=(x_k):\lim_{r}h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}\left(u\Delta^{n}x-Le\right)|}{\rho}\right)\right]^{p_k}=0,\right.$$
$$\left.\text{for some } L,\ \rho>0,\ \text{uniformly in } m\right\},$$

$$\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{0}=\left\{x=(x_k):\lim_{r}h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}\left(u\Delta^{n}x\right)|}{\rho}\right)\right]^{p_k}=0,\right.$$
$$\left.\text{for some } \rho>0,\ \text{uniformly in } m\right\},$$

$$\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{\infty}=\left\{x=(x_k):\sup_{r,m}h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}\left(u\Delta^{n}x\right)|}{\rho}\right)\right]^{p_k}<\infty,\right.$$
$$\left.\text{for some } \rho>0\right\}.$$

If $M_k=M$ for all $k\in N$ and $n=1$, then these spaces reduce to those defined and studied by Bataineh and Azar [2]. When $u=e, M_k=M$ for all $k\in N$ and $n=0$, then these spaces reduce to those defined and studied by Savaş and Rhoades [22]. Also some sequence spaces are obtained by specializing $\theta=(k_r), \Omega=(M_k), n\in N$ and $p=(p_k)$. For instance, if $u=e$, $\Delta^{n}x_k=x_k$ and $p_k=1$ for all $k\in N$, then we get the spaces $\left[w^{\theta},\Omega\right]_{\sigma}$, $\left[w^{\theta},\Omega,\right]_{\sigma}^{0}$ and $\left[w^{\theta},\Omega\right]_{\sigma}^{\infty}$. If $x\in\left[w^{\theta},\Omega\right]_{\sigma}$, we say that x is lacunary $[w]_{\sigma}-$*convergence* with respect to the sequence of Orlicz functions $\Omega=(M_k)$.

If $u=e$, $\Delta^{n}x_k=x_k$ and $p_k=1$ for all $k\in N$, $\theta=(2^r)$, $M_k(x)=x_k$ and $p_k=1$ for all $k\in N$, then $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}=[w]_{\sigma}$ which were defined and studied by Mursaleen et.al [17]. If $u=e$, $\Delta^{n}x_k=x_k$ and $p_k=1$ for all $k\in N$, $\theta=(2^r)$, $M_k(x)=x_k$ and $p_k=1$ for all $k\in N$, $\sigma(n)=n+1$, then $\left[\widehat{w}^{\theta},\Omega,u,\Delta^{n},p\right]=[\widehat{w}]$ which were defined and studied by Das and Sahoo [4] . If $u=e$, $\Delta^{n}x_k=x_k$ for all $k\in N$ and $\theta=(2^r)$,then $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}=\left[w^{\theta},\Omega,p\right]_{\sigma}$, $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{0}=\left[w^{\theta},\Omega,p\right]_{\sigma}^{0}$ and $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{\infty}=\left[w^{\theta},\Omega,p\right]_{\sigma}^{\infty}$.

2 Main Results

We proved the following theorems.

Theorem 1 *For any sequence of Orlicz functions $\Omega=(M_k)$ and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}$, $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{0}$ and $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{\infty}$ are linear spaces over the set of complex numbers C.*

Proof. We will prove the result only for $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{0}$. The others can be treated similarly. Let $x,y\in\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{0}$ and $\alpha,\beta\in C$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r}h_{r}^{-1}\sum_{k\in I_{r}}\left[M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}x\right)\right|}{\rho_{1}}\right)\right]^{p_{k}}=0,\text{ uniformly in }m$$

and

$$\lim_{r}h_{r}^{-1}\sum_{k\in I_{r}}\left[M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}y\right)\right|}{\rho_{2}}\right)\right]^{p_{k}}=0,\text{ uniformly in }m.$$

Let $\rho_3=\max\left(2\left|\alpha\right|\rho_1,2\left|\beta\right|\rho_2\right)$. Since M_k is non-decreasing and convex for all $k\in N$,

$$\begin{aligned}
&h_{r}^{-1}\sum_{k\in I_{r}}\left[M_{k}\left(\frac{\left|t_{km}\left(\alpha u\Delta^{n}x+\beta u\Delta^{n}y\right)\right|}{\rho_{3}}\right)\right]^{p_{k}}\\
&\leq h_{r}^{-1}\sum_{k\in I_{r}}\frac{1}{2^{p_{k}}}\left[M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}x\right)\right|}{\rho_{1}}\right)+M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}y\right)\right|}{\rho_{2}}\right)\right]^{p_{k}}\\
&\leq h_{r}^{-1}\sum_{k\in I_{r}}\left[M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}x\right)\right|}{\rho_{1}}\right)+M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}y\right)\right|}{\rho_{2}}\right)\right]^{p_{k}}\\
&\leq Dh_{r}^{-1}\sum_{k\in I_{r}}\left[M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}x\right)\right|}{\rho_{1}}\right)\right]^{p_{k}}+\\
&\qquad Dh_{r}^{-1}\sum_{k\in I_{r}}\left[M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}y\right)\right|}{\rho_{2}}\right)\right]^{p_{k}}\to 0,\text{ as }r\to\infty,\text{ uniformly in }m,
\end{aligned}$$

where $D=\max\left(1,2^{H-1}\right)$, $H=\sup_k p_k<\infty$.

Therefore $\alpha x+\beta y\in\left[w^{\theta},\Omega,p,u,\Delta^{n}\right]_{\sigma}^{0}$. This completes the proof. ■

Theorem 2 *For any sequence of Orlicz functions $\Omega=(M_k)$ and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $\left[w^{\theta},\Omega,u,\Delta^{n},p\right]_{\sigma}^{0}$ is a topological linear space, paranormed by*

$$h\left(x\right)=\sum_{i=1}^{n}\left|u_{i}x_{i}\right|+$$

$$\inf\left\{\rho^{p_{r}/H}:\ \left(h_{r}^{-1}\sum_{k\in I_{r}}\left[M_{k}\left(\frac{\left|t_{km}\left(u\Delta^{n}x\right)\right|}{\rho}\right)\right]^{p_{k}}\right)^{1/H}\leq 1,\ r,m=1,2,\ldots\right\},$$

where $H=\max\left(1,\sup_k p_k<\infty\right)$.

Proof. Clearly $h(x) = h(-x)$. It is trivial that $\Delta^n x_k = 0$ for $x = 0$. Since $M_k(0) = 0$ for all $k \in N$, we get $\inf\left\{\rho^{p_r/H}\right\} = 0$. Therefore $h(0) = 0$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n x)|}{\rho_1}\right)\right]^{p_k}\right)^{1/H} \leq 1,\ r,m = 1,2,\ldots$$

and

$$\left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n y)|}{\rho_2}\right)\right]^{p_k}\right)^{1/H} \leq 1,\ r,m = 1,2,\ldots.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n(x+y))|}{\rho}\right)\right]^{p_k}\right)^{1/H}$$

$$= \left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n(x+y))|}{\rho_1+\rho_2}\right)\right]^{p_k}\right)^{1/H}$$

$$\leq \left(h_r^{-1}\sum_{k\in I_r}\left[\frac{\rho_1}{\rho_1+\rho_2}M_k\left(\frac{|t_{km}(u\Delta^n x)|}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2}M_k\left(\frac{|t_{km}(u\Delta^n y)|}{\rho_1}\right)\right]^{p_k}\right)^{1/H}$$

by Minkowski's inequality

$$\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n x)|}{\rho_1}\right)\right]^{p_k}\right)^{1/H} +$$

$$\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n y)|}{\rho_2}\right)\right]^{p_k}\right)^{1/H} \leq 1.$$

Since the $\rho's$ are non-negative, so we have

$$\inf\left\{\rho^{p_r/H}:\ \left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n(x+y))|}{\rho}\right)\right]^{p_k}\right)^{1/H} \leq 1,\ r,m = 1,2,\ldots\right\}$$

$$\leq \inf\left\{\rho_1^{p_r/H}:\ \left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n x)|}{\rho_1}\right)\right]^{p_k}\right)^{1/H} \leq 1,\ r,m = 1,2,\ldots\right\}$$

$$+\inf\left\{\rho_2^{p_r/H}:\ \left(h_r^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}(u\Delta^n y)|}{\rho_2}\right)\right]^{p_k}\right)^{1/H} \leq 1,\ r,m = 1,2,\ldots\right\}$$

Therefore $h(x+y) \leq h(x) + h(y)$. We now show that the scalar multiplication is continuous. Whenever $\eta \to 0$ and $x \to 0$, imply $h(\eta x) \to 0$. Also $x \to 0$

imply $h(\eta x) \to 0$. Now we show that $\eta \to 0$ and x fixed imply $h(\eta x) \to 0$. Without loss of generality let $|\eta| < 1$. Then the required proof follows from the following inequality:

$$h(\eta x) = \sum_{i=1}^{n} |\eta u_i x_i|$$

$$+ \inf \left\{ \rho^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km} (\eta u \Delta^n x)|}{\rho} \right) \right]^{p_k} \right)^{1/H} \leq 1,\ r, m = 1, 2, \ldots \right\}$$

$$\leq |\eta| \sum_{i=1}^{n} |u_i x_i|$$

$$+ \inf \left\{ (|\eta| t)^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km} (\eta u \Delta^n x)|}{t} \right) \right]^{p_k} \right)^{1/H} \leq 1,\ r, m = 1, 2, \ldots \right\}$$

$$\left(\text{where } \frac{\rho}{|\eta|} = t \right)$$

$$\leq \max \left(|\eta|, \sup |\eta|^{\frac{H}{M}} \right) \cdot$$

$$\left\{ \sum_{i=1}^{n} |u_i x_i| + \inf \left\{ t^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km} (\eta u \Delta^n x)|}{t} \right) \right]^{p_k} \right)^{1/H} \leq 1,\ r, m = 1, 2, \ldots \right\} \right\}$$

$$= \max \left(|\eta|, \sup |\eta|^{\frac{H}{M}} \right) h(x) \to 0, \text{ as } \eta \to 0.$$

This completes the proof. ■

The following result follows by a straightforward calculation using the Δ_2-condition.

Lemma 3 *For any sequence of Orlicz functions $\Omega = (M_k)$ which satisfy the Δ_2-condition for all $k \in N$ and let $0 < \delta < 1$. Then for each $x \geq \delta$, $M_k(x) < Kx\delta^{-1} M_k(2)$ for some constant $K > 0$.*

Theorem 4 *For any sequence of Orlicz functions $\Omega = (M_k)$ which satisfy the Δ_2-condition for all $k \in N$,*

$$\left[w^\theta, u, \Delta^n, p \right]_\sigma \subset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma .$$

Proof. Let $x \in \left[w^\theta, u, \Delta^n, p \right]_\sigma$. Then we have

$$A_r = h_r^{-1} \sum_{k \in I_r} \left[|t_{km} (u \Delta^n x - Le)| \right]^{p_k} \to 0, \text{ as } r \to \infty \text{ for some } L, \text{ uniformly in } m.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \varepsilon$ for $0 \leq t \leq \delta$. Then we can write

$$
\begin{aligned}
& h_r^{-1} \sum_{k \in I_r} \left[M_k \left| t_{km} \left(u \Delta^n x - Le \right) \right| \right]^{p_k} \\
&= h_r^{-1} \sum_{k \in I_r, \left| t_{km} \left(u \Delta^n x - Le \right) \right| \leq \delta} \left[M_k \left(\left| t_{km} \left(u \Delta^n x - Le \right) \right| \right) \right]^{p_k} \\
&\quad + h_r^{-1} \sum_{k \in I_r, \left| t_{km} \left(u \Delta^n x - Le \right) \right| > \delta} \left[M_k \left(\left| t_{km} \left(u \Delta^n x - Le \right) \right| \right) \right]^{p_k} \\
&< h_r^{-1} h_r \max \left(\varepsilon^h, \varepsilon^H \right) + h_r^{-1} \max \left(1, K \delta^{-1} M_k \left(2 \right) \right)^H h_r A_r, \text{ by Lemma 3.}
\end{aligned}
$$

Letting $r \to \infty$, it follows that $x \in \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma$. ■

Theorem 5 *Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$. Then for any sequence of Orlicz functions $\Omega = (M_k)$, $[w, \Omega, u, \Delta^n, p]_\sigma \subset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma$, $[w, \Omega, u, \Delta^n, p]_\sigma^0 \subset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^0$ and $[w, \Omega, u, \Delta^n, p]_\sigma^\infty \subset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^\infty$, where*

$$
[w, \Omega, u, \Delta^n, p]_\sigma = \left\{ x = (x_k) : \lim_s s^{-1} \sum_{k=1}^{s} \left[M_k \left(\frac{\left| t_{km} \left(u \Delta^n x - Le \right) \right|}{\rho} \right) \right]^{p_k} = 0, \right.
$$
$$
\left. \text{for some } L,\ \rho > 0, \text{ uniformly in } m \right\},
$$

$$
[w, \Omega, u, \Delta^n, p]_\sigma^0 = \left\{ x = (x_k) : \lim_s s^{-1} \sum_{k=1}^{s} \left[M_k \left(\frac{\left| t_{km} \left(u \Delta^n x \right) \right|}{\rho} \right) \right]^{p_k} = 0, \right.
$$
$$
\left. \text{for some } \rho > 0, \text{ uniformly in } m \right\},
$$

$$
\left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^\infty = \left\{ x = (x_k) : \sup_{s,m} s^{-1} \sum_{k=1}^{s} \left[M_k \left(\frac{\left| t_{km} \left(u \Delta^n x \right) \right|}{\rho} \right) \right]^{p_k} < \infty, \right.
$$
$$
\left. \text{for some } \rho > 0 \right\}.
$$

Proof. We will prove $[w, \Omega, u, \Delta^n, p]_\sigma \subset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma$ only. The others can be treated similarly. It is sufficient to show that $[w, \Omega, u, \Delta^n, p]_\sigma^0 \subset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^0$, the general inclusion follows by linearity. Suppose that $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \delta$ for all $r \geq 1$. Then for $x \in [w, \Omega, u, \Delta^n, p]_\sigma^0$, we write

$$B_r = h_r^{-1} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k}$$
$$= h_r^{-1} \sum_{k=1}^{k_r} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k} - h_r^{-1} \sum_{k=1}^{k_{r-1}} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k}$$
$$= \frac{k_r}{h_r} \left(k_r^{-1} \sum_{k=1}^{k_r} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k} \right)$$
$$- \frac{k_{r-1}}{h_r} \left(k_{r-1}^{-1} \sum_{k=1}^{k_{r-1}} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k} \right).$$

Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$, $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$. The terms $k_r^{-1} \sum_{k=1}^{k_r} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k}$ and $k_{r-1}^{-1} \sum_{k=1}^{k_{r-1}} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k}$ both converge to zero uniformly in m, and it follows that B_r converges to zero as $r \to \infty$, uniformly in m, that is, $x \in \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^0$. This completes the proof. ■

Theorem 6 *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r q_r < \infty$. Then for any sequence of Orlicz functions $\Omega = (M_k)$, $[w, \Omega, u, \Delta^n, p]_\sigma \supset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma$, $[w, \Omega, u, \Delta^n, p]_\sigma^0 \supset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^0$ and $[w, \Omega, u, \Delta^n, p]_\sigma^\infty \supset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^\infty$.*

Proof. We will prove $[w, \Omega, u, \Delta^n, p]_\sigma \supset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma$ only. The others can be treated similarly. It is sufficient to show that $[w, \Omega, u, \Delta^n, p]_\sigma^0 \supset \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^0$, the general inclusion follows by linearity. Suppose that $\limsup_r q_r < \infty$, then there exists $C > 0$ such that $q_r < C$ for all $r \geq 1$. Let $x \in \left[w^\theta, \Omega, u, \Delta^n, p \right]_\sigma^0$ and $\varepsilon > 0$. Then there exists $R > 0$ such that for every $j \geq R$ and all m,

$$B_j = h_j^{-1} \sum_{k \in I_j} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k} < \varepsilon.$$

We can also find $K > 0$ such that $B_j < K$ for all $j = 1, 2, \ldots$. Now let s be any integer with $k_{r-1} < s \leq k_r$, where $r > R$. Then

$$s^{-1} \sum_{k=1}^{s} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k} \leq k_{r-1}^{-1} \sum_{k=1}^{k_r} \left[M_k \left(\frac{|t_{km}(u\Delta^n x)|}{\rho} \right) \right]^{p_k}$$

$$= k_{r-1}^{-1}\left\{\sum_{k\in I_1}\left[M_k\left(\frac{|t_{km}\left(u\Delta^n x\right)|}{\rho}\right)\right]^{p_k} + \sum_{k\in I_2}\left[M_k\left(\frac{|t_{km}\left(u\Delta^n x\right)|}{\rho}\right)\right]^{p_k} + \ldots + \sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}\left(u\Delta^n x\right)|}{\rho}\right)\right]^{p_k}\right\}$$

$$= k_{r-1}^{-1}k_1k_1^{-1}\sum_{k\in I_1}\left[M_k\left(\frac{|t_{km}\left(u\Delta^n x\right)|}{\rho}\right)\right]^{p_k} + k_{r-1}^{-1}\left(k_2-k_1\right)\left(k_2-k_1\right)^{-1}\sum_{k\in I_2}\left[M_k\left(\frac{|t_{km}\left(u\Delta^n x\right)|}{\rho}\right)\right]^{p_k} + \ldots +$$

$$k_{r-1}^{-1}\left(k_R-k_{R-1}\right)\left(k_R-k_{R-1}\right)^{-1}\sum_{k\in I_R}\left[M_k\left(\frac{|t_{km}\left(u\Delta^n x\right)|}{\rho}\right)\right]^{p_k} + \ldots + k_{r-1}^{-1}\left(k_r-k_{r-1}\right)\left(k_r-k_{r-1}\right)^{-1}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{km}\left(u\Delta^n x\right)|}{\rho}\right)\right]^{p_k}$$

$$= k_{r-1}^{-1}k_1B_1 + k_{r-1}^{-1}\left(k_2-k_1\right)B_2 + \ldots + k_{r-1}^{-1}\left(k_R-k_{R-1}\right)B_R + \ldots + k_{r-1}^{-1}\left(k_r-k_{r-1}\right)B_r$$

$$\leq \left(\sup_{j\geq 1}B_j\right)k_{r-1}^{-1}k_R + \left(\sup_{j\geq R}B_j\right)k_{r-1}^{-1}\left(k_r-k_R\right) < Kk_{r-1}^{-1}k_R + \varepsilon C.$$

Since $k_{r-1}\to\infty$ as $n\to\infty$, it follows that $s^{-1}\sum_{k=1}^{s}\left[M_k\left(\frac{|t_{km}(u\Delta^n x)|}{\rho}\right)\right]^{p_k}\to 0$ uniformly in m, and consequently $x\in[w,\Omega,u,\Delta^n,p]_\sigma^0$. This completes the proof. ■

Theorem 7 *Let $\theta=(k_r)$ be a lacunary sequence with $1<\liminf_r q_r\leq\limsup_r q_r<\infty$. Then for any sequence of Orlicz functions $\Omega=(M_k)$,*

$$[w,\Omega,u,\Delta^n,p]_\sigma = \left[w^\theta,\Omega,u,\Delta^n,p\right]_\sigma.$$

Proof. It follows from Theorem 5 and Theorem 6. ■

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SOME CLASSES OF GENERALIZED DIFFERENCE PARANORMED SEQUENCE SPACES ASSOCIATED WITH MULTIPLIER SEQUENCES

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Abstract: Let $\Lambda = (\lambda_k)$ be a sequence of non-zero complex numbers. In this paper we introduce generalized difference sequence spaces associated with multiplier sequence $\Lambda = (\lambda_k)$ and study their different properties. We also introduce $\Delta_{\Lambda}^{m,n}$-statistical convergence and strongly $\Delta_{\Lambda}^{m,n}(p)$-Cesaro summable and give some relations between them.

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Key Words: Multiplier sequence; paranorm; completeness; difference sequence.

1.Introduction

Throughout the article w, c_o, c and ℓ_∞ denote the spaces of all, null, convergent and bounded sequences, respectively. The studies on difference sequence spaces was initiated by Kizmaz [11]. He studied the spaces

$$Z(\Delta) = \{x = (x_k) \in w : \Delta x = (\Delta x_k) \in Z\}$$

for $Z = c_o, c$ and ℓ_∞. It was shown by him that these spaces are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k| .$$

The notion was further generalized by Et and Colak [4] as follows:

Let $m \geq 0$ be an integer, then

$$Z(\Delta^m) = \{x = (x_k) \in w : \Delta^m x = (\Delta^m x_k) \in Z\}$$

for $Z = c_o, c$ and ℓ_∞, where $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$, $\Delta^0 x_k = x_k$, for all $k \in N$. It was shown by them that these spaces are Banach spaces, normed by

$$\|x\|_{\Delta^m} = \sum_{k=1}^{m} |x_k| + \sup_k |\Delta^m x_k| .$$

The generalized difference operator $\Delta^m x_k$ has the following binomial representation

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} \qquad \ldots(1)$$

Later on the notion was further investigated by Et and Esi [3], Tripathy [17] and many others.

The scope for the studies on sequence spaces was extended on introducing the notion of associated multiplier sequences. Goes and Goes [9] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence E with the help of the multiplier sequences $\left(k^{-1}\right)$ and (k), respectively. Kamthan [10] used the multiplier sequence $(k!)$. We shall use a general multiplier sequence fort he sequence spaces introduced in this article. Throughout the article, $p=\left(p_k\right)$ is a sequence of positive real numbers.

Let $\Lambda=\left(\lambda_k\right)$ be a sequence of non-zero scalars. Then for E a sequence space, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence $\Lambda=\left(\lambda_k\right)$ is defined as

$$E(\Lambda)=\left\{x=\left(x_k\right)\in w:\left(\lambda_k x_k\right)\in E\right\}.$$

2.Definitions and Preliminaries

A sequence space E is said to be solid (or normal) if $\alpha x=\left(\alpha_k x_k\right)\in E$, whenever $x=\left(x_k\right)\in E$, for all sequences $\alpha=\left(\alpha_k\right)$ of scalars with $\left|\alpha_k\right|\leq 1$ for all $k\in N$.

A sequence space E is said to be symmetric if $\left(x_{\pi(k)}\right)\in E$, whenever $x=\left(x_k\right)\in E$, where $\pi(k)$ is a permutation on N.

Let $\Lambda=\left(\lambda_k\right)$ be a given multiplier sequence, then we have the following known difference sequence spaces:

$$c_o(\Delta,\Lambda,p)=\left\{x=\left(x_k\right)\in w:\ \left|\lambda_k\Delta x_k\right|^{p_k}\to 0,\ as\ k\to\infty\right\},$$

$$c(\Delta,\Lambda,p)=\left\{x=\left(x_k\right)\in w:\ \left|\lambda_k\left(\Delta x_k-L\right)\right|^{p_k}\to 0,\ as\ k\to\infty,\ for\ some\ L\right\}$$

and

$$l_\infty(\Delta,\Lambda,p)=\left\{x=\left(x_k\right)\in w:\ \sup_k\left|\lambda_k\Delta x_k\right|^{p_k}<\infty\right\}.$$

Now we introduce the following generalized difference sequence spaces associated with a multiplier sequence $\Lambda=\left(\lambda_k\right)$:

$$V_o\left(\Delta^m,\Lambda,p\right)=\left\{x=\left(x_k\right)\in w:\ \left|\lambda_k\Delta^m x_{k+n}\right|^{p_k}\to 0,\ as\ k\to\infty,\ uniformly\ in\ n\right\},$$

$$V_1\left(\Delta^m,\Lambda,p\right)=\left\{x=\left(x_k\right)\in w:\ \left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k}\to 0,\ as\ k\to\infty,\ uniformly\ in\ n,\ for\ some\ L\right\}$$

and

$$V_\infty\left(\Delta^m,\Lambda,p\right)=\left\{x=\left(x_k\right)\in w:\ \sup_{k,n}\left|\lambda_k\Delta^m x_{k+n}\right|^{p_k}<\infty\right\}.$$

We get particular cases of the above sequence spaces by restricting some of the parameters m, n, p and $\Lambda = (\lambda_k)$. Some examples are below:

When $m = n = 0$ and $\Lambda = e = (1,1,1,...)$, we obtain the sequence spaces $c_o(p)$, $c(p)$ and $\ell_\infty(p)$ studied by Maddox [13] , Lascarides [12] and others.

When $m = n = 0$, $\Lambda = e = (1,1,1,...)$ and $p_k = 1$ for all $k \in N$, we obtain the sequence spaces of null, convergent and bounded sequences, respectively.

When $m = n = 0$, we obtain the sequence spaces $c_o(\Delta, \Lambda, p)$, $c(\Delta, \Lambda, p)$ and $\ell_\infty(\Delta, \Lambda, p)$.

Let $p = (p_k)$ be a bounded sequence of positive real numbers. Let $H = \sup_k p_k$ and $C = \max(1, 2^{H-1})$. Then, we have (see for instance Maddox [14]).

$$|x_k + y_k|^{p_k} \le C\left(|x_k|^{p_k} + |y_k|^{p_k}\right) \qquad \ldots(2)$$

Remark. Let $H = \sup_k p_k < \infty$. Then the spaces $c_o(p)$ and $c(p)$ are paranormed spaces, paranormed by

$$g(x) = \sup_k |x_k|^{\frac{p_k}{M}}, \text{ where } M = \max\{1, H\}.$$

The space $\ell_\infty(p)$ is paranormed by g, if $\inf_k p_k > 0$ (see for instance Maddox [13]).

3. Main Results

The proof of the following result is routine verification, so we omit it.

Theorem 3.1. Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then,

(***a***) $V_o(\Delta^m, \Lambda, p)$, $V_1(\Delta^m, \Lambda, p)$ and $V_\infty(\Delta^m, \Lambda, p)$ are linear spaces over the complex field **C**.

(***b***) $V_o(\Delta^m, \Lambda, p) \subset V_1(\Delta^m, \Lambda, p) \subset V_\infty(\Delta^m, \Lambda, p)$.

Theorem 3.2. Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then,

(***a***) The spaces $V_o(\Delta^m, \Lambda, p)$ and $V_1(\Delta^m, \Lambda, p)$ are complete linear topological spaces, paranormed by

$$g(x) = \sum_{i=1}^{m} |\lambda_k x_k| + \sup_{k,n} \left|\lambda_k \Delta^m x_{k+n}\right|^{p_k/M}$$

where $M = \max(1, \sup_k p_k = H)$.

(***b***) The space $V_\infty(\Delta^m, \Lambda, p)$ is a complete paranormed space, paranormed by g, if $0 < \inf_k p_k$.

Proof. (***a***) Clearly $g(\theta) = 0, g(-x) = g(x)$ and by Minkowski's inequality

$g(x+y) \le g(x)+g(y)$. We now show that the scalar multiplication is continuous. Whenever $\eta \to 0$ and $x \to \theta$ imply $g(\eta x) \to 0$. Also $x \to \theta$ imply $g(\eta x) \to 0$. Now, we show that $\eta \to 0$ and x fixed imply $g(\eta x) \to 0$. Without loss of generality let $|\eta| < 1$. Then, the required proof follows from the following inequality:

$$\begin{aligned} g(\eta x) &= \sum_{i=1}^{m} |\eta \lambda_k x_k| + \sup_{k,n} \left|\eta \lambda_k \Delta^m x_{k+n}\right|^{p_k/M} \\ &\le |\eta| \sum_{i=1}^{m} |\lambda_k x_k| + \max\left\{|\eta|, |\eta|^{H/M}\right\} \sup_{k,n} \left|\lambda_k \Delta^m x_{k+n}\right|^{p_k/M} \\ &\le \max\left\{|\eta|, |\eta|^{H/M}\right\} g(x) \to 0, \ as \ \eta \to 0. \end{aligned}$$

Let (x^s) be a Cauchy sequence in $c_o(\Delta^m, \Lambda, p)$. Then, $g(x^s - x^t) \to 0, \ as \ s,t \to \infty$. For a given $\varepsilon > 0$, let $n_o = n_o(\varepsilon)$ be such that

$$\sum_{i=1}^{m} \left|\lambda_k \left(x_k^s - x_k^t\right)\right| + \sup_{k,n} \left|\lambda_k \Delta^m \left(x_{k+n}^s - x_{k+n}^t\right)\right|^{p_k/M} < \varepsilon, \text{ for all } s,t \ge n_o \qquad \ldots(3)$$

Hence, $\sum_{i=1}^{m} \left|\lambda_k \left(x_k^s - x_k^t\right)\right| < \varepsilon$, for all $s,t \ge n_o$. This implies that $(\lambda_k x_k^s)$ is a Cauchy sequence for each $k = 1,2,...,m$. So, $(\lambda_k x_k^s)$ is converges in **C** for each $k = 1,2,...,m$. Let $\lim_s \lambda_k x_k^s = y_k$ for each $k = 1,2,...,m$ and let $\lim_s x_k^s = x_k$, say
where

$$x_k = y_k \lambda_k^{-1}, \text{ for each } k = 1,2,...,m. \qquad \ldots(4)$$

From (3), we have $\sup_{k,n} \left|\lambda_k \Delta^m \left(x_{k+n}^s - x_{k+n}^t\right)\right|^{p_k/M} < \varepsilon$, for all $s,t \ge n_o$. This implies that $(\lambda_k \Delta^m x_k^s)$ is a Cauchy sequence in **C** for each $k \in N$. Hence $(\lambda_k \Delta^m x_k^s)$ is converges for each $k \in N$. Let $\lim_s \lambda_k \Delta^m x_k^s = z_k$, for each $k \in N$.
Let $\lim_s \Delta^m x_k^s = y_k = z_k \lambda_k^{-1}$, for each $k \in N$. $\qquad \ldots(5)$

Hence from (1),(4) and (5), it follows that $\lim_s x_{m+1}^s = x_{m+1}$. Proceeding in this way inductively, we have $\lim_s x_k^s = x_k$, for each $k \in N$. By (3) we have

$$\lim_t \left[\sum_{i=1}^{m} \left|\lambda_k \left(x_k^s - x_k^t\right)\right| + \sup_{k,n} \left|\lambda_k \Delta^m \left(x_{k+n}^s - x_{k+n}^t\right)\right|^{p_k/M}\right] < \varepsilon, \text{ for all } s \ge n_o.$$

So,

$$\left[\sum_{i=1}^{m} \left|\lambda_k \left(x_k^s - x_k\right)\right| + \sup_{k,n} \left|\lambda_k \Delta^m \left(x_{k+n}^s - x_{k+n}\right)\right|^{p_k/M}\right] < \varepsilon, \text{ for all } s \ge n_o \ .$$

This implies that $(x^s - x) \in c_o(\Delta^m, \Lambda, p)$. Since $c_o(\Delta^m, \Lambda, p)$ is a linear space, so we have $x = x^s - (x^s - x) \in c_o(\Delta^m, \Lambda, p)$. This completes the proof.

(*b*) The proof follows by using the technique applied for establishing the part (a) and the Remark.

Theorem 3.3. (*a*) Let $0 < p_k \le q_k \le 1$ for all $k \in N$. Then, $V_\infty(\Delta^m, \Lambda, q)$ is a subset of $V_\infty(\Delta^m, \Lambda, p)$.

(*b*) If $\inf_k q_k > 0$, then $V_\infty(\Delta^m, \Lambda, q)$ is a closed subset of $V_\infty(\Delta^m, \Lambda, p)$

Proof.(*a*) Let $x \in l_\infty(\Delta^m, \Lambda, q)$. Then there exists a constant $T > 1$ such that

$$\left|\lambda_k \Delta^m x_{k+n}\right|^{q_k} \le T, \text{ for every } n \in N$$

and so

$$\left|\lambda_k \Delta^m x_{k+n}\right|^{p_k} \le T, \text{ for every } n \in N. \text{ Thus, } x \in V_\infty(\Delta^m, \Lambda, p).$$

(*b*) The proof follows from Theorem 3.3(*a*) and Theorem 3.2(*b*).

Theorem 3.4.(*a*) Let $0 < \inf_k p_k \le p_k \le 1$. Then, $V_1(\Delta^m, \Lambda, p) \subset V_1(\Delta^m, \Lambda)$.

(*b*) Let $1 \le p_k \le \sup_k p_k < \infty$. Then, $V_1(\Delta^m, \Lambda, p) \supset V_1(\Delta^m, \Lambda)$.

(*c*) Let $m_1 \le m_2$. Then $V_1(\Delta^{m_2}, \Lambda, p) \subset V_1(\Delta^{m_1}, \Lambda, p)$.

Proof.(*a*) Let $0 < \inf_k p_k \le p_k \le 1$ and $x = (x_k) \in V_1(\Delta^m, \Lambda, p)$. Then there exists L such that

$$\left|\lambda_k (\Delta^m x_{k+n} - L)\right| \le \left|\lambda_k (\Delta^m x_{k+n} - L)\right|^{p_k}.$$

Hence $x = (x_k) \in V_1(\Delta^m, \Lambda)$.

(*b*) Let $1 \le p_k \le \sup_k p_k < \infty$ and $x = (x_k) \in V_1(\Delta^m, \Lambda)$. Then for each $0 < \varepsilon < 1$, there exists a positive integer k_o such that $\left|\lambda_k (\Delta^m x_{k+n} - L)\right| < \varepsilon < 1$, for all $k > k_o$ and for all $n \in N$. This implies that

$$\left|\lambda_k (\Delta^m x_{k+n} - L)\right|^{p_k} \le \left|\lambda_k (\Delta^m x_{k+n} - L)\right| < \varepsilon, \text{ for all } k > k_o \text{ and for all } n \in N.$$

Hence $x = (x_k) \in V_1(\Delta^m, \Lambda, p)$.

(c) The proof is a routine verification.

Theorem 3.5. The spaces $V_0(\Delta^m, \Lambda, p)$, $V_1(\Delta^m, \Lambda, p)$ and $V_\infty(\Delta^m, \Lambda, p)$ are not solid in general.

Proof. To show this, consider the following example.

Example 1. Let $\lambda_k = k$ and $p_k = k$ for all $k \in N$. Then consider the sequence $x = (x_k) = (2)$ and $m = 1$. Then $x \in V_0(\Delta^m, \Lambda, p)$. Now consider the sequence $\alpha = (\alpha_k)$ defined by $\alpha_k = (-1)^k$ for all $k \in N$. Then αx neither belongs to $V_o(\Delta^m, \Lambda, p)$ nor to $V_1(\Delta^m, \Lambda, p)$ nor to $V_\infty(\Delta^m, \Lambda, p)$.

Theorem 3.6. The spaces $V_0\left(\Delta^m,\Lambda,p\right)$, $V_1\left(\Delta^m,\Lambda,p\right)$ and $V_\infty\left(\Delta^m,\Lambda,p\right)$ are not symmetric spaces in general.

Proof. To show this, consider the following example.

Example 2. Let $\lambda_k = k^{-1}$, $p_k = k$ for all $k \in N$ and $m = 1$. Then the sequence $x = \left(x_k\right) = \left(k\right)$ is in $V_0\left(\Delta^m,\Lambda,p\right)$. Consider the following sequence $y = \left(y_k\right)$ which is a rearrangement of the sequence $x = \left(x_k\right) = \left(k\right)$ defined as

$y = \left(y_k\right) = \left(x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, x_{64}, x_{11}, ...\right)$. Then, the sequence $y = \left(y_k\right)$ neither belongs to $V_o\left(\Delta^m,\Lambda,p\right)$ nor to $V_1\left(\Delta^m,\Lambda,p\right)$ nor to $V_\infty\left(\Delta^m,\Lambda,p\right)$.

4. Statistical convergence

A complex number sequence $x = \left(x_k\right)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n}\left|\left\{k \le n: \ \left|x_k - L\right| \ge \varepsilon\right\}\right| = 0,$$

where the vertical bars indicate the number of elements in the closed set. In this case we write $stat - \lim x_k = L$.

The idea of the statistical convergence for sequences of real number was introduced by Fast [6] and Schoenberg [16] at the initial stage. Later on it was studied from sequence space point of view and linked with summability methods by Salat [15], Fridy [7], Fridy and Orhan [8], Esi and Tripathy [2] and many others.

A complex number sequence $x = \left(x_k\right)$ is said to be $\Delta_\Lambda^{m,n}$-statistically convergent to the number L if for every $\varepsilon > 0$ and fixed $m \in N$,

$$\lim_t \frac{1}{t}\left|\left\{k \le t: \ \left|\lambda_k \Delta^m x_{k+n} - l\right| \ge \varepsilon\right\}\right| = 0, \text{ uniformly in } n.$$

In this case we write $\Delta_\Lambda^{m,n} - stat - \lim x_k = L$ and by $S\left(\Delta_\Lambda^{m,n}\right)$, we denote the class of all $\Delta_\Lambda^{m,n}$-statistically convergent sequences.

When $m = n = 0$ and $\Lambda = e = \left(1,1,1,...\right)$, the space $S\left(\Delta_\Lambda^{m,n}\right)$ represents the ordinary statistical convergence.

When $n = 0$, the space $S\left(\Delta_\Lambda^{m,n}\right)$ represent the space $S\left(\Delta_\Lambda^m\right)$, which was defined and studied by Esi and Tripathy [2].

When $n = 0$ and $\Lambda = e = \left(1,1,1,...\right)$, the space $S\left(\Delta_\Lambda^{m,n}\right)$ becomes the generalized difference statistically convergent sequence space $S\left(\Delta^m\right)$ defined and studied by Et and Nuray [5].

Let $x=(x_k)$ be a complex number sequence and $p=(p_k)$ be a bounded sequence of positive real numbers. The sequence $x=(x_k)$ is said to be strongly $(\Delta_\Lambda^{m,n},(p))$ -Cesaro summable if there is a complex number L such that

$$\lim_t \frac{1}{t}\sum_{k=1}^{t}\left|\lambda_k\Delta^m x_{k+n}-L\right|^{p_k}=0, \text{ uniformly in } n.$$

We denote the strongly $(\Delta_\Lambda^{m,n},(p))$ -Cesaro summable sequences by $w\left[\Delta_\Lambda^{m,n},(p)\right]$.

When $m=n=0$ and $\Lambda=e=(1,1,1,...)$, then we obtain the familiar space of strongly p -Cesaro sequences [1].

The proof of the following theorem is easy, so omitted.

Theorem 4.1. Let $\lambda_k\neq 0$ for all $k\in N$. Then,

(***a***) The sequence space $w\left[\Delta_\Lambda^{m,n},(p)\right]$ is complete linear topological space, paranormed by

$$g(x)=\sum_{k=1}^{m}\left|\lambda_k x_k\right|+\sup_{t,n}\left(\frac{1}{t}\sum_{k=1}^{t}\left|\lambda_k\Delta^m x_{k+n}\right|^{p_k}\right)^{1/M}$$

where $M=(1,\sup_k p_k)$.

(***b***) The sequence space $w\left[\Delta_\Lambda^{m,n},p\right]$ normed *BK* space with the norm

$$\|x\|_1=\sum_{k=1}^{m}\left|\lambda_k x_k\right|+\sup_{t,n}\left(\frac{1}{t}\sum_{k=1}^{t}\left|\lambda_k\Delta^m x_{k+n}\right|^{p}\right)^{1/p}, \text{ for } 1\le p<\infty$$

and the sequence space $w\left[\Delta_\Lambda^{m,n},p\right]$ is a complete p -normed space with the norm

$$\|x\|_1=\sum_{k=1}^{m}\left|\lambda_k x_k\right|^{p}+\sup_{t,n}\frac{1}{t}\sum_{k=1}^{t}\left|\lambda_k\Delta^m x_{k+n}\right|^{p}, \text{ for } 0<p<1.$$

Now, we will define an another norm on the sequence space $w\left[\Delta_\Lambda^{m,n},p\right]$ in follows theorem.

Theorem 4.2. Let $\lambda_k\neq 0$ for all $k\in N$ and $k\in\left[2^r,2^{r+1}\right)$. Then, the sequence space $w\left[\Delta_\Lambda^{m,n},p\right]$ normed *BK* space with the norm

$$\|x\|_2 = \sum_{k=1}^{m}|\lambda_k x_k| + \sup_{r,n}\left(\frac{1}{2^r}\sum_r\left|\lambda_k\Delta^m x_{k+n}\right|^p\right)^{1/p}, \text{ for } 1\le p<\infty$$

and the sequence space $w\left[\Delta_\Lambda^{m,n}, p\right]$ is a complete p-normed space with the norm

$$\|x\|_2 = \sum_{k=1}^{m}|\lambda_k x_k|^p + \sup_{r,n}\frac{1}{2^r}\sum_r\left|\lambda_k\Delta^m x_{k+n}\right|^p, \text{ for } 0<p<1.$$

Result 4.3. The two norms $\|x\|_1$ and $\|x\|_2$ are equivalent defined in the Theorem 4.1. and Theorem 4.2.

Now, we give some inclusion relations between the spaces $w\left[\Delta_\Lambda^{m,n},(p)\right]$ and $S\left(\Delta_\Lambda^{m,n}\right)$.

Theorem 4.4. Let $0<\inf_k p_k \le \sup_k p_k<\infty$. Then

(***a***) If $x=(x_k)\in w\left[\Delta_\Lambda^{m,n},(p)\right]$, then $x=(x_k)\in S\left(\Delta_\Lambda^{m,n}\right)$,

(***b***) If $x=(x_k)\in \ell_\infty\left(\Delta^m,\Lambda,p\right)\cap S\left(\Delta_\Lambda^{m,n}\right)$, then $x=(x_k)\in w\left[\Delta_\Lambda^{m,n},(p)\right]$,

(***c***) $\ell_\infty\left(\Delta^m,\Lambda,p\right)\cap S\left(\Delta_\Lambda^{m,n}\right)=\ell_\infty\left(\Delta^m,\Lambda,p\right)\cap w\left[\Delta_\Lambda^{m,n},(p)\right]$.

Proof .(*a*) Let $\varepsilon>0$ and $x=(x_k)\in w\left[\Delta_\Lambda^{m,n},(p)\right]$. Then, we have

$$\sum_{k=1}^{t}\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k} = \sum_{\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|\ge\varepsilon}^{t}\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k} + \sum_{\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|<\varepsilon}^{t}\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k}$$

$$\ge \sum_{\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|\ge\varepsilon}^{t}\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k}$$

$$\ge\left|\left\{k\le t:\ \left|\lambda_k\Delta^m x_{k+n}-L\right|\ge\varepsilon\right\}\right|\varepsilon^{p_k}.$$

$$\ge\left|\left\{k\le t:\ \left|\lambda_k\Delta^m x_{k+n}-L\right|\ge\varepsilon\right\}\right|\min\left(\varepsilon^h,\varepsilon^H\right),$$

where $0<\inf_k p_k=h\le p_k\le\sup_k p_k=H<\infty$. It follows that $x=(x_k)\in S\left(\Delta_\Lambda^{m,n}\right)$.

(***b***) Suppose that $x=(x_k)\in \ell_\infty\left(\Delta^m,\Lambda,p\right)\cap S\left(\Delta_\Lambda^{m,n}\right)$. Set $K=\sup_{k,n}\left|\lambda_k\Delta^m x_{k+n}\right|^{p_k}+L$.

$$\sum_{k=1}^{t}\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k} = \sum_{\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|\geq\varepsilon}^{t}\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k} + \sum_{\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|<\varepsilon}^{t}\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|^{p_k}$$

$$\leq \sum_{\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|\geq\varepsilon}^{t}\max\left(K^h, K^H\right) + \sum_{\left|\lambda_k\left(\Delta^m x_{k+n}-L\right)\right|<\varepsilon}^{t}\varepsilon^{p_k}$$

$$\leq \max\left(K^h, K^H\right)\left\{k\leq t:\ \left|\lambda_k\Delta^m x_{k+n}-l\right|\geq\varepsilon\right\} + \max\left(\varepsilon^h, \varepsilon^H\right).$$

It follows that $x=(x_k)\in w\left[\Delta_{\Lambda}^{m,n}(p)\right]$.

(***c***) Follows from (*a*) and (*b*).

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The Lyapunov stability for the ε - revised dynamics of the rigid body with three linear controls

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Abstract. In this paper we introduce the ε - revised system associated to a Hamilton - Poisson system. The ε - revised dynamical system of the rigid body with three linear controls is defined and its geometrical properties and dynamical stability of equilibrium points are investigated. [1]

1 Introduction

It is well known that many dynamical systems can be formulated using a Poisson structure (see for instance, R. Abraham and J. E. Marsden [1] and M. Puta [11]).

The metriplectic systems and the almost metriplectic systems are investigated in a series of papers (see P. J. Morrison [8]; D. Fish [2]; J. E. Marsden [7]; J.-P. Ortega and V. Planas - Bielsa [9]). An interesting class of almost metriplectic systems are so-called the revised dynamical systems associated to Hamilton-Poisson systems (see Gh. Ivan and D. Opriş [5];MR 2006m:53130).

The control of the rotation rigid body is one of the problems with a large applicability. The dynamics of the rigid body with three linear controls has been studied by M. Puta and D. Comănescu in [12] (Zbl 1024.70002).

2 Almost metriplectic systems

Let M be a smooth manifold of dimension n and let $C^\infty(M)$ be the ring of smooth real-valued functions on M.

A *Leibniz manifold* is a pair $(M, [\cdot,\cdot])$, where $[\cdot,\cdot]$ is a Leibniz bracket on M, that is $[\cdot,\cdot] : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ is a $\mathbf{R}$ - bilinear operation satisfying the conditions:

(i) the *left Leibniz rule*:

$$[f_1 \cdot f_2, f_3] = [f_1, f_3] \cdot f_2 + f_1 \cdot [f_2, f_3] \quad \text{for all} \quad f_1, f_2, f_3 \in C^\infty(M);$$

(ii) the *right Leibniz rule*:

$$[f_1, f_2 \cdot f_3] = [f_1, f_2] \cdot f_3 + f_2 \cdot [f_1, f_3] \quad \text{for all} \quad f_3, f_3, f_3 \in C^\infty(M)$$

where "$\cdot$" denotes the ordinary multiplication of functions.

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Let P and $\mathbf{g}$ be two tensor fields of type $(2,0)$ on M and $\varepsilon \in \mathbf{R}$ be a parameter. We define the map $[\cdot,(\cdot,\cdot)]_\varepsilon : C^\infty(M) \times (C^\infty(M) \times C^\infty(M)) \to C^\infty(M)$ by:

$$[f,(h_1,h_2)]_\varepsilon = P(df,dh_1) + \varepsilon \mathbf{g}(df,dh_2), \quad \text{for all} \quad f,h_1,h_2 \in C^\infty(M). \tag{1}$$

Proposition 2.1. *The map* $[\cdot,(\cdot,\cdot)]_\varepsilon$ *given by* (1) *satisfy the following relations:*

(i) $[af_1+bf_2,(h_1,h_2)]_\varepsilon = a[f_1,(h_1,h_2)]_\varepsilon + b[f_2,(h_1,h_2)]_\varepsilon$;

(ii) $[f,a(h_1,h_2)+b(h_1',h_2')]_\varepsilon = a[f,(h_1,h_2)]_\varepsilon + b[f,(h_1',h_2')]_\varepsilon$;

(iii) $[ff_1,(h_1,h_2)]_\varepsilon = f[f_1,(h_1,h_2)]_\varepsilon + f_1[f,(h_1,h_2)]_\varepsilon$;

(iv) $[f,h(h_1,h_2)]_\varepsilon = h[f,(h_1,h_2)]_\varepsilon + h_1 P(df,dh) + \varepsilon h_2 \mathbf{g}(df,dh)$,

for all $f,f_1,f_2,h_1,h_2,h_1',h_2' \in C^\infty(M)$ *and* $a,b \in \mathbf{R}$.

Proof. Applying the properties of the differential of functions and using that P and $\mathbf{g}$ are $\mathbf{R}$- bilinear maps, it is easy to establish the relations $(i)-(iv)$. □

We consider the map $[[\cdot,\cdot]]_\varepsilon : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined by:

$$[[f,h]]_\varepsilon = [f,(h,h)]_\varepsilon, \quad \text{for all} \quad f,h \in C^\infty(M). \tag{2}$$

Therefore, the map $[[\cdot,\cdot]]_\varepsilon : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is given by:

$$[[f,h]]_\varepsilon = P(df,dh) + \varepsilon \mathbf{g}(df,dh), \quad \text{for all} \quad f,h \in C^\infty(M). \tag{3}$$

Proposition 2.2. *The bracket* $[[\cdot,\cdot]]_\varepsilon$ *on* M *given by* (3) *verify the right Leibniz rule:*

$$[[f,hh']]_\varepsilon = h[[f,h']]_\varepsilon + h'[[f,h]]_\varepsilon, \quad \textit{for all} \quad f,h,h' \in C^\infty(M).$$

Proof. Indeed, $[[f,hh']]_\varepsilon = [f,(hh',hh')]_\varepsilon = [f,h(h',h')]_\varepsilon$. Putting $h_1 = h_2 = h'$ in the relation (iv) from Proposition 2.1, we have:
$[f,h(h',h')]_\varepsilon = h[f,(h',h')] + h'P(df,dh) + \varepsilon h'\mathbf{g}(df,dh) = h[f,(h',h')]_\varepsilon + h'(P(df,dh) + \varepsilon \mathbf{g}(df,dh)) = h[f,(h',h')]_\varepsilon + h'[f,(h,h)]_\varepsilon = h[[f,h']]_\varepsilon + h'[[f,h]]_\varepsilon$. □

By Proposition 2.1.(i),(ii) and (iii) and Proposition 2.2, we have that $[[\cdot,\cdot]]_\varepsilon$ given by (3) is a Leibniz bracket on M. Hence, $[[\cdot,\cdot]]_\varepsilon$ defines a Leibniz structure on M and $(M,P,\mathbf{g},[[\cdot,\cdot]]_\varepsilon)$ is a Leibniz manifold for each $\varepsilon \in \mathbf{R}$.

A Leibniz manifold $(M,P,\mathbf{g},[[\cdot,\cdot]]_\varepsilon)$ such that P is a skewsymmetric tensor field and $\mathbf{g}$ is a symmetric tensor field is called *almost metriplectic manifold*. In other words, given a skewsymmetric tensor field P of type $(2,0)$ and a symmetric tensor field $\mathbf{g}$ of type $(2,0)$ on a manifold M, we can define an almost metriplectic structure on M.

If the tensor field P is Poisson and the tensor field $\mathbf{g}$ is nondegenerate, then $(M,P,\mathbf{g},[[\cdot,\cdot]]_\varepsilon)$ is a *metriplectic manifold*, see [9].

Proposition 2.3. *Let* $(M,P,\mathbf{g},[[\cdot,\cdot]]_\varepsilon)$ *be an almost metriplectic manifold. If there exist* $h_1,h_2 \in C^\infty(M)$ *such that* $P(df,dh_2)=0$ *and* $\mathbf{g}(df,dh_1)=0$ *for all* $f \in C^\infty(M)$, *then the bracket* $[[\cdot,\cdot]]_\varepsilon$ *given by* (3) *satisfies the relation:*

$$[[f,h_1+h_2]]_\varepsilon = [f,(h_1,h_2)]_\varepsilon, \quad \text{for all} \quad f \in C^\infty(M). \tag{4}$$

Proof. Indeed, $[[f, h_1+h_2]]_\varepsilon = P(df, d(h_1+h_2))+\varepsilon\mathbf{g}(df, d(h_1+h_2)) = P(df, dh_1+dh_2)+\varepsilon\mathbf{g}(df, dh_1+dh_2) = P(df, dh_1)+P(df, dh_2)+\varepsilon\mathbf{g}(df, dh_1)+\varepsilon\mathbf{g}(df, dh_2) = P(df, dh_1)+\varepsilon\mathbf{g}(df, dh_2) = [f, (h_1, h_2)]_\varepsilon$. □

Let $(M, P, \mathbf{g}, [[\cdot, \cdot]]_\varepsilon)$ be an almost metriplectic manifold and let $h_1, h_2 \in C^\infty(M)$ two functions such that $P(df, dh_2) = 0$ and $\mathbf{g}(df, dh_1) = 0$ for all $f \in C^\infty(M)$. The vector field $X_{h_1h_2}$ given by:

$$X_{h_1h_2}(f) = [[f, h_1 + h_2]]_\varepsilon \quad \text{for any} \quad f \in C^\infty(M)$$

is called the *Leibniz vector field* associated to triple (h_1, h_2, ε) on M.

Taking account into Proposition 2.3 and (1), $X_{h_1h_2}$ is given by:

$$X_{h_1h_2}(f) = [f, (h_1, h_2)]_\varepsilon = P(df, dh_1) + \varepsilon\mathbf{g}(df, dh_2), \quad \text{for all} \quad f \in C^\infty(M). \tag{5}$$

In local coordinates on M, the differential system given by:

$$\dot{x}^i = [[x^i, h_1 + h_2]]_\varepsilon = [x^i, (h_1, h_2)]_\varepsilon, \quad \text{where} \tag{6}$$

$$[x^i, (h_1, h_2)]_\varepsilon = X_{h_1h_2}(x^i) = P^{ij}\frac{\partial h_1}{\partial x^j} + \varepsilon g^{ij}\frac{\partial h_2}{\partial x^j}, \ i, j = \overline{1, n} \tag{7}$$

with $P^{ij} = P(dx^i, dx^j)$ and $g^{ij} = \mathbf{g}(dx^i, dx^j)$, is called the *almost metriplectic system* on M associated to Leibniz vector field $X_{h_1h_2}$ with the bracket $[[\cdot, \cdot]]_\varepsilon$.

We denote the matrices of P and $\mathbf{g}$ respectively by $P = (P^{ij})$ and $g = (g^{ij})$.

Proposition 2.4. *For a skewsymmetric tensor P of type $(2, 0)$ on a manifold M and two functions $h_1, h_2 \in C^\infty(M)$ such that $P(df, dh_2) = 0$ for all $f \in C^\infty(M)$, there exists a symmetric tensor $\mathbf{g}$ of type $(2, 0)$ on M such that $\mathbf{g}(df, dh_1) = 0$ for all $f \in C^\infty(M)$ and $(M, P, \mathbf{g}, [[\cdot, \cdot]]_\varepsilon)$ is an almost metriplectic manifold.*

Proof. Let $g = (g^{ij})$ the matrix which must to be determined. Then $g^{ij}\frac{\partial h_1}{\partial x^j} = 0, \ i, j = \overline{1, n}$.

In a chart U such that $\frac{\partial h_1}{\partial x^j}(x) \neq 0$, the components g^{ij} are given by:

$$g^{ii}(x) = -\sum_{k=1, \ k\neq i}^{n}(\frac{\partial h_1}{\partial x^k})^2, \qquad g^{ij}(x) = \frac{\partial h_1}{\partial x^i}\frac{\partial h_1}{\partial x^j}, \quad \text{for} \ \ i \neq j. \tag{8}$$

Applying now Proposition 2.3 we obtain the result. □

Let be a Hamilton-Poisson system on M described by the Poisson tensor P having the matrix $P = (P^{ij})$ and by the Hamiltonian function $h_1 \in C^\infty(M)$ with the Casimir function $h_2 \in C^\infty(M)$ (i.e. $P^{ij}\frac{\partial h_2}{\partial x^j} = 0$ for $i, j = \overline{1, n}$). The differential equations of the Hamilton-Poisson system are the following:

$$\dot{x}^i = P^{ij}\frac{\partial h_1}{\partial x^j}, \quad i, j = \overline{1, n}. \tag{9}$$

Using (8), we determine $g = (g^{ij})$ and we have $g^{ij}\frac{\partial h_1}{\partial x^j} = 0$ for $i, j = \overline{1, n}$.

Applying now Proposition 2.4, for each $\varepsilon \in \mathbf{R}$, we obtain an almost metriplectic structure on M associated to system (9). The differential system associated to this structure is called the ε - *revised system* of the Hamilton - Poisson system.

Hence, the ε - revised system of the Hamilton - Poisson system defined by (9) is:

$$\dot{x}^i = P^{ij}\frac{\partial h_1}{\partial x^j} + \varepsilon g^{ij}\frac{\partial h_2}{\partial x^j}, \quad i,j = \overline{1,n}. \tag{10}$$

The terms $g^{ij}\frac{\partial h_2}{\partial x^j}$ from (10) describe a *cube perturbation* of the system (9).

Remark 2.1. For $\varepsilon = 0$, the system (10) coincides with (9). □

3 The ε - revised system associated to rigid body with three linear controls

The rigid body equations with three linear controls (see [12]) are given by:

$$\left\{\begin{array}{lcl} \dot{x}^1 &=& (a_3-a_2)x^2x^3 + cx^2 - bx^3 \\ \dot{x}^2 &=& (a_1-a_3)x^1x^3 - cx^1 + ax^3 \\ \dot{x}^3 &=& (a_2-a_1)x^1x^2 + bx^1 - ax^2 \end{array}\right. \tag{11}$$

where $x(t) = (x^1(t), x^2(t), x^3(t)) \in \mathbf{R}^3$ and $a_1 = \dfrac{1}{I_1},\ a_2 = \dfrac{1}{I_2},\ a_3 = \dfrac{1}{I_3}$ with $I_1 > I_2 > I_3 > 0$ (I_1, I_2, I_3 being the principal moments of inertia of the body) and $a, b, c \in \mathbf{R}$ are feedback parameters. We have $0 < a_1 < a_2 < a_3$.

The dynamics (11) is described by the Poisson tensor Π and the Hamiltonian $H \in C^\infty(\mathbf{R}^3)$ given by:

$$\Pi(x) = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{pmatrix}, \tag{12}$$

$$H(x) = \frac{1}{2}[a_1(x^1)^2 + a_2(x^2)^2 + a_3(x^3)^2] + ax^1 + bx^2 + cx^3. \tag{13}$$

Using (12) and (13), the dynamics (11) can be written in the matrix form:

$$\dot{x}(t) = \Pi(x(t)) \cdot \nabla H(x(t)), \tag{14}$$

where $\dot{x}(t) = (\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t))^T$ and $\nabla H(x(t))$ is the gradient of the Hamiltonian function H with respect to the canonical metric on $\mathbf{R}^3$.

Therefore, the dynamics (11) has the Hamilton-Poisson formulation $(\mathbf{R}^3, \Pi, H)$, where Π and H are given by (12) and (13).

The function $C \in C^\infty(\mathbf{R}^3)$ given by:

$$C(x) = \frac{1}{2}[(x^1)^2 + (x^2)^2 + (x^3)^2] \tag{15}$$

is a Casimir of the configuration $(\mathbf{R}^3, \Pi)$, i.e. $C(x) \cdot \nabla H(x) = O$.

Applying (8) for $P = \Pi, h_1(x) = H(x)$ and $h_2(x) = C(x)$, the tensor $\mathbf{g}$ is:

$$g = \begin{pmatrix} -(a_2x^2+b)^2-(a_3x^3+c)^2 & (a_1x^1+a)(a_2x^2+b) & (a_1x^1+a)(a_3x^3+c) \\ (a_1x^1+a)(a_2x^2+b) & -(a_1x^1+a)^2-(a_3x^3+c)^2 & (a_2x^2+b)(a_3x^3+c) \\ (a_1x^1+a)(a_3x^3+c) & (a_2x^2+b)(a_3x^3+c) & -(a_1x^1+a)^2-(a_2x^2+b)^2 \end{pmatrix}$$

since $\dfrac{\partial h_1}{\partial x^1} = a_1x^1 + a, \quad \dfrac{\partial h_1}{\partial x^2} = a_2x^2 + b, \quad \dfrac{\partial h_1}{\partial x^3} = a_3x^3 + c.$

We have

$$g(x) \cdot \nabla h_2(x) = (v_1(x), v_2(x), v_3(x))^T, \quad \text{where} \tag{16}$$

$$\begin{cases} v_1(x) = -[(a_2x^2+b)^2 + (a_3x^3+c)^2]x^1 + (a_1x^1+a)[(a_2x^2+b)x^2 + (a_3x^3+c)x^3] \\ v_2(x) = -[(a_1x^1+a)^2 + (a_3x^3+c)^2]x^2 + (a_2x^2+b)[(a_1x^1+a)x^1 + (a_3x^3+c)x^3] \\ v_3(x) = -[(a_1x^1+a)^2 + (a_2x^2+b)^2]x^3 + (a_3x^3+c)[(a_1x^1+a)x^1 + (a_2x^2+b)x^2] \end{cases}$$

The ε - revised system associated to dynamics (11) is:

$$\begin{cases} \dot{x}^1 = [(a_3-a_2)x^2x^3 + cx^2 - bx^3] + \varepsilon v_1(x) \\ \dot{x}^2 = [(a_1-a_3)x^1x^3 - cx^1 + ax^3] + \varepsilon v_2(x) \\ \dot{x}^3 = [(a_2-a_1)x^1x^2 + bx^1 - ax^2] + \varepsilon v_3(x) \end{cases} \tag{17}$$

The differential system (17) is called the ε - *revised system* of the rigid body with three linear controls. Taking $a = b = c = 0$ and $\varepsilon = 1$ in (17), we obtain the revised system of the free rigid body, see [5].

Vector writing of the dynamics (17). We introduce the following notations: $\mathbf{x} = (x^1, x^2, x^3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{a} = (a, b, c)$, $\mathbf{m}(\mathbf{x}) = (a_1x^1 + a, a_2x^2 + b, a_3x^3 + c)$.

For all $\mathbf{u} = (u_1, u_2, u_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbf{R}^3$, the following relation holds:

$$\mathbf{u} \cdot [\mathbf{w} \times (\mathbf{u} \times \mathbf{w})] = (\mathbf{u} \times \mathbf{w})^2, \quad \text{with} \quad (\mathbf{u} \times \mathbf{w})^2 = (\mathbf{u} \times \mathbf{w}) \cdot (\mathbf{u} \times \mathbf{w}) \tag{18}$$

where " $\times$ " and " $\cdot$ " denote the cross product resp. inner product in $\mathbf{R}^3$; that is:
$\mathbf{u} \times \mathbf{w} = (u_2w_3 - u_3w_2, u_3w_1 - u_1w_3, u_1w_2 - u_2w_1), \quad \mathbf{u} \cdot \mathbf{w} = u_1w_1 + u_2w_2 + u_3w_3.$

With the above notations, the dynamics (11) has the vector form:

$$\dot{\mathbf{x}} = \mathbf{x} \times \mathbf{m}(\mathbf{x}). \tag{19}$$

It is not hard to verify the following equality:

$$\mathbf{v} = (\mathbf{x} \times \mathbf{m}(\mathbf{x})) \times \mathbf{m}(\mathbf{x}). \tag{20}$$

Using (18), (19) and (17), the ε- revised system (27) is written in the vector form:

$$\dot{\mathbf{x}} = \mathbf{x} \times \mathbf{m}(\mathbf{x}) + \varepsilon[(\mathbf{x} \times \mathbf{m}(\mathbf{x})) \times \mathbf{m}(\mathbf{x})]. \tag{21}$$

4 The equilibrium points of the ε - revised system

The equilibrium points of the system (19) are solutions of the vector equation:

$$\mathbf{x} \times \mathbf{m}(\mathbf{x}) = \mathbf{0}. \tag{22}$$

The equilibrium points of the system (21) are solutions of the vector equation:

$$\mathbf{x} \times \mathbf{m}(\mathbf{x}) + \varepsilon[(\mathbf{x} \times \mathbf{m}(\mathbf{x})) \times \mathbf{m}(\mathbf{x})] = \mathbf{0}. \tag{23}$$

Theorem 4.1. *The Hamilton - Poisson system* (11) *and its revised system* (17) *have the same equilibrium points.*

Proof. Let $\mathbf{x}_0$ be an equilibrium point for (11). Then $\mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0) = \mathbf{0}$. We have that $\mathbf{x}_0$ is a solution of equation (23), since $x_0 \times \mathbf{m}(\mathbf{x}_0) + (\mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0)) \times \mathbf{m}(\mathbf{x}_0) = \mathbf{0} + \mathbf{0} \times \mathbf{m}(\mathbf{x}_0) = \mathbf{0}$. Hence $\mathbf{x}_0$ is an equilibrium point of (17).

Conversely, let $\mathbf{x}_0$ be an equilibrium point for (17). Using (22) it follows

(a) $\quad \mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0) + \varepsilon[\mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0)) \times \mathbf{m}(\mathbf{x}_0)] = \mathbf{0}$

The relation (a) can be written in the form:

(b) $\quad \mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0) - \varepsilon[\mathbf{m}(\mathbf{x}_0) \times (\mathbf{x}_0) \times \mathbf{m}(\mathbf{x}_0))] = \mathbf{0}$

Multiplying the relation (b) with the vector $\mathbf{x}_0$, we obtain

(c) $\quad \mathbf{x}_0 \cdot (\mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0)) - \varepsilon \mathbf{x}_0 \cdot [\mathbf{m}(\mathbf{x}_0) \times (\mathbf{x}_0) \times \mathbf{m}(\mathbf{x}_0))] = \mathbf{0}.$

Using the equality (18), the relation (c) is equivalent with

(d) $\quad -\varepsilon(\mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0))^2 = \mathbf{0}.$

From (d) follows $\mathbf{x}_0 \times \mathbf{m}(\mathbf{x}_0) = \mathbf{0}$, i.e. $\mathbf{x}_0$ is an equilibrium point for (17). □

Proposition 4.1. ([12]) *The equilibrium points of the Hamilton - Poisson system* (11) *are the following:*

(i) $\quad e_1 = (0,0,0);$ $\quad (ii)$ $\quad e_2 = (\dfrac{a}{\lambda - a_1}, \dfrac{b}{\lambda - a_2}, \dfrac{c}{\lambda - a_3})$ *for* $\lambda \in \mathbf{R} \backslash \{a_1, a_2, a_3\}$;

(iii) $\quad e_3 = (\alpha, -\dfrac{b}{a_2 - a_1}, \dfrac{c}{a_1 - a_3})$ *for* $\alpha \in \mathbf{R}$, *if* $a = 0$;

(iv) $\quad e_4 = (\dfrac{a}{a_2 - a_1}, \alpha, -\dfrac{c}{a_3 - a_2})$ *for* $\alpha \in \mathbf{R}$, *if* $b = 0$;

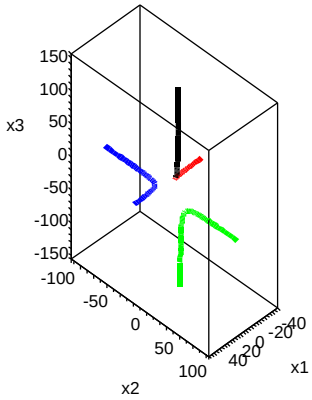
(v) $\quad e_5 = (-\dfrac{a}{a_1 - a_2}, \dfrac{b}{a_3 - a_2}, \alpha)$ *for* $\alpha \in \mathbf{R}$, *if* $c = 0$.

By Theorem 4.1, the equilibrium points of the ε- revised system (17) are $e_1, ..., e_5$ indicated in the Proposition 4.1.

In the sequel, we give a graphic representation of the set of equilibrium points in various cases.

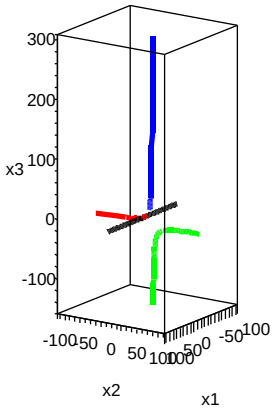
I. In this case, we consider the constants:

$$a_1 = 0.2, \quad a_2 = 0.4, \quad a_3 = 0.6 \quad \text{and} \quad a = 1, \quad b = 2, \quad c = 3.$$



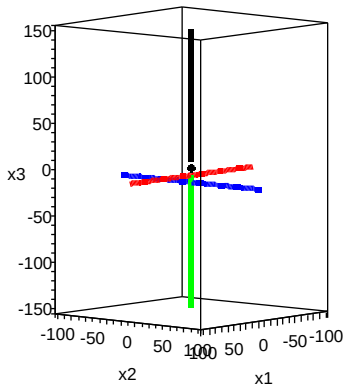
II. In this case, the constants are:

$$a_1 = 0.2, \quad a_2 = 0.4, \quad a_3 = 0.6 \quad \text{and} \quad a = 0, \quad b = 2, \quad c = 3.$$



III. In this case, the constants are:

$$a_1 = 0.2, \quad a_2 = 0.4, \quad a_3 = 0.6 \quad \text{and} \quad a = 0, \quad b = 0, \quad c = 3.$$



It is well-known that the dynamics (11) have the first integrals H and C given by (13) and (15). These first integrals may be written thus:

$$H(x^1, x^2, x^3) = \frac{1}{2}\mathbf{x} \cdot \mathbf{I}^{-1}\mathbf{x} + \mathbf{a} \cdot \mathbf{x} \quad \text{and} \quad C(x^1, x^2, x^3) = \frac{1}{2}\mathbf{x}^2 \tag{24}$$

where $\mathbf{I}$ is inertia tensor and $\mathbf{I}^{-1}$ is its inverse. We have:

$$\frac{dH}{dt}(\mathbf{x}) = \mathbf{m}(\mathbf{x}) \cdot \dot{\mathbf{x}} \quad \text{and} \quad \frac{dC}{dt}(\mathbf{x}) = \mathbf{x} \cdot \dot{\mathbf{x}}. \tag{25}$$

Theorem 4.2. (i) *For each $\varepsilon \in \mathbf{R}$, the function H given by* (13) *is a first integral for the ε- revised system* (17).

(ii) *If $\mathbf{x} : \mathbf{R} \to \mathbf{R}^3$ is a solution of the ε- revised system, then:*

$$\frac{d}{dt}(\frac{1}{2}\mathbf{x}^2) = -\varepsilon(\mathbf{x} \times \mathbf{m}(\mathbf{x}))^2. \tag{26}$$

(iii) *For $\varepsilon \in \mathbf{R}^*$, the function C is not a first integral for the ε- revised system.*

Proof. (i) Multiplying the relation (21) with the vector $\mathbf{m}(\mathbf{x})$, we have: $\mathbf{m}(\mathbf{x}) \cdot \dot{\mathbf{x}} = \mathbf{m}(\mathbf{x}) \cdot (\mathbf{x} \times \mathbf{m}(\mathbf{x})) + \varepsilon \mathbf{m}(\mathbf{x}) \cdot [(\mathbf{x} \times \mathbf{m}(\mathbf{x})) \times \mathbf{m}(\mathbf{x})] = \mathbf{0}$. Applying now (25), we obtain $\dfrac{dH}{dt} = \mathbf{m}(\mathbf{x}) \cdot \dot{\mathbf{x}} = \mathbf{0}$. Hence H is a first integral for the system (17).

(ii) Multiplying the relation (21) with the vector $\mathbf{x}$, we have $\mathbf{x} \cdot \dot{\mathbf{x}} = \mathbf{x} \cdot (\mathbf{x} \times \mathbf{m}(\mathbf{x})) + \varepsilon \mathbf{x} \cdot [(\mathbf{x} \times \mathbf{m}(\mathbf{x})) \times \mathbf{m}(\mathbf{x})] = -\varepsilon \mathbf{x} \cdot [\mathbf{m}(\mathbf{x}) \times (\mathbf{x} \times \mathbf{m}(\mathbf{x}))]$.

Using now the equality (18), we obtain $\mathbf{x} \cdot \dot{\mathbf{x}} = -\varepsilon(\mathbf{x} \times \mathbf{m}(\mathbf{x}))^2$. Then, we have $\dfrac{d}{dt}(\dfrac{1}{2}\mathbf{x}^2) = \mathbf{x} \cdot \dot{\mathbf{x}} = -(\mathbf{x} \times \mathbf{m}(\mathbf{x}))^2$.

(iii) This assertion follows from the second relation of (25) and (ii). □

Remark 4.1. *The function H given by* (13) *can be put in the equivalent form:*

$$H(x^1,x^2,x^3)=\frac{1}{2}[a_1(x^1+\frac{a}{a_1})^2+a_2(x^2+\frac{b}{a_2})^2+a_3(x^3+\frac{c}{a_3})^2]-\frac{1}{2}(\frac{a^2}{a_1}+\frac{b^2}{a_2}+\frac{c^2}{a_3}). \quad (27)$$

For a given $k\in\mathbf{R}$, the geometrical image of the surface $H(x^1,x^2,x^3)=k$, is an ellipsoid since $a_1>0,a_2>0,a_3>0$. □

Proposition 4.2. *The set of equilibrium points which belong to the ellipsoid $H(x^1,x^2,x^3)=k$ is finite.*

Proof. Following the description given in Proposition 4.1, we remark that:

(*i*) the points e_3 (similarly, e_4 and e_5) make a straight line; the intersection between a straight line and an ellipsoid have at most two points; we deduce that on the chosen ellipsoid there exist at most two points of the form e_3.

(*ii*) the points e_2 can be obtained by solving with respect λ, the equation:

$$\frac{1}{2}[a_1(\frac{a}{\lambda-a_1})^2+a_2(\frac{b}{\lambda-a_2})^2+a_3(\frac{c}{\lambda-a_3})^2]+\frac{a^2}{\lambda-a_1}+\frac{b^2}{\lambda-a_2}+\frac{c^2}{\lambda-a_3}=k.$$

The above equation is equivalent with the determination of roots of a polynomial of degree at most 6; therefore on the chosen ellipsoid there exist at most 6 equilibrium points of the form e_2. □

5 The behavior of the solutions of the ε - revised system

Theorem 5.1.*(i) The solutions of the ε-revised system are bounded.*

(ii) The maximal solutions of the ε-revised system are globally solutions.

Proof.(i) Given a solution of (17), there exists k such that its trajectory lie on the ellipsoid $H(x^1,x^2,x^3)=k$. From this, deduce that all solutions are bounded.

(ii) Let $\mathbf{x}:(m,M)\subset\mathbf{R}\to\mathbf{R}^3$ be a maximal solution. We assume that $\mathbf{x}$ is not globally. It follows $m>-\infty$ or $M<\infty$. In these situations, we known that there exists $k\in\mathbf{R}$ such that $H(x^1,x^2,x^3)=k$ for all $t\in\mathbf{R}$ and the graph of the solution is contained in a compact domain. According with [6] (theorem 3.2.5, p.141) we obtain a contradiction with the fact that $\mathbf{x}$ admit a prolongation on the right or the left (also, can be applied the theorem of Chilingworth, see theorem 1.0.3, p.7 in [3]). □

In the sequel we study the asymptotic behavior of the globally solutions of (17).

Denote by $\mathbf{E}$ the set of equilibrium points of the ε-revised system (17) and by Γ the trajectory of a solution $\mathbf{x}:\mathbf{R}\to\mathbf{R}^3$ of (17). By theory of differential equations (see [10] p. 174-176), the ω-limit set and α-limit set of Γ are:

$$\omega(\Gamma)=\{\mathbf{y}\in\mathbf{R}^3\mid\exists t_n\to\infty,\ \mathbf{x}(t_n)\to\mathbf{y}\};\quad \alpha(\Gamma)=\{\mathbf{z}\in\mathbf{R}^3\mid\exists t_n\to-\infty,\ \mathbf{x}(t_n)\to\mathbf{z}\}.$$

Theorem 5.2. *Let $\boldsymbol{x} : \mathbf{R} \to \mathbf{R}^3$ be a solution of the ε-revised system with $\varepsilon \neq 0$. There exist the equilibrium points $\boldsymbol{x}_m, \boldsymbol{x}_M \in \mathbf{R}^3$ of (17) such that $\lim_{t\to-\infty} \boldsymbol{x}(t) = \boldsymbol{x}_M$ and $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_m$.*

Proof. The theorem is proved in the following steps:

(i) $\alpha(\Gamma) \neq \emptyset,\ \omega(\Gamma) \neq \emptyset$; (ii) $\alpha(\Gamma) \bigcap \omega(\Gamma) \subset \mathbf{E}$; (iii) $card(\alpha(\Gamma)) = card(\omega(\Gamma)) = 1$.

Taking account into that each solution is bounded (hence it is contained in a compact domain) and applying theorem 1, p. 175 in [10], we obtain the assertions (i).

(ii) For demonstration consider the case when $\varepsilon > 0$. Using the relation (26), we deduce that the function $t \to \mathbf{x}^2(t)$ is a strictly decreasing function. Being bounded it follows that there exists $\lim_{t\to\infty} \mathbf{x}^2(t) = L$ and L is finite.

For each $\mathbf{y} \in \omega(\Gamma)$ there exists the sequence $t_n \to \infty$ such that $\mathbf{x}(t_n) \to \mathbf{y}$. Then $\mathbf{x}^2(t_n) \to \mathbf{y}^2$ and hence $\mathbf{y}^2 = L$.

By theorem 2, p.176 in [10], we have that the trajectory $\Gamma_{\mathbf{y}}$ of the solution $\mathbf{x}_{\mathbf{y}}$ which verifies the initial condition $\mathbf{x}_{\mathbf{y}}(0) = \mathbf{y}$, satisfies the relation $\Gamma_{\mathbf{y}} \subset \omega(\Gamma)$.

If we assume that $\mathbf{y}$ is not an equilibrium point, then we deduce (using (26)) that for $t > 0$ we have $\mathbf{x}^2_{\mathbf{y}}(t) < L$ and this is in contradiction with the above result. Therefore, we have $\omega(\Gamma) \subset \mathbf{E}$. Similarly, we prove that $\alpha(\Gamma) \subset \mathbf{E}$. Hence the assertion (ii) holds.

The case $\varepsilon < 0$ is similar.

(iii) There exists a constant k such that the sets $\alpha(\Gamma)$ and $\omega(\Gamma)$ are included in the ellipsoid $H(x^1, x^2, x^3) = k$. By (ii), we deduce that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are included in the set of equilibrium points which lies of the above ellipsoid. On the other hand, applying Proposition 4.2 and using the fact that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are connected (see theorem 1, p.175 in [10]), we obtain that $\alpha(\Gamma)$ and $\omega(\Gamma)$ are formed by only one element. □

Remark 5.1. *Using (26) it is easy to observe that the following assertions hold:*

(i) if $\varepsilon > 0$ then $\boldsymbol{x}^2_M > \boldsymbol{x}^2_m$; (ii) if $\varepsilon < 0$ then $\boldsymbol{x}^2_M < \boldsymbol{x}^2_m$. □

As an immediate consequence we obtain the following theorem.

Theorem 5.3. *If $\varepsilon \neq 0$, then for each solution $\mathbf{x} : \mathbf{R} \to \mathbf{R}^3$ of (17), we have:*

(i) *if* $t \to \infty \Rightarrow d(\mathbf{x}(t), \mathbf{E}) \to 0$; (ii) *if* $t \to -\infty \Rightarrow d(\mathbf{x}(t), \mathbf{E}) \to 0$. □

Remark 5.2. From Theorem 5.3 follows that the set $\mathbf{E}$ is an attracting set (see definition 2, p.178 in [10]) and also is a reppeling set (see [3], p.34). Thus, the space $\mathbf{R}^3$ is simultaneously a domain of attraction and a domain of repulsion of $\mathbf{E}$. □

6 The Lyapunov stability of equilibrium points of the ε - revised system in the case $\varepsilon > 0$

The stability of the point $e_1 = (0, 0, 0)$. We have the following results.

Theorem 6.1. *The equilibrium point e_1 is Lyapunov stable.*

Proof. Let $\gamma > 0$, $t_0 \in \mathbf{R}$ and $\mathbf{x}_0 \in \mathbf{R}^3$ such that $|\mathbf{x}_0| < \gamma$, where $|\cdot|$ denotes the euclidian norm in $\mathbf{R}^3$. Denote by $t \to \mathbf{x}(t, t_0, \mathbf{x}_0)$ the solution of the ε-revised system which verifies the initial condition $\mathbf{x}(0, t_0, \mathbf{x}_0) = \mathbf{x}_0$.

Using the relation $|\mathbf{x}(t, t_0, \mathbf{x}_0)| = \sqrt{\mathbf{x}^2(t, t_0, \mathbf{x}_0)}$ and according with the relation (26), we observe that the function $t \to \mathbf{x}(t, t_0, \mathbf{x}_0)$ is a decreasing function and hence we have:

$$|\mathbf{x}(t, t_0, \mathbf{x}_0)| \leq |\mathbf{x}_0| < \gamma \quad \texttt{for} \quad t > t_0.$$

Then (see [4], p.22) we have that e_1 is a Lyapunov stable equilibrium point. □

Remark 6.1. *The equilibrium point e_1 is not asymptotical stable.*

Indeed, if $a = b = c = 0$ then the coordinates axis are formed from equilibrium points. If al least of one of the numbers a, b, c is non null, then:

$$\texttt{if} \quad |\lambda| \to \infty \Rightarrow \left(\frac{a}{\lambda - a_1}, \frac{b}{\lambda - a_2}, \frac{c}{\lambda - a_3}\right) \to (0, 0, 0).$$

Hence, in all neighbourhood of e_1 there exist an infinity of equilibrium points. □

The stability of the point $\overline{\mathbf{x}_0} = \left(-\frac{a}{a_1}, -\frac{b}{a_2}, -\frac{c}{a_3}\right)$. The equilibrium point $\overline{\mathbf{x}_0}$ is an equilibrium point of the form e_2 and it is obtained for $\lambda = 0$.

Theorem 6.2. *The equilibrium point $\overline{\boldsymbol{x}_0}$ is Lyapunov stable.*

Proof. Using the relation (27) and the inequality $0 < a_1 < a_2 < a_3$, we deduce:

$$\frac{a_1}{2}|\mathbf{x} - \overline{\mathbf{x}_0}| \leq H(\mathbf{x}) - H(\overline{\mathbf{x}_0}) \leq \frac{a_3}{2}|\mathbf{x} - \overline{\mathbf{x}_0}|$$

For $t_0 \in \mathbf{R}$ and $\mathbf{x}_0 \in \mathbf{R}^3$ denote with $\mathbf{x}(t, t_0, \mathbf{x}_0)$ a solution of ε-revised system which verifies the initial condition $\mathbf{x}(0, t_0, \mathbf{x}_0) = \mathbf{x}_0$.

Let $\gamma > 0$ and $\delta(\gamma) = \frac{2\gamma}{a_1}$. Let $\mathbf{x}_0 \in \mathbf{R}^3$ such that:

$$H(\mathbf{x}_0) - H(\overline{\mathbf{x}_0}) \leq \delta(\gamma).$$

From the fact that H is a first integral we deduce that:

$$H(\mathbf{x}(t, t_0, \mathbf{x}_0) - H(\overline{\mathbf{x}_0}) = H(\mathbf{x}_0) - H(\overline{\mathbf{x}_0}).$$

Hence for all $t \in \mathbf{R}$, we have that $\frac{a_1}{2}|\mathbf{x} - \overline{\mathbf{x}_0}| \leq \delta(\gamma)$ and $\overline{\mathbf{x}_0}$ is Lyapunov stable. □

Remark 6.2. *The equilibrium point $\overline{\boldsymbol{x}_0}$ realizes the absolute minimum of H.* □

The unstability of equilibrium points of the form e_2 with $\lambda \in (0, a_1)$. For the demonstration of this results we use the Theorem 6.3 and Lemma 6.1.

Theorem 6.3. *If $\boldsymbol{x}_0 \in \mathbf{E}$ such that there exists $\boldsymbol{y} \in \mathbf{E}$ with the properties:*

$$(i) \quad H(\boldsymbol{y}) = H(\boldsymbol{x}_0) \quad \textit{and} \quad (ii) \quad |\boldsymbol{y}| < |\boldsymbol{x}_0|$$

then $\boldsymbol{x}_0$ *is an unstable equilibrium point.*

Proof. For $k \in \mathbf{R}$ denote by $\mathbf{E}_k = \{\mathbf{x} \in \mathbf{E} \,/\, H(x) = k\}$. The set $\mathbf{E}_{H(\mathbf{x}_0)}$ is finite (by Proposition 4.2). We denote:

$$\gamma_0 = \min\{|\mathbf{x} - \mathbf{x}_0| \mid \mathbf{x} \in E_{H(\mathbf{x}_0)} \setminus \{\mathbf{x}_0\}\}.$$

Let $\mathbf{z} \in \mathbf{R}^3$ such that $H(\mathbf{z}) = H(\mathbf{x}_0)$ and $|\mathbf{z}| < |\mathbf{x}_0|$. Then $\lim_{t\to\infty} \mathbf{x}(t, 0, \mathbf{z}) \in \mathbf{E}$ and $\texttt{if } t > 0 \Rightarrow |\mathbf{x}(t, 0, \mathbf{z})| < |\mathbf{z}|$. We deduce that there exists $t_z > 0$ such that $|\mathbf{x}(t, 0, \mathbf{z}) - \mathbf{x}_0| > \frac{\gamma_0}{2}$ $\texttt{if } t > t_z$. It follows that $\mathbf{x}_0$ is unstable. □

We assume that $(a, b, c) \neq (0, 0, 0)$ and we introduce the notation:

$$e_{2\lambda} = (\frac{a}{\lambda - a_1}, \frac{b}{\lambda - a_2}, \frac{c}{\lambda - a_3}) \text{ for all } \lambda \in \mathbf{R} - \{a_1, a_2, a_3\}.$$

Lemma 6.1. *(i) If* $\sigma < \mu < a_1$, *then* $|e_{2\sigma}| < |e_{2\mu}|$.
(ii) If $\sigma, \mu > 0$, $(\frac{\mu}{\sigma})^2 > \frac{a_3}{a_1}$ *and* $H(e_{2\sigma}) = H(e_{2\mu})$, *then* $|e_{2\sigma}| > |e_{2\mu}|$.
(iii) If $0 < \sigma < a_1 < a_3 < \mu$ *and* $H(e_{2\sigma}) = H(e_{2\mu})$, *then* $|e_{2\sigma}| > |e_{2\mu}|$.

Proof. (i) Consider the function $g : (-\infty, a_1) \to \mathbf{R}$ given by:

$$g(\lambda) = (\frac{a}{\lambda - a_1})^2 + (\frac{b}{\lambda - a_2})^2 + (\frac{c}{\lambda - a_3})^2.$$

The derivative of the function g is:

$$g'(\lambda) = -\frac{2a^2}{(\lambda - a_1)^3} - \frac{2b^2}{(\lambda - a_2)^3} - \frac{2c^2}{(\lambda - a_3)^3}$$

We observe that $g'(\lambda) > 0$. Then g is a strictly increasing function. We have $g(\sigma) = |e_{2\sigma}|^2$, $g(\mu) = |e_{2\mu}|^2$ and we obtain the desired result.

(ii) From hypothesis $H(e_{2\sigma}) = H(e_{2\mu})$ follows that there exists a constant $q > 0$ with the following properties:

$$\frac{1}{a_1}\frac{a^2}{(\sigma - a_1)^2} + \frac{1}{a_2}\frac{b^2}{(\sigma - a_2)^2} + \frac{1}{a_3}\frac{c^2}{(\sigma - a_3)^2} = \frac{q}{\sigma^2}$$

$$\frac{1}{a_1}\frac{a^2}{(\mu - a_1)^2} + \frac{1}{a_2}\frac{b^2}{(\mu - a_2)^2} + \frac{1}{a_3}\frac{c^2}{(\mu - a_3)^2} = \frac{q}{\mu^2}.$$

Using $a_1 < a_2 < a_3$, we obtain $|e_{2\sigma}|^2 > \frac{a_1 q}{\sigma}$, $|e_{2\mu}|^2 < \frac{a_3 q}{\mu}$ and (ii) holds.
(iii) This assertion follows immediately from (ii). □

Theorem 6.4. *The equilibrium point* $e_{2\lambda}$ *with* $0 < \lambda < a_1$ *is unstable.*

Proof. Let the function $h : (-\infty, a_1) \bigcup (a_3, \infty) \to \mathbf{R}$ given by $h(\sigma) = H(e_{2\sigma})$. Using the relation (27) for H, we find:

$$h(\sigma) = \frac{\sigma^2}{2}[\frac{a^2}{a_1(\sigma - a_1)^2} + \frac{b^2}{a_2(\sigma - a_2)^2} + \frac{c^2}{a_3(\sigma - a_3)^2}] - \frac{1}{2}(\frac{a^2}{a_1} + \frac{b^2}{a_2} + \frac{c^2}{a_3}).$$

The function h have the following properties:
(i) 0 is an absolute minimum point;
(ii) $\lim_{\sigma\to-\infty} h(\sigma) = \lim_{\sigma\to\infty} h(\sigma) = 0$; (iii) $\lim_{\sigma\to a_1} h(\sigma) = \lim_{\sigma\to a_3} h(\sigma) = \infty$;
(iv) h is strictly decreasing on $(-\infty, 0)$, strictly increasing on $(0, a_1)$ and strictly decreasing on (a_3, ∞).

The demonstrations divided on three cases.

(I) Assume that $h(\lambda) < 0$. Then there exists $\sigma < 0 < \lambda < a_1$ such that $h(\lambda) = h(\sigma)$ and imply $H(e_{2\lambda}) = H(e_{2\sigma})$. Hence $e_{2\lambda}$ and $e_{2\sigma}$ belong to same ellipsoid. On the other hand, by Lemma 6.1 (ii), follows $|e_{2\sigma}| < |e_{2\lambda}|$. Applying now Theorem 6.3, deduce that $e_{2\lambda}$ is an unstable equilibrium point.

(II) Assume that $h(\lambda) = 0$ we have $H(e_{2\lambda}) = H(0,0,0)$ and it is clearly that $|(0,0,0)| < |e_{2\lambda}|$. By Theorem 6.3 we find the desired result.

(III) Assume that $h(\lambda) > 0$. Then there exists $\sigma > a_3$ such that $h(\lambda) = h(\sigma)$ and hence $H(e_{2\lambda}) = H(e_{2\sigma})$. Applying Lemma 6.1 (iii) follows $|e_{2\lambda}| > |e_{2\sigma}|$ and by Theorem 6.3 we deduce that $e_{2\lambda}$ is unstable. □

The stability of equilibrium points of the form e_2 with $\lambda < 0$.

Theorem 6.5. *For $\lambda < 0$, the equilibrium points e_2 are Lyapunov stables.*

Proof. Let $\lambda < 0$ and the equilibrium point $\mathbf{x}_0 = (\frac{a}{\lambda-a_1}, \frac{b}{\lambda-a_2}, \frac{c}{\lambda-a_3})$ of the form e_2. It is well-known that the study of stability of $\mathbf{x}_0$ in the Lyapunov sense is equivalent with the study of stability of the null solution $(0,0,0)$ for the differential system obtained from the ε-revised system by transformation of variables $\mathbf{z} = \mathbf{x} - \mathbf{x}_0$.

The system obtained in this manner is called the *perturbed ε-revised system.*

Consider the function $K : \mathbf{R}^3 \to \mathbf{R}$ given by

$$K(\mathbf{z}) = \frac{1}{2}\mathbf{z}\cdot\mathbf{I}^{-1}\mathbf{z} - \frac{\lambda}{2}\mathbf{z}^2.$$

Since the tensor $\mathbf{I}^{-1}$ is strictly positive definite and $\lambda < 0$ we obtain that K is a quadratic form strictly positive definite.

We prove that if $\mathbf{z} : \mathbf{R} \to \mathbf{R}^3$ is a solution for the perturbed ε-revised system, then:

$$\frac{d}{dt}K(\mathbf{z}(t)) < 0.$$

By a direct computation and taking account into the relations (24) we have:

$$K(\mathbf{z}) = H(\mathbf{x}) - \lambda C(\mathbf{x}) - \frac{1}{2}\mathbf{x}_0\cdot\mathbf{I}^{-1}\mathbf{x}_0 - \mathbf{a}\cdot\mathbf{x}_0 + \frac{\lambda}{2}\mathbf{x}_0^2.$$

Applying now Theorem 4.2, we obtain $\frac{d}{dt}K(\mathbf{z}(t)) = \varepsilon\lambda(\mathbf{x}\times\mathbf{m}(\mathbf{x}))^2$ and follows that $\frac{d}{dt}K(\mathbf{z}(t)) < 0$, since $\varepsilon > 0,\ \lambda < 0$.

It is easy to see that $K^*(t) = K(\mathbf{z}(t))$ is a strictly decreasing function. By theorem 1.1, p.21 in the paper [4] we deduce that $\mathbf{x}_0$ is Lyapunov stable. □

Conclusion - the stability problem for the Hamilton Poisson system (11) **versus the ε- revised system** (17) **with** $\varepsilon > 0$

Concerning to the equilibrium points of the system (11) are established the following results (see, theorem 1.1, [12]):

(i) e_1, e_3 *and* e_5 *are Lyapunov stables;*

(ii) e_2 *are Lyapunov stables for* $\lambda \in (-\infty, a_1) \cup (a_3, \infty)$; (iii) e_4 *are unstables.*

For the stability of equilibrium points of the ε- revised system (17) with $\varepsilon > 0$ have proved the following assertions:

(1) e_1 *is Lyapunov stable;*

(2) e_2 *are Lyapunov stables if* $\lambda \leq 0$ *and unstables if* $0 < \lambda < a_1$.

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SOME RESULTS ON THE STABILITY OF QUASI-LINEAR DYNAMIC SYSTEMS ON TIME SCALES

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Abstract. It is shown that we study some results on the stability of Quasi-linear dynamic systems on time scales characterized completely by its linear part.

1. Introduction

The theory of dynamic equations on time scales (aka measure chains) was introduced by Hilger [6] with the motivation of providing a unified approach to continuous and discrete analysis. The generalized derivative or Hilger derivative $f^{\triangle}(t)$ of a function $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is a so-called "time scale" (an arbitrary closed nonempty subset of $\mathbb{R}$) becomes the usual derivative when $\mathbb{T} = \mathbb{R}$, that is $f^{\triangle}(t) = f'(t)$. On the other hand, if $\mathbb{T} = \mathbb{Z}$, then $f^{\triangle}(t)$ reduces to the usual forward difference, that is $f^{\triangle}(t) = \triangle f(t)$. This theory not only brought equations leading to new applications. Also, this theory allows one to get some insight into and better understanding of the subtle differences between discrete and continuous systems [1, 3].

DaCunha [4] have introduced stability for time varying linear dynamic systems on time scales. He introduced the unified theorems of uniform stability and uniform exponential stability of linear systems on time scales, as well as illustrations of these theorems in examples, and demonstrated how the quadratic Lyapunov function developed, it can also be used to determine instability of a system.

In this paper, we study some results on the stability of Quasi-linear dynamic systems on time scales characterized completely by its linear part.

Now, first we mention without proof several foundational definitions and result from the calculus on time scales in an excellent introductory text by Bohner and Peterson [2, 3].

2. General Definitions

A *time scale* $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$. Thus time scales can be any of the usual integer subsets (e.g. $\mathbb{Z}$ or $\mathbb{N}$), the entire real line $\mathbb{R}$, or any combination of discrete points unioned with continuous intervals. The majority of research on time scales so far has focused on expanding and generalizing the vast suite of tools available to the differential and difference equation theorist. We now briefly outline the portions of the time scales theory that are needed for this paper to be as self-contained as is practically possible.

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The *forward jump operator* of $\mathbb{T}$, $\sigma(t) : \mathbb{T} \to \mathbb{T}$, is given by $\sigma(t) = \inf_{s\in\mathbb{T}} \{s > t\}$. The *backward jump operator* of $\mathbb{T}$, $\rho(t) : \mathbb{T} \to \mathbb{T}$, is given by $\rho(t) = \inf_{s\in\mathbb{T}} \{s < t\}$. The *graininess function* $\mu(t) : \mathbb{T} \to [0,\infty)$ is given by $\mu(t) = \sigma(t) - t$. Here we adopt the conventions $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(t) = t$ if $\mathbb{T}$ has a maximum element t), and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(t) = t$ if $\mathbb{T}$ has a minimum element t). For notational purposes, the intersection of a real interval $[a, b]$ with a time scale $\mathbb{T}$ is denoted by $[a, b] \cap \mathbb{T}$: $[a, b]_{\mathbb{T}}$.

A point $t \in \mathbb{T}$ is *right-scattered* if $\sigma(t) > t$ and *right dense* if $\sigma(t) = t$. A point $t \in \mathbb{T}$ is *left-scattered* if $\rho(t) < t$ and *left dense* if $\rho(t) = t$ If t is both left-scattered and right-scattered, we say t is *isolated.* If t is both left-dense and right-dense, we say t is dense. The set $\mathbb{T}^k$ is defined as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}$ -$\{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then the composition $f(\sigma(t))$ is often denoted by $f^\sigma(t)$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, define $f^\triangle(t)$ as the number (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of f such that

$$\left|[f(\sigma(t)) - f(s))] - f^\triangle(t)\,[\sigma(t) - s]\right| \le \varepsilon\,|\sigma(t) - s|\,, \qquad \forall s \in U.$$

The $f^\triangle : \mathbb{T}^k \to \mathbb{R}$ is called the *delta derivative* or the *Hilger derivative* of f on $\mathbb{T}^k$. We say f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}^k$.

The following theorem establishes several important observations regarding delta derivatives.

Theorem 1. *Suppose $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$.*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is delta differentiable at t and $f^\triangle(t) = \frac{f^\sigma(t)-f(t)}{\mu(t)}$.

(iii) If t is right-dense, then f is delta differentiable at t if and only if $\lim_{t\to s} \frac{f(t)-f(s)}{t-s}$ exists. In this case, $f^\triangle(t) = \lim_{t\to s} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t) = f(t) + \mu(t)f^\triangle(t)$.

2.1. **Generalized exponential Functions.** The function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$, and this concept motivates the definition of the following sets:

$\Re = \left\{p : \mathbb{T} \to \mathbb{R} : p \in C_{rd}(\mathbb{T}) \text{ and } 1 + \mu(t)p(t) \neq 0\ \forall t \in \mathbb{T}^k\right\}$,

$\Re^+ = \left\{p \in \Re : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^k\right\}$.

The function $p : \mathbb{T} \to \mathbb{R}$ is *uniformly regressive* on $\mathbb{T}$ there exists a positive constant δ such that $0 < \delta^{-1} \le |1 + \mu(t)p(t)|\,,\ t \in \mathbb{T}^k$. A matrix is regressive if and only if all of its eigenvalues are in $\Re$. Equivalently, the matrix $A(t)$ is regressive if and only if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$.

If $p \in \Re$, then we define the *generalized time scale exponential function* by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \triangle \tau\right) \quad \text{for all } s, t \in \mathbb{T}$$

The following theorem is a compilation of properties of $e_p(t, s)$ (some of which are counterintuitive) that we need in the main body of the paper.

Theorem 2. *The function $e_p(t, s)$ has the following properties:*

(i) If $p \in \Re$, then $e_p(t, r)e_p(r, s) = e_p(t, s)$ for all $r, s, t \in \mathbb{T}$.

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$.

(iii) If $p \in \Re^+$, *then* $e_p(t, t_0) > 0$ *for all* $t \in \mathbb{T}$.

(iv) If $1 + \mu(t)p(t) < 0$ *for some* $t \in \mathbb{T}^k$, *then* $e_p(t, t_0)e_p(\sigma(t), t_0) < 0$.

(v) If $\mathbb{T} = \mathbb{R}$, *then* $e_p(t, s) = e^{\int_s^t p(\tau)d\tau}$. *Moreover, If* p *is constant, then* $e_p(t, s) = e^{p(t-s)}$.

(vi) If $\mathbb{T} = \mathbb{Z}$, *then* $e_p(t, s) = \Pi_{\tau=s}^{t-1}(1 + p(\tau))$. *Moreover, If* $\mathbb{T} = h\mathbb{Z}$, *with* $h > 0$ *and* p *is constant, then* $e_p(t, s) = (1 + hp)^{\frac{(t-s)}{h}}$.

If $p \in \Re$ *and* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous, then the dynamic equation*

$$y^{\triangle}(t) = p(t)y(t) + f(t) \tag{2.1}$$

is called regressive.

Theorem 3. *If* $p, q \in \Re$, *then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$;

(iv) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;

(v) $e_p(t, s)e_p(s, \tau) = e_p(t, \tau)$;

(vi) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$;

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$;

Theorem 4. *(Variation of constants). Let* $t_0 \in \mathbb{T}$ *and* $y(t_0) = y_0 \in \mathbb{R}$.*Then the regressive IVP (2.1) has a unique solution* $y : \mathbb{T} \to \mathbb{R}^n$ *given by*

$$y(t) = y_0 e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau) \triangle \tau.$$

We say the $n \times 1$-vector-valued system

$$y^{\triangle}(t) = A(t)y(t) + f(t) \tag{2.2}$$

is *regressive* provided $A \in \Re$ and $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous vector-valued function.

Let $t_0 \in \mathbb{T}$ and assume that $A \in \Re$ is an $n \times n$-matrix-valued function. The unique matrix-valued solution to the IVP

$$Y^{\triangle}(t) = A(t)Y(t), \quad Y(t_0) = I_n \tag{2.3}$$

where I_n is the $n \times n$-identity matrix, is called the *transition matrix* and it is denoted by $\Phi_A(t, t_0)$.

In this paper, we denote the solution to (2.3) as $\Phi_A(t, t_0)$ when $A(t)$ is time varying and denote the solution as $e_A(t, t_0) \equiv \Phi_A(t, t_0)$ (the *matrix exponential*, as in [3]) only when $A(t) \equiv A$ is a constant matrix. Also, if $A(t)$ is a function on $\mathbb{T}$ and the time scale matrix exponential function is a function on some other time scale $\mathbb{S}$, then $A(t)$ is constant with respect to $e_{A(t)}(\tau, s)$, for all $\tau, s \in \mathbb{S}$ and $t \in \mathbb{T}$. The following lemma lists some properties of the transition matrix.

Theorem 5. *Suppose* $A, B \in \Re$ *are matrix-valued functions on* $\mathbb{T}$.

(i) Then the semigroup property $\Phi_A(t, r)\Phi_A(r, s) = \Phi_A(t, s)$ *is satisfies for all* $r, s, t \in \mathbb{T}$.

(ii) $\Phi_A(\sigma(t), s) = (1 + \mu(t)p(t))\Phi_A(t, s)$.

(iii) If $\mathbb{T}=\mathbb{R}$ *and* A *is constant, then* $\Phi_A(t,s)=e_A(t,s)=e^{A(t-s)}$.

(iv) If $\mathbb{T}=h\mathbb{Z}$, *with* $h>0$ *and* A *is constant, then* $\Phi_A(t,s)=e_A(t,s)=(1+hp)^{\frac{(t-s)}{h}}$.

We now present a theorem that guarantees a unique solution to the regressive $n\times 1$-vector-valued dynamic IVP (2.2).

Theorem 6. *(Variation of constants). Let* $t_0\in\mathbb{T}$ *and* $y(t_0)=y_0\in\mathbb{R}^n$. *Then the regressive IVP (2.2) has a unique solution* $y:\mathbb{T}\to\mathbb{R}^n$ *given by*

$$y(t)=y_0\Phi_A(t,t_0)+\int_{t_0}^{t}\Phi_A(t,\sigma(\tau))f(\tau)\triangle\tau. \tag{2.4}$$

Beside the calculus on the time scales, we talk about several foundational definitions and result from the stability for time varying linear dynamic systems on time scales in an excellent introductory text by DaCunha [4].

3. Stability

We now define the concepts of uniform stability and uniform exponential stability. These two concepts involve the boundedness of the solutions of the regressive time varying linear dynamic equation

$$x^{\triangle}(t)=A(t)x(t),\qquad x(t_0)=x_0,\quad t_0\in\mathbb{T}. \tag{3.1}$$

Definition 1. *The time varying linear dynamic equation (3.1) is uniformly stable if there exists a finite constant* $K>0$ *such that for any* t_0 *and* $x(t_0)$, *the corresponding solution satisfies*

$$\|x(t)\|\le K\|x(t_0)\|,\quad t\ge t_0. \tag{3.2}$$

For the next definition, we define a stability property that not only concerns the boundedness of a solution to (3.1), but also the asymptotic characteristics of the solutions as well. If the solution to (3.1) posses the following stability property, then the solution approach zero exponentially as $t\to\infty$ (i.e. the norms of the solutions are bounded above by a decaying exponential function).

Definition 2. *The time varying linear dynamic equation (3.1) is called uniformly exponentially stable if there exists constants* $K,\alpha>0$ *with* $-\alpha\in\Re^+$ *such that for any* t_0 *and* $x(t_0)$, *the corresponding solution satisfies*

$$\|x(t)\|\le K\|x(t_0)\|e_{-\alpha}(t,t_0),\quad t\ge t_0. \tag{3.3}$$

It is obvious by inspection of the previous definitions that we must have $K\ge 1$. By using the word uniform, it is implied that the choice of K does not depend on the initial time t_0.

The last stability definition given uses a uniformity condition to conclude exponential stability.

Definition 3. *The linear state equation (3.1) is defined to be uniformly asymptotically stable if it is uniformly stable and given any* $\delta>0$, *there exists a* $T>0$ *so that for any* t_0 *and* $x(t_0)$, *the corresponding solution* $x(t)$ *satisfies*

$$\|x(t)\|\le\delta\|x(t_0)\|,\quad t\ge t_0+T. \tag{3.4}$$

It is noted that the time $\mathbb{T}$ that must pass before the norm of the solution satisfies (3.4) and the constant $\delta > 0$ is independent of the initial time t_0.

Theorem 7. *The time varying linear dynamic equation (3.1) is uniformly stable if and only if there exists a $K > 0$ such that*
$\|\Phi_A(t, t_0)\| \leq K$
for all $t \geq t_0$, with $t, t_0 \in \mathbb{T}$.

Theorem 8. *The time varying linear dynamic equation (3.1) is uniformly exponentially stable if and only if there exists $K, \alpha > 0$ with $-\alpha \in \Re^+$ such that*
$\|\Phi_A(t, t_0)\| \leq K e_{-\alpha}(t, t_0)$
for all $t \geq t_0$, with $t, t_0 \in \mathbb{T}$.

3.1. **Perturbation results.** It is also useful to consider state equations that are "close" to another linear state equation that is uniformly stable. In [7, 8], as well as [9], if the stability of system (3.1) has been determined by an appropriate Lyapunov function, then certain conditions on the peturbation matrix $F(t)$ guarantee stability of the perturbed linear system

$$z^{\triangle}(t) = [A(t) + F(t)]\, z(t). \tag{3.5}$$

DaCunha [4, Theorem 5.1] obtained result about the uniform stability for the perturbed system (3.5) under the condition

$$\int_{\tau}^{\infty} \|F(s)\| \triangle s \leq \beta.$$

for some $\beta \geq 0$ in the following theorem.

Theorem 9. *Suppose the linear state equations (3.1) is uniformly stable. Then the perturbed linear dynamic equation (3.5) is uniformly stable if there exists some $\beta \geq 0$ such that for all τ*

$$\int_{\tau}^{\infty} \|F(s)\| \triangle s \leq \beta. \tag{3.6}$$

Let $\mathbb{R}^n$ be the n-dimesional space of complex column vectors. Let $|.|$ denote any norm of a vector the associated induced norm of a square matrix.

Consider the system of nonlinear ordinary differential equations

$$x^{\triangle}(t) = f(t, x), \tag{3.7}$$

where $f : [0, \infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is a rd-continuous function.

We are interested in the application of this notion to the equation

$$x^{\triangle}(t) = A(t)x(t) + g(t, x), \tag{3.8}$$

where $A \in \Re$ is an $n \times n$ matrix-valued function on $\mathbb{T}$, $g : [0, \infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ is a rd-continuous and $B_\eta(0) = \{x \in \mathbb{R}^n : \|x\| < \eta\}$ for $\eta > 0$.

Throughout the paper, we shall assume that g satisfies the following condition of *quasilinearity*

$$|g(t, x(t))| \leq \gamma \frac{\|x(t)\|}{e_\alpha(\sigma(t), t)}, \text{ for } t \geq 0, \ \left\| \frac{x(t)}{e_\alpha(\sigma(t), t)} \right\| < c, \tag{3.9}$$

DaCunha [4] showed that the solution (3.1) is exponentially stable if and only if there exists $K, \alpha > 0$ with $-\alpha \in \Re^+$ such that

$$\|\Phi_A(t, t_0)\| \leq K e_{-\alpha}(t, t_0) \tag{3.10}$$

for all $t \geq t_0$, with $t, t_0 \in \mathbb{T}$.

The following some results on the stability of the solution of the quasilinear equation (3.8) and perturbed linear equation (3.5).

4. Main Results

Theorem 10 ([10])**.** *Suppose the linear state equations (3.1) is uniformly exponentially stable. Then the perturbed linear dynamic equation (3.5) is uniformly exponentially stable if there exists some $\beta \geq 0$ and $\alpha > 0$ with $-\alpha \in \Re^+$ such that for all $\tau \in \mathbb{T}$*

$$\int_{\tau}^{\infty} \frac{\|F(s)\|}{e_{-\alpha}(\sigma(s), s)} \triangle s \leq \beta. \tag{4.1}$$

Proof. For any t_0 and $z(t_0) = z_0$, by theorem 4 the solution of (3.5) satisfies

$$z(t) = \Phi_A(t, t_0) z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s)) F(s) z(s) \triangle s, \tag{4.2}$$

where $\Phi_A(t, t_0)$ is the transition matrix for system (3.1). By the uniform exponentially stability of (3.1), there exists constants $K, \alpha > 0$ with $-\alpha \in \Re^+$ such that $\|\Phi_A(t, \tau)\| \leq K e_{-\alpha}(t, \tau)$, for all $t, \tau \in \mathbb{T}$ with $t \geq \tau$. By taking the norms of both sides of (4.2), we have

$$\|z(t)\| \leq K \|z_0\| e_{-\alpha}(t, t_0) + K \int_{t_0}^{t} e_{-\alpha}(t, \sigma(s)) \|F(s)\| \|z(s)\| \triangle s, \quad t \geq t_0.$$

Dividing by $e_{-\alpha}(t, t_0)$ on both sides, we have

$$\begin{aligned} \tfrac{\|z(t)\|}{e_{-\alpha}(t,t_0)} &\leq K \|z_0\| + K \int_{t_0}^{t} \tfrac{e_{-\alpha}(t,\sigma(s))}{e_{-\alpha}(t,t_0)} \|F(s)\| \tfrac{\|z(s)\|}{e_{-\alpha}(s,t_0)} e_{-\alpha}(s, t_0) \triangle s, \\ &\leq K \|z_0\| + K \int_{t_0}^{t} \tfrac{\|F(s)\|}{e_{-\alpha}(\sigma(s),s)} \tfrac{\|z(s)\|}{e_{-\alpha}(s,t_0)} \triangle s, \end{aligned}$$

Letting $u(t) = \frac{\|z(t)\|}{e_{-\alpha}(t,t_0)}$, we have

$$\|u(t)\| \leq K \|z_0\| + K \int_{t_0}^{t} \tfrac{\|F(s)\|}{e_{-\alpha}(\sigma(s),s)} \|u(s)\| \triangle s$$

By Gronwall's Inequality in [3] , a result in [5], and the inequality (4.1), we obtain

$$\begin{aligned} &\leq K \|z_0\| e_{K \frac{\|F(s)\|}{e_{-\alpha}(\sigma(s),s)}}(t, t_0) \\ &\leq K \|z_0\| \exp\left(\int_{t_0}^{t} \tfrac{Log(1+\mu(s) K \frac{\|F(s)\|}{e_{-\alpha}(\sigma(s),s)})}{\mu(s)} \triangle s \right) \end{aligned}$$

$$\leq K\|z_0\| \exp\left(\int_{t_0}^{\infty} \frac{Log(1+\mu(s)K\frac{\|F(s)\|}{e_{-\alpha}(\sigma(s),s)})}{\mu(s)} \triangle s\right)$$
$$\leq K\|z_0\| \exp\left(K\int_{t_0}^{\infty} \frac{\|F(s)\|}{e_{-\alpha}(\sigma(s),s)} \triangle s\right)$$
$$\leq K\|z_0\| e^{K\beta}, \qquad t \geq t_0.$$

Thus

$\|z(t)\| \leq \gamma_1 \|z_0\| e_{-\alpha}(t,t_0) \qquad t \geq t_0.$

where $\gamma_1 = Ke^{K\beta}$. Hence the state equation (3.5) is uniformly exponentially stable. □

Theorem 11. *Suppose that the linear state equations (3.1) is uniformly exponentially stable and satisfies below conditions*

(i) $|g(t,x)| \leq r(t)x(t)$ *for some positive function* $r : \mathbb{T} \to \mathbb{R}$ *and for all* $t \in \mathbb{T}$,

(ii) $\int_{\tau}^{\infty} \frac{\|r(s)\|}{e_{-\alpha}(\sigma(s),s)} \triangle s \leq \beta < \infty, \quad t \geq t_0 \geq 0.$

Then the dynamic equation (3.8) is uniformly exponentially stable if there exists some $\beta \geq 0$ *and* $\alpha > 0$ *with* $-\alpha \in \Re^+$ *such that for all* $\tau \in \mathbb{T}$.

Proof. it is clear that we can have the same theorem 10. □

Lemma 1. *If* $-\alpha \in \Re^+$, *then* $-\beta \in \Re^+$ *with* $\beta = \alpha - \gamma K$, $\gamma \geq 0$, $K > 0$.

Proof. Since $-\alpha$ is positive regressive, then $1 - \mu\alpha > 0$.

$1 - \mu\beta = 1 - \mu\alpha + \mu\gamma K > 0$. Thus $-\beta \in \Re^+$. □

Theorem 12. *Suppose that there exist positive constants* $K > 0$, $\alpha > 0$ *such that condition (3.10) holds, and let* g *satisfy the inequality (3.9)*

$$|g(t,x)| \leq \gamma \frac{\|x(t)\|}{e_\alpha(\sigma(t),t)}, \text{ for } t \geq 0, \quad \left\|\frac{x}{e_\alpha(\sigma(t),t)}\right\| < c,$$

where $\gamma < K^{-1}\alpha$ *and* $0 < c < \eta$. *Then every solution* x *of (3.8) for which* $\|x(t_0)\| < K^{-1}c$ *for some* $t_0 \geq 0$ *is defined for all* $t \geq t_0$ *and satisfies*

$$\|x(t)\| \leq K\|x(t_0)\| e_{-\beta}(t,t_0), \quad t \geq t_0.$$

where $\beta = \alpha - \gamma K > 0$.

Proof. From the usual integral equation (3.8)

$$x(t) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(s))g(s,x(s)) \triangle s,$$

we obtain

$$\|x(t)\| \leq Ke_{-\alpha}(t,t_0)\|x_0\| + \gamma K\int_{t_0}^{t} e_{-\alpha}(t,\sigma(s))\frac{\|x(s)\|}{e_\alpha(\sigma(s),s)} \triangle s.$$

$$e_\alpha(t,t_0)\|x(t)\| \leq K\|x_0\| + \gamma K\int_{t_0}^{t} \frac{e_{-\alpha}(t,\sigma(s))e_\alpha(t,t_0)}{e_\alpha(\sigma(s),s)e_\alpha(s,t_0)} e_\alpha(s,t_0)\|x(s)\| \triangle s.$$

$$e_\alpha(t,t_0)\,\|x(t)\| \le K\,\|x_0\| + \gamma K \int_{t_0}^{t} e_\alpha(s,t_0)\,\|x(s)\| \triangle s.$$

Thus the scalar function $\omega(t) = e_\alpha(t,t_0)\,\|x(t)\|$ satisfies the inequality

$$\omega(t) = K\omega(t_0) + \gamma K \int_{t_0}^{t} \omega(s) \triangle s.$$

By Gronwall's Inequality in [3] this implies

$$\omega(t) = K\omega(t_0) e_{\gamma K}(t,t_0)$$

for $t \ge t_0$, and hence

$$\|x(t)\| \le \|x(t_0)\|\,\gamma e_{-\beta}(t,t_0), \quad t \ge t_0.$$

where $\beta = \alpha - \gamma K > 0$. □

Corollary 1. *Suppose that there exist constants $K > 0$, $\alpha > 0$ such that condition (3.10) holds, and let g satisfy the inequality*

$$|g(t,x)| \le \gamma\,\|x(t)\|\,, \ \textit{for } t \ge 0, \ \ \|x(t)\| < c,$$

where $\gamma < K^{-1}\alpha$ and $0 < c < \eta$. Then every solution x of (3.8) for which $\|x(t_0)\| < K^{-1}c$ for some $t_0 \ge 0$ is defined for all $t \ge t_0$ and satisfies

$$\|x(t)\| \le K\,\|x(t_0)\|\,e^{-\beta(t-t_0)}, \quad t \ge t_0.$$

where $\beta = \alpha - \gamma K > 0$.

Proof. If we take $\mathbb{T} = \mathbb{R}$, then the proof is easily follows from Theorem 12. □

Let $A \in \Re$ be an $n \times n$ matrix-valued function on $\mathbb{T}$, suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous and $B(t)$ is a rd-continuous $n \times n$ matrix. Now, we consider the equation in $\mathbb{R}^n$

$$x^{\triangle}(t) = A(t)x(t) + B(t)x(t) + f(t,x(t)),\ x(t_0) = x_0,\ t \in \mathbb{T} \tag{4.3}$$

Theorem 13. *Suppose that the linear state equations (3.1) is uniformly exponentially stable and satisfies below conditions*

(i) $\|B(t)\| \le \frac{\delta}{e_{-\alpha}(\sigma(t),t)}, \quad t \ge t_0$

(ii)$\|f(t,x(t)\| \le \frac{\delta}{e_{-\alpha}(\sigma(t),t)}\,\|x(t)\|, \quad t \ge t_0 \ge 0.$

Then the dynamic equation (4.3) is uniformly exponentially stable.

Proof. For any t_0 and $x(t_0) = x_0$, by Theorem 4 the solution of (4.3) satisfies

$$x(t) = \Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(s))\,[B(s)x(s) + f(s,x(s))] \triangle s, \tag{4.4}$$

where $\Phi_A(t,t_0)$ is the transition matrix for system (3.1). By the uniform exponentially stability of (3.1), there exists constants $K, \alpha > 0$ with $-\alpha \in \Re^+$ such that $\|\Phi_A(t,\tau)\| \le K e_{-\alpha}(t,\tau)$, for all $t,\tau \in \mathbb{T}$ with $t \ge \tau$. By taking the norms of both sides of (4.4), we have

$$\|x(t)\| \le K\,\|x_0\|\,e_{-\alpha}(t,t_0) + 2\delta K \int_{t_0}^{t} e_{-\alpha}(t,\sigma(s)) e_{-\alpha}(\sigma(s),s)\,\|x(s)\| \triangle s, \quad t \ge t_0.$$

Dividing by $e_{-\alpha}(t,t_0)$ on both sides, we have

$$\frac{\|x(t)\|}{e_{-\alpha}(t,t_0)} \leq K\|x_0\| + 2\delta K\int_{t_0}^{t} \frac{e_{-\alpha}(t,\sigma(s))e_{-\alpha}(\sigma(s),s)}{e_{-\alpha}(t,t_0)} e_{-\alpha}(s,t_0)\frac{\|x(s)\|}{e_{-\alpha}(s,t_0)} \triangle s,$$

$$= K\|x_0\| + 2\delta K\int_{t_0}^{t} \frac{\|x(s)\|}{e_{-\alpha}(s,t_0)} \triangle s,$$

Letting $u(t) = \frac{\|x(t)\|}{e_{-\alpha}(t,t_0)}$, we have

$$\|u(t)\| \leq K\|u_0\| + 2\delta K\int_{t_0}^{t} \|u(s)\| \triangle s$$

By Gronwall's Inequality in [3] , a result in [5], we obtain

$$\leq K\|x_0\| e_{2\delta K}(t,t_0)$$

$$\|x(t)\| \leq K\|x_0\| e_{2\delta K}(t,t_0)e_{-\alpha}(t,t_0), \quad t \geq t_0.$$

$$\|x(t)\| \leq K\|x_0\| e_{-\lambda}(t,t_0) \quad t \geq t_0.$$

where $\lambda = \alpha - 2\delta K$. If δ are small enough, the quantity $\lambda = \alpha - 2\delta K$ is positive and we have the required estimate for $t_0 \leq t \leq t_1$. Hence the state equation (4.3) is uniformly exponentially stable. □

Corollary 2. *Consider the equation in* $\mathbb{R}^n$

$$x^{\triangle}(t) = A(t)x(t) + B(t)x(t) + f(t,x(t)), x(t_0) = x_0, t \in \mathbb{R}.$$

A is a constant $n \times n$-matrix with eigenvalues which have all nonpositive real part; $B(t)$ is a continuous $n \times n$-matrix with the property

$$\lim_{t\to\infty} \|B(t)\| = 0.$$

The vector function $f(t,x)$ is continuous in t and x and Lipschitz-continuous in x in a neighbourhood of $x = 0$; moreover we have

$$\lim_{\|x\|\to 0} \frac{\|f(t,x(t))\|}{\|x\|} = 0 \text{ uniformly in } t$$

Then there exist positive constant C, t_0, δ, μ such that $\|x_0\| < \delta$ implies

$$\|x(t)\| \leq C\|x_0\| e^{-\mu(t-t_0)} \quad t \geq t_0$$

The solution $x = 0$ is asymptotically stable and the attraction is exponential in a δ-neighbourhood of $x = 0$.

Proof. If we take $\mathbb{T} = \mathbb{R}$, then the proof is easily follows from Theorem 13. □

Now, with the aid of previous method we consider stability of nonhomogeneous linear dynamic systems

$$z^{\triangle}(t) = A(t)z(t) + f(t) \tag{4.5}$$

Theorem 14. *Suppose the linear state equations (3.1) is uniformly stable. Then the nonhomogeneous linear dynamic equation (4.5) is uniformly stable if there exists some $\beta \geq 0$ such that for all $\tau \in \mathbb{T}$*

$$\int_{\tau}^{\infty} \|f(s)\| \triangle s \leq \beta. \tag{4.6}$$

Proof. For any t_0 and $z(t_0) = z_0$, by theorem 4 the solution of (4.5) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))f(s) \triangle s, \tag{4.7}$$

where $\Phi_A(t, t_0)$ is the transition matrix for system (3.1). By the uniformly stability of (3.1), there exists constants $\gamma > 0$ with such that $\|\Phi_A(t, \tau)\| \leq \gamma$, for all $t, \tau \in \mathbb{T}$ with $t \geq \tau$. By taking the norms of both sides of (4.7), we have

$$\begin{aligned}
\|z(t)\| &\leq \gamma \|z_0\| + \int_{t_0}^{t} \gamma \|f(s)\| \triangle s, \quad t \geq t_0. \\
&\leq \gamma \|z_0\| + \int_{t_0}^{\infty} \gamma \|f(s)\| \triangle s, \\
&\leq \gamma \|z_0\| + \gamma\beta, \\
&\text{let } \beta = \beta^* \|z_0\| \text{ so that } \beta^* > 0 \\
\|z(t)\| &\leq \gamma \|z_0\| + \gamma\beta^* \|z_0\| \\
&\leq \gamma \|z_0\| (1 + \beta^*) \\
&\leq \gamma^* \|z_0\|
\end{aligned}$$

where $\gamma^* = \gamma(1 + \beta^*)$. Then (4.5) is uniformly stable. □

Theorem 15. *Suppose the linear state equations (3.1) is uniformly exponentially stable. Then the perturbed linear dynamic equation (4.5) is uniformly exponentially stable if there exists some $\beta \geq 0$ and $\lambda > 0$ with $-\lambda \in \Re^+$ such that for all $\tau \in \mathbb{T}$*

$$\int_{\tau}^{\infty} \frac{\|f(s)\|}{e_{-\lambda}(\sigma(s), t_0)} \triangle s \leq \beta. \tag{4.8}$$

Proof. For any t_0 and $z(t_0) = z_0$, by theorem 4 the solution of (4.5) satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, \sigma(s))f(s) \triangle s, \tag{4.9}$$

where $\Phi_A(t, t_0)$ is the transition matrix for system (3.1). By the uniform exponentially stability of (3.1), there exists constants $\gamma, \lambda > 0$ with $-\lambda \in \Re^+$ such that $\|\Phi_A(t, \tau)\| \leq \gamma e_{-\lambda}(t, \tau)$, for all $t, \tau \in \mathbb{T}$ with $t \geq \tau$. By taking the norms of both sides of (4.9), we have

$$\|z(t)\| \leq \gamma \|z_0\| e_{-\lambda}(t, t_0) + \int_{t_0}^{t} \gamma e_{-\lambda}(t, \sigma(s)) \|f(s)\| \triangle s, \quad t \geq t_0.$$

Dividing by $e_{-\lambda}(t, t_0)$ on both sides, we have

$$\begin{aligned}
\tfrac{\|z(t)\|}{e_{-\lambda}(t,t_0)} &\leq \gamma \|z_0\| + \gamma \int_{t_0}^{t} \tfrac{e_{-\lambda}(t,\sigma(s))}{e_{-\lambda}(t,t_0)} \|f(s)\| \triangle s, \\
&= \gamma \|z_0\| + \gamma \int_{t_0}^{t} \tfrac{\|f(s)\|}{e_{-\lambda}(\sigma(s),t_0)} \triangle s,
\end{aligned}$$

$$\begin{aligned}
&\leq \gamma \|z_0\| + \gamma \int_{t_0}^{\infty} \tfrac{\|f(s)\|}{e_{-\lambda}(\sigma(s),t_0)} \triangle s \\
&\leq \gamma \|z_0\| + \gamma\beta, \\
&\text{let } \beta = \beta^* \|z_0\| \text{ so that } \beta^* > 0 \\
\tfrac{\|z(t)\|}{e_{-\lambda}(t,t_0)} &\leq \gamma \|z_0\| + \gamma\beta^* \|z_0\| \\
&\leq \gamma \|z_0\| (1+\beta^*) \\
&\leq \gamma^* \|z_0\| \\
\|z(t)\| &\leq \gamma^* \|z_0\| e_{-\lambda}(t,t_0)
\end{aligned}$$

where $\gamma^* = \gamma(1+\beta^*)$. Then (4.5) is uniformly exponentially stable. □

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A NOTE ON THE FOURIER TRANSFORM OF FERMIONIC p-ADIC INTEGRAL ON $\mathbb{Z}_p$

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Abstract. The fermionic p-adic integral ($=I_{-1}$-integral) is defined by T.Kim in the previous paper, see [2]. In this paper we consider I_{-1}-Fourier transform and investigate some properties which are related to this transform.

§1. Introduction

Let us denote $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ sets of positive integers, integers, rational and complex numbers respectively. Let p be a prime number and $x \in \mathbb{Q}$. Then $x = p^{\nu(x)}\frac{m}{n}$, where $m, n, \nu = \nu(x) \in \mathbb{Z}$, m and n are not divisible by p. Let $|x|_p = p^{-\nu(x)}$ and $|0|_p = 0$. Then $|x|_p$ is valuation on $\mathbb{Q}$ satisfying

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

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Completion of $\mathbb{Q}$ with respect to $|\cdot|$ is denoted by $\mathbb{Q}_p$ and called the field of p-adic rational numbers. $\mathbb{C}_p$ is the completion of algebraic closure of $\mathbb{Q}_p$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p | |x|_p \leq 1\}$ is called the ring of p-adic rational integers. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$ if the difference quotients

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. cf. [1-6].

For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{x=0}^{p^n-1} f(x)(-1)^x, \quad \text{cf. [2,3,4].} \tag{1}$$

Let C_{p^n} be the cyclic group consisting of all p^n-th roots of unity in $\mathbb{C}_p$ for any $n \geq 0$ and T_p be the direct limit of $\mathbb{C}_{p^n}$ with respect to the natural morphism, hence T_p is the union of all $\mathbb{C}_{p^n}$ with discrete topology. U_p denotes the group of all principal units in $\mathbb{C}_p$. Note that $T_p \subset U_p$. Define $\phi_\alpha : \mathbb{Z}_p \to (\mathbb{C}_p^\times, X)$ by $\phi_\alpha(z) = \alpha^z$ for $\alpha \in U_p(\text{or} T_p)$, see[5,6]. If $\alpha \in T_p$, then $\phi_\alpha(z)$ is locally constant function. If $\alpha \in U_p$, then $\phi_\alpha(t)$ is locally analytic function in $\mathbb{C}_p$. The purpose of this paper is to study the Fourier transform of fermionic p-adic invariant integral on $\mathbb{Z}_p$. Finally we will investigate some properties related to this transform.

§2. I_{-1}-Fourier transform on $\mathbb{Z}_p$

Let $W(T_p)$ denote the space of all functions $h : w \longmapsto h_w$ from T_p to $\mathbb{C}_p$. Throughout this paper, $\sum_w$ will means $\lim_{n\to\infty} \sum_{w\in\mathbb{C}_{p^n}}$. We define the Fourier transform for the fermionic p-adic invariant integral on $\mathbb{Z}_p$(=I_{-1}-Fourier transform) as follows:

$$\widehat{f_{w}}_{-1} = I_{-1}(f\phi_w) = \int_{\mathbb{Z}_p} f(x) w^x d\mu_{-1}(x),$$

for all $w \in T_p$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$ and $Lip(\mathbb{Z}_p)$ the space of Lipschitz functions on $\mathbb{Z}_p$. Let U be a non-empty open subset of U_p. Then $\{\phi_\alpha | \alpha \in U\}$ has dense linear span in $C(\mathbb{Z}_p)$ and $UD(\mathbb{Z}_p)$. Since we note that

$$\lim_{\beta\to\alpha} \frac{\phi_\beta(z) - \phi_\alpha(z)}{\beta - \alpha} = \frac{z}{\alpha}\phi_\alpha(z), \quad \text{for } \alpha \in U.$$

For $x \in \mathbb{Z}_p$, let $\chi_{x,n} = char(x + p^n\mathbb{Z}_p),\ n \geq 0$ be a characteristic function. Then

$$I_1(char(\mathbb{Z}_p)) = \int_{\mathbb{Z}_p} d\mu_{-1}(x) = 1,$$

and

$$I_{-1}(char(a+p^n\mathbb{Z}_p)) = I_{-1}(\chi_{x,n}) = \int_{a+p^n\mathbb{Z}_p} d\mu_{-1}(x) = (-1)^a.$$

It is easy to see that

$$\sum_w w^{-x}\phi_w = \sum_{i=0}^{p^n-1} w^{-xi}w^i = \sum_{i=0}^{p^n-1} w^{i(1-x)} = p^n\chi_{x,n}.$$

Thus, we have $\chi_{x,n} = \frac{1}{p^n}\sum_w w^{-x}\phi_w$.

$$\begin{aligned} I_{-1}(f\chi_{x,n}) - \frac{f(x)}{p^n} &= I_{-1}(f\frac{1}{p^n}\sum_w w^{-x}\phi_w) - \frac{f(x)}{p^n} \\ &= \frac{1}{p^n}\sum_w I_{-1}(f\phi_w)\phi_{w^{-1}}(x) - \frac{f(x)}{p^n} \\ &= \frac{1}{p^n}\left(\sum_w (\widehat{f_w})_{-1}\phi_{w^{-1}}(x) - f\right). \end{aligned} \tag{2}$$

From (2) we derive

$$\left|\sum_w (\widehat{f_w})_{-1}\phi_{w^{-1}}(x) - f\right|_p \leq M\frac{1}{p^n}$$

for some constant M. Hence, we see that

$$\sum_w (\widehat{f_w})_{-1}\phi_{w^{-1}}(x) = f. \tag{3}$$

Let $f, g \in UD(\mathbb{Z}_p)$. Then we define the convolution associated with I_{-1}-integral by

$$(f*g)_{-1} = \sum_w (\widehat{f_w})_{-1}(\widehat{g_w})_{-1}\phi_{w^{-1}}.$$

It is easy to see that $(f*g)_{-1} \in Lip(\mathbb{Z}_p)$. We now define bilinear map(continuous system) as follows:

$$* : UD(\mathbb{Z}_p) \times UD(\mathbb{Z}_p) \text{ by } (f,g) \longmapsto (f*g)_{-1}.$$

Since $UD(\mathbb{Z}_p)$ is closed in $Lip(\mathbb{Z}_p)$ and I_{-1} is continuous function. We note that $\{\phi_\alpha | \alpha \in U_p\}$ has dense linear span in $C(\mathbb{Z}_p), UD(\mathbb{Z}_p)$. Let $f = \phi_\alpha, g = \phi_\beta$ for $\alpha, \beta \in U_p \setminus T_p$, $\alpha \neq \beta$. From the definition of I_{-1}-integral, we can derive

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \text{ where } f_1(x) = f(x+1) \tag{4}$$

By (4), we easily see that

$$(\widehat{f_w})_{-1} = I_{-1}(f\phi_w) = \frac{2}{\alpha w+1},$$
$$(\widehat{g_w})_{-1} = I_{-1}(g\phi_w) = \frac{2}{\beta w+1},\ w \in T_p. \tag{5}$$

From (5), we note that

$$\begin{aligned}(f*g)_{-1} &= \sum_w (\widehat{f_w})_{-1}(\widehat{g_w})_{-1}\phi_{w^{-1}} \\ &= \sum_w \frac{4}{(\alpha w+1)(\beta w+1)}\phi_{w^{-1}} \\ &= \frac{2}{\beta-\alpha}(\alpha f-\beta g) \in UD(\mathbb{Z}_p).\end{aligned}$$

Let $((\widehat{f*g})_w)_{-1} = I_{-1}((f*g)_{-1}\phi_{w^{-1}})$. Then

$$\begin{aligned}((\widehat{f*g})_w)_{-1} &= I_{-1}(\frac{2}{\beta-\alpha}(\alpha f-\beta g)\phi_{w^{-1}}) \\ &= \frac{2}{\beta-\alpha}(\alpha(\widehat{f_w})_{-1}-\beta(\widehat{g_w})_{-1}) \\ &= \frac{2}{\beta-\alpha}\frac{2(\alpha-\beta)}{(\alpha w+1)(\beta w+1)} \\ &= -\left(\frac{2}{\alpha w+1}\right)\left(\frac{2}{\beta w+1}\right) \\ &= -(\widehat{f_w})_{-1}(\widehat{g_w})_{-1}.\end{aligned}$$

Therefore we obtain the following:

Theorem 1. *For $w \in T_p$, we have*

$$((\widehat{f*g})_w)_{-1} = -(\widehat{f_w})_{-1}(\widehat{g_w})_{-1}.$$

Let $Int(\mathbb{Z}_p) = \{f \in UD(\mathbb{Z}_p) | f' = 0\}$ be a $*$-ideal of $UD(\mathbb{Z}_p)$. We define the following induced convolution as follows:

$$\otimes : UD(\mathbb{Z}_p)/Int(\mathbb{Z}_p) \to C(\mathbb{Z}_p) \text{ with } (f*g)_{-1} = -f' \otimes g' \text{ for } f, g \in UD(\mathbb{Z}_p).$$

In [1], it is easy to show the following theorem:

Corollary 2. *For $f, g \in UD(\mathbb{Z}_p)$, we have*

$$I_{-1}^{(z)}(f \otimes g'(z)(-1)^z) = -I_{-1}^{(z)}(f(z)(-1)^z)I_{-1}(g(z)(-1)^z).$$

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On the relative strength of two absolute summability methods

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Abstract

In this paper, a theorem dealing with $\varphi - |A|_k$ summability method, which generalizes a theorem of Bor [1] on $|R, p_n|_k$ summability method, has been proved.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad (n \to \infty), \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)\,. \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} \ |t_n - t_{n-1}|^k < \infty. \tag{3}$$

Key Words: Absolute matrix summability, infinite series, relative strength.

2000 AMS Subject Classification: 40D25, 40F05, 40G99.

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{4}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{5}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $a_{nv} = \frac{p_v}{P_n}$, then $|A|_k$ summability is the same as $|R, p_n|_k$ summability.
Let (φ_n) be a sequence of positive real numbers. We say that the series $\sum a_n$ is summable $\varphi - |A|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty. \tag{6}$$

If we take $\varphi_n = n$ for all values of n, then $\varphi - |A|_k$ summability is the same as $|A|_k$ summability.
Before stating the main theorem we must first introduce some further notations.
Given a normal matrix $A = (a_{nv})$, we associate two lover semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{8}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{9}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{10}$$

If A is a normal matrix, then $A' = (a'_{nv})$ will denote the inverse of A. Clearly if A is normal then, $\hat{A} = (\hat{a}_{nv})$ is normal and it has two-sided inverse $\hat{A}' = (\hat{a}'_{nv})$, which is also normal (see [2]).
The following result dealing with the relative strength of two absolute summability methods was given by Bor [1].

Theorem A. Let $k > 1$. In order that

$$|R, p_n|_k \Rightarrow |R, q_n|_k \tag{11}$$

it is necessary that

$$\frac{q_n \ P_n}{p_n \ Q_n} = O(1). \tag{12}$$

If we suppose that

$$\sum_{n=v}^{\infty} \frac{n^{k-1} \ q_n^k}{Q_n^k \ Q_{n-1}} = O\left(\frac{v^{k-1} \ q_v^{k-1}}{Q_v^k}\right) \tag{13}$$

then (12) is also sufficient.
Remark. If we take $k = 1$, then condition (13) is obvious.

2. Main Theorem

The aim of this paper is to generalize Theorem A for the $\varphi - |A|_k$ and $\varphi - |B|_k$ summabilities. Therefore we shall prove the following theorem.

Theorem. Let $k > 1$, $A = (a_{nv})$ and $B = (b_{nv})$ be two positive normal matrices. In order that

$$\varphi - |A|_k \Rightarrow \varphi - |B|_k \tag{14}$$

it is necessary that

$$b_{nn} = O(a_{nn}). \tag{15}$$

If we suppose that

$$b_{n-1,v} \geq b_{nv}, \quad for \quad n \geq v + 1, \tag{16}$$

$$\bar{a}_{n0} = 1, \quad \bar{b}_{n0} = 1, \quad n = 0, 1, 2, ..., \tag{17}$$

$$a_{vv} - a_{v+1,v} = O(a_{vv}\ a_{v+1,v+1}), \tag{18}$$

$$\sum_{v=1}^{n-1}(b_{vv}\ \hat{b}_{n,v+1}) = O(b_{nn}), \tag{19}$$

$$\sum_{n=v+1}^{m+1}(\varphi_n\ b_{nn})^{k-1}\ \hat{b}_{n,v+1} = O(\varphi_v^{k-1}\ b_{vv}^{k-1}), \tag{20}$$

$$\sum_{n=v+1}^{m+1}(\varphi_n\ b_{nn})^{k-1}\ \left|\Delta_v \hat{b}_{nv}\right| = O(\varphi_v^{k-1}\ b_{vv}^{k}), \tag{21}$$

$$\sum_{v=r+2}^{n}\hat{b}_{nv}\ \ |\hat{a}'_{vr}| = O(\hat{b}_{n,r+1}), \tag{22}$$

then (15) is also sufficient.

It should be noted that if we take $\varphi_n = n$, $a_{nv} = \frac{p_v}{P_n}$, $b_{nv} = \frac{q_v}{Q_n}$ in this Theorem, then we get Theorem A.

We need the following lemma for the proof of our theorem.

Lemma ([1]). Let $k \geq 1$ and let $A = (a_{nv})$ be an infinite matrix. In order that $A\ \epsilon\ (l^k; l^k)$ it is necessary that

$$a_{nv} = O(1)\ \ (all\ n,\ v). \tag{23}$$

3. Proof of the Theorem

Necessity. Now, let (x_n) and (y_n) be denote the A-transform and B-transform of the series $\sum a_n$, respectively. Then we have, by (9) and (10)

$$\bar{\Delta}x_n = \sum_{v=0}^{n}\hat{a}_{nv}a_v \quad and \quad \bar{\Delta}y_n = \sum_{v=0}^{n}\hat{b}_{nv}a_v$$

which implies that

$$a_v = \sum_{r=0}^{v}\hat{a}'_{vr}\ \overline{\Delta}x_r. \tag{24}$$

In this case

$$\bar{\Delta}y_n = \sum_{v=0}^{n} \hat{b}_{nv} a_v = \sum_{v=0}^{n} \hat{b}_{nv} \sum_{r=0}^{v} \hat{a}'_{vr} \bar{\Delta}x_r.$$

On the other hand, since

$$\hat{b}_{n0} = \bar{b}_{n0} - \bar{b}_{n-1,0}$$

by (17), we have that

$$\begin{aligned}
\bar{\Delta}y_n &= \sum_{v=1}^{n} \hat{b}_{nv} \left\{ \sum_{r=0}^{v} \hat{a}'_{vr} \ \bar{\Delta}x_r \right\} \\
&= \sum_{v=1}^{n} \hat{b}_{nv} \left\{ \hat{a}'_{vv} \ \bar{\Delta}x_v + \hat{a}'_{v,v-1} \ \bar{\Delta}x_{v-1} + \sum_{r=0}^{v-2} \hat{a}'_{vr} \ \bar{\Delta}x_r \right\} \\
&= \sum_{v=1}^{n} \hat{b}_{nv} \ \hat{a}'_{vv} \ \bar{\Delta}x_v + \sum_{v=1}^{n} \hat{b}_{nv} \ \hat{a}'_{v,v-1} \ \bar{\Delta}x_{v-1} + \sum_{v=1}^{n} \hat{b}_{nv} \sum_{r=0}^{v-2} \hat{a}'_{vr} \ \bar{\Delta}x_r \\
&= \hat{b}_{nn} \ \hat{a}'_{nn} \ \bar{\Delta}x_n + \sum_{v=1}^{n-1} (\hat{b}_{nv} \ \hat{a}'_{vv} + \ \hat{b}_{n,v+1} \ \hat{a}'_{v+1,v}) \ \bar{\Delta}x_v \\
&\quad + \sum_{r=0}^{n-2} \bar{\Delta}x_r \sum_{v=r+2}^{n} \hat{b}_{nv} \ \hat{a}'_{vr}. \qquad (25)
\end{aligned}$$

By considering the equality

$$\sum_{k=v}^{n} \hat{a}'_{nk} \ \hat{a}_{kv} = \delta_{nv},$$

where δ_{nv} is the Kronocker delta, we have that

$$\begin{aligned}
\hat{b}_{nv} \ \hat{a}'_{vv} + \hat{b}_{n,v+1} \ \hat{a}'_{v+1,v} &= \frac{\hat{b}_{nv}}{\hat{a}_{vv}} + \hat{b}_{n,v+1} \ (-\frac{\hat{a}_{v+1,v}}{\hat{a}_{vv} \ \hat{a}_{v+1,v+1}}) \\
&= \frac{\hat{b}_{nv}}{a_{vv}} - \frac{\hat{b}_{n,v+1} \ (\bar{a}_{v+1,v} - \bar{a}_{v,v})}{a_{vv} \ a_{v+1,v+1}} \\
&= \frac{\hat{b}_{nv}}{a_{vv}} - \frac{\hat{b}_{n,v+1} \ (a_{v+1,v+1} + a_{v+1,v} - a_{vv})}{a_{vv} \ a_{v+1,v+1}} \\
&= \frac{\Delta_v \hat{b}_{nv}}{a_{vv}} + \hat{b}_{n,v+1} \ \frac{a_{vv} - a_{v+1,v}}{a_{vv} \ a_{v+1,v+1}}
\end{aligned}$$

and so

$$\begin{aligned}
\bar{\Delta}y_n &= \frac{b_{nn}}{a_{nn}} \ \bar{\Delta}x_n + \sum_{v=1}^{n-1} \ \frac{\Delta_v \hat{b}_{nv}}{a_{vv}} \ \bar{\Delta}x_v + \sum_{v=1}^{n-1} \hat{b}_{n,v+1} \ \frac{a_{vv} - a_{v+1,v}}{a_{vv} \ a_{v+1,v+1}} \ \bar{\Delta}x_v + \sum_{r=0}^{n-2} \bar{\Delta}x_r \sum_{v=r+2}^{n} \hat{b}_{nv} \ \hat{a}'_{vr} \\
&= T_n(1) + T_n(2) + T_n(3) + T_n(4), \ say.
\end{aligned}$$

Now, from (23) we can write down the matrix transforming $\left(\varphi_n^{1-\frac{1}{k}}\ \bar{\Delta}x_n\right)$ into $\left(\varphi_n^{1-\frac{1}{k}}\ \bar{\Delta}y_n\right)$. The assertion (14) is equivalent to the assertion that this matrix $\epsilon\left(l^k;l^k\right)$. Hence, by the Lemma, a necessary condition for (14) is that the elements of this matrix should be bounded, and this gives the result that (15) is necessary.

Sufficiency. Suppose the conditions are satisfied. Then, since

$$|T_n(1)+T_n(2)+T_n(3)+T_n(4)|^k \leq 4^k\left(|T_n(1)|^k+|T_n(2)|^k+|T_n(3)|^k+|T_n(3)|^k\right)$$

to complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}\varphi_n^{k-1}\,|T_n(i)|^k<\infty \quad for \quad i=1,2,3,4.$$

Firstly, we have

$$\begin{aligned}\sum_{n=1}^{m}\varphi_n^{k-1}\,|T_n(1)|^k &= \sum_{n=1}^{m}\varphi_n^{k-1}\ \left|\frac{b_{nn}}{a_{nn}}\ \bar{\Delta}x_n\right|^k\\ &= O(1)\sum_{n=1}^{m}\varphi_n^{k-1}\ \left|\bar{\Delta}x_n\right|^k\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

in view of the hypotheses of the Theorem.

Applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{k-1}\,|T_n(2)|^k &= \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{v=1}^{n-1}\ \frac{\Delta_v\hat{b}_{nv}}{a_{vv}}\ \bar{\Delta}x_v\right|\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}\ \frac{\left|\Delta_v\hat{b}_{nv}\right|}{a_{vv}^k}\ \left|\bar{\Delta}x_v\right|^k\right\}\ \left\{\sum_{v=1}^{n-1}\left|\Delta_v\hat{b}_{nv}\right|\ \right\}^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\ b_{nn}^{k-1}\sum_{v=1}^{n-1}\ \frac{\left|\Delta_v\hat{b}_{nv}\right|}{a_{vv}^k}\ \left|\bar{\Delta}x_v\right|^k\\ &= O(1)\sum_{v=1}^{m}\frac{\left|\bar{\Delta}x_v\right|^k}{a_{vv}^k}\ \sum_{n=v+1}^{m+1}(\varphi_n b_{nn})^{k-1}\left|\Delta_v\hat{b}_{nv}\right|\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\ \left(\frac{b_{vv}}{a_{vv}}\right)^k\ \left|\bar{\Delta}x_v\right|^k\\ &= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\left|\bar{\Delta}x_v\right|^k\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of the Theorem.

Also

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}\left|T_n(3)\right|^k &= \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{v=1}^{n-1}\hat{b}_{n,v+1}\ \frac{a_{vv}-a_{v+1,v}}{a_{vv}\ a_{v+1,v+1}}\ \bar{\Delta}x_v\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{b}_{n,v+1}\ \left|\bar{\Delta}x_v\right|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{b}_{n,v+1}\ \left|\bar{\Delta}x_v\right|^k\frac{b_{vv}}{b_{vv}^k}\right\}\left\{\sum_{v=1}^{n-1}\hat{b}_{n,v+1}\ b_{vv}\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}(\varphi_n\ b_{nn})^{k-1}\sum_{v=1}^{n-1}\hat{b}_{n,v+1}\ \left|\bar{\Delta}x_v\right|^k\ b_{vv}^{1-k} \\
&= O(1)\sum_{v=1}^{m}b_{vv}^{1-k}\ \left|\bar{\Delta}x_v\right|^k\sum_{n=v+1}^{m}(\varphi_n\ b_{nn})^{k-1}\ \hat{b}_{n,v+1} \\
&= O(1)\sum_{v=1}^{m}\varphi_v^{k-1}\ \left|\bar{\Delta}x_v\right|^k \\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem.

Finally, as in $T_n(3)$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\varphi_n^{k-1}\left|T_n(4)\right|^k &= \sum_{n=2}^{m+1}\varphi_n^{k-1}\left|\sum_{r=0}^{n-2}\bar{\Delta}x_r\sum_{v=r+2}^{n}\hat{b}_{nv}\ \hat{a}'_{vr}\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{r=0}^{n-2}\left|\bar{\Delta}x_r\right|\sum_{v=r+2}^{n}\hat{b}_{nv}\ \left|\hat{a}'_{vr}\right|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{k-1}\left\{\sum_{r=0}^{n-2}\left|\bar{\Delta}x_r\right|\ \hat{b}_{n,r+1}\right\}^k = O(1)\quad as\quad m\to\infty.
\end{aligned}$$

by virtue of the hypotheses of the Theorem.

Therefore, we have that

$$\sum_{n=1}^{m}n^{k-1}\left|T_n(i)\right|^k = O(1)\quad as\quad m\to\infty,\quad for\quad i=1,2,3,4.$$

This completes the proof of the Theorem.

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Algebraic Multigrid Preconditioner for a Finite Element Method in TM Electromagnetic Scattering

K. Kim*, K. H. Leem†, G. Pelekanos†, and M. Song†

Abstract — The finite element method (FEM) is applied on an arbitrarily shaped and perfectly conducted cylindrical scatterer whose cross section and material properties are uniform along its infinite axis, say the z axis here. The scatterer is enclosed within a fictitious boundary via a simple first order absorbing boundary condition (ABC). The Algebraic Multigrid (AMG) method is employed as a preconditioner in order to accelerate the convergence rate of the Krylov iterations. Our experimental results suggest much faster convergence compared to the non preconditioned Krylov subspace solver, and hence significant reduction to the overall computational time.

Keywords:Algebraic Multigrid; Finite elements; Absorbing boundary condition.

1 Introduction

It is widely known that the implementation of the standard finite element method (FEM) produces a sparse coefficient matrix. Due to significant growth of computer time and memory during the elimination process direct methods are not preferable for solving large sparse linear systems. Hence systems of this kind require iterative methods, such as Krylov subspace methods [15, 17], for their solution. The convergence rate of an iterative method depends on the spectral properties of the coefficient matrix. Preconditioning is a well-known technique used to accelerate the convergence of iterative methods [15, 17]. Hence, developing efficient preconditioners has been one of the major research interests in many applications [4, 8, 12, 13, 14]. In this work, scattering by a conducting cylinder is considered and the FEM is used to construct the resulting linear system. In our numerical experiments, BiCGSTAB (BiConjugate Gradient Stablized) is used as a choice of a linear solver and AMG (Algebraic Multigrid) method is employed as a preconditioner for accelerating the convergence.

AMG has been developed to solve large problems posed on unstructured grids since it doesn't require geometric grid information. To date, many AMG based on element interpolation and smooth aggregation have been proposed [3, 6, 18] and

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are shown to be effective and robust. A variant of smoothed aggregation AMG preconditioner was developed by Leem et al [13] to solve large, sparse, and real-valued saddle point systems from meshfree discretizations. In [14], the smoothed sggregation AMG preconditioner was modified to successfully accelerate the convergence of large, dense, and complex-valued linear systems encountered in scattering by dielectric objects.

It is important to mention here that since we are dealing with an open-region scattering problem, the infinite region exterior to the scatterer must be truncated with an artificial boundary. In order to obtain a unique finite element solution a boundary condition will be introduced. The purpose of this condition, which is called absorbing boundary condition (ABC), is to allow the scattered wave to propagate to infinity without causing any reflections back to the object. The (ABCs) are applied at the artificial boundary and hence their use does not introduce additional unknowns and retains sparsity. In this work, for simplicity, we will use first-order (ABCs) [9].

We organize our paper as follows. In section 2, we formulate the problem and summarize the well known finite elements steps required for the generation of the corresponding linear system. In section 3, we discuss the construction of the AMG preconditioner. Numerical experiments that yield a dramatic reduction to the number iterations and computational time are presented in section 4.

2 Formulation of the problem

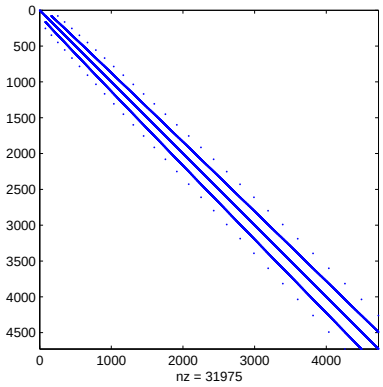
It is well known that the propagation of time-harmonic fields in a homogeneous medium, in the presence of a perfect conductor D, is modeled by the exterior boundary value problem (direct obstacle scattering problem)

$$\triangle_2 u_z(x) + k^2 u_z(x) = 0, \quad x \in \Re^2 \setminus \bar{D} \tag{2.1}$$

$$u_z(x) + u^i(x) = 0, \quad x \in \partial D \tag{2.2}$$

where k is a real positive wavenumber and u^i is a given incident plane wave polarized along the $z-$direction, that in the presence of D will generate the scattered field u_z which will also be polarized along the $z-$ direction.
As indicated in the introduction the infinite region exterior to the scatterer needs to be truncated by the introduction of a fictitious boundary. Consequently, a boundary condition must be applied at this boundary so the field scattered by the cylinder must continue propagating toward infinity without disturbance. In other words, such a condition should minimize any reflections from the boundary. This kind of boundary conditions are called absorbing boundary conditions (ABC). As indicated in [10] an ideal boundary condition is one that possesses zero reflection for all angles of incidence. However, absorbing boundary conditions lead to localized relations between the boundary fields (i.e. they are approximate) hence zero reflection for all incident angles is impossible.
For simplicity in the present problem we will be using a first order ABC. According

Figure 1: Sparsity Pattern of the coefficient matrix K when $n = 4727$.



to Bayliss at al. [1, 2], the (ABC) employed here is given by

$$\frac{\partial u_z}{\partial n} + (ik + \frac{k(s)}{2\rho})u_z = 0 \tag{2.3}$$

where n denotes the outward unit vector normal to the artificial boundary, s is the arc length measured along the boundary, and $k(s)$ is the curvature of the boundary at s.

Details about the major steps of the solution of the problem above can be found in [9] and include discretization of the domain using linear triangular elements, generation of proper mesh data, construction of the elemental equations, assembly of these element matrices and vectors into the global matrix and right hand side vector, imposition of Dirichlet boundary conditions on the surface of the cylinder, and finally solution of the matrix system to obtain the total acoustic field at the nodes of the domain.

3 Smoothed Aggregation AMG Preconditioner

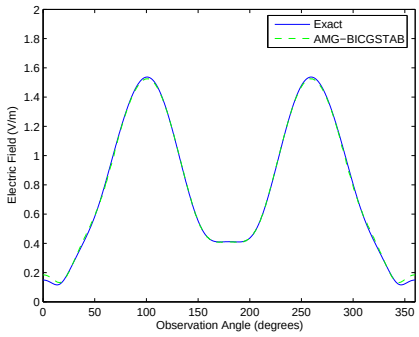
We begin our discussion by introducing the basic framework of AMG. Consider the complex linear system

$$K\mathbf{u} = \mathbf{b}, \tag{3.4}$$

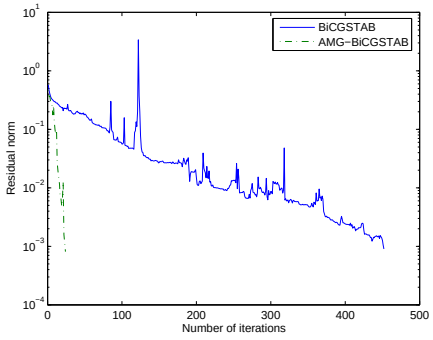
where K is a large, and sparse $n \times n$ matrix. AMG preconditioner will be constructed based on the coefficient matrix K. Note that AMG does not require to access the physical grids of problems. With "grids" we mean sets of indices of the unknown variables. Hence the grid set for (3.4) is $\Omega = \{1, 2, \cdots, n\}$, since the unknown

Table 1: Size of Coarse-grid Operator

n	K_0	K_1	K_2	K_3
4727	4727	659	133	
18563	18563	2454	349	
73569	73569	9036	952	361



(a) Comparison of exact and approximate solutions for $n = 4,727$



(b) Convergence history for $n = 4,727$

Figure 2: Solution Comparison (left) and Convergence History (right)

vector $\mathbf{u}$ in (3.4) has components $u_1, u_2, ..., u_n$. The graph of the matrix K can be defined as follows: A node in the graph represents a row and the normalized edge weights $\overline{\omega}_{ij}$ can be computed as follows:

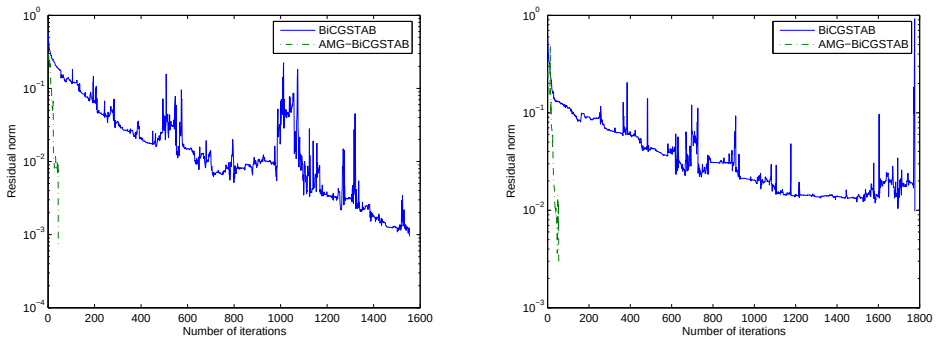
$$\overline{\omega}_{ij} = |k_{ij}|/\sqrt{|k_{ii}| \cdot |k_{jj}|}. \tag{3.5}$$

The main idea of AMG is to remove the smooth error by coarse grid correction, where smooth error is the error not eliminated by relaxation on the fine grid, which also can be characterized by small residuals [5].

In order to develop the multi-grid algorithm, we consider the sets of grids in each level. The number 0 stands for the finest-grid level. Then the numbers $1, 2, \cdots, l_{max}$ represent the corresponding coarse-grid levels. Hence, the original equation (3.4) can be written as $K_0\mathbf{u}_0 = \mathbf{b}_0$ and the set of finest grid set is $\Omega_0 = \Omega$.

AMG can be implemented in two main phases, so called the setup phase and the solve phase. The setup phase includes the following tasks:

- Create the coarse grid sets Ω_{l+1}.
- Construct interpolation operator I_{l+1}^{l}, and restriction operators I_{l}^{l+1}.
- Construct the coarse grid operator K_{l+1}.



(a) $n = 18,563$ (b) $n = 73,569$

Figure 3: Convergence Histories

In general, the restriction operator I_l^{l+1} is defined by the transpose of the interpolation operator I_{l+1}^l, i.e., $I_l^{l+1} = (I_{l+1}^l)^T$ and the coarse grid operator K_{l+1} is constructed from the fine grid operator K_l by the Galerkin approach:

$$K_{l+1} = I_l^{l+1} \, K_l \, I_{l+1}^l, \tag{3.6}$$

so that AMG satisfies the principle that the coarse-grid problem needs to provide a good approximation to fine-grid error in the range of interpolation [5].

We will now provide more information about the setup process. Two main tasks in the setup phase are to find a suitable coarsening strategy and an effective interpolation operator I_{l+1}^l. The creation of the coarse-grid sets Ω_l, where $l = 1, 2, \cdots, l_{max}$ is based on a combinatorial clustering algorithm developed by Vaněk, Mandel and Brezina in [18] with normalized edge weights, $\overline{\omega}_{ij}$.

The first step of their coarsening algorithm iterates through the nodes $\Omega_0 = \{1, 2, \cdots, n\}$ creating clusters $\{j \mid \overline{\omega}_{ij} \geq \eta_1\}$ for a given tolerance $\eta_1 > 0$, provided no node in $\{j \mid \overline{\omega}_{ij} \geq \eta_1\}$ is already a cluster. Two nodes i and j are said to be strongly connected if $\overline{\omega}_{ij} \geq \eta_1$. In the second step, unassigned nodes are assigned to a cluster from step one to which the node is strongly connected, if any. In the last step, the remaining nodes are assigned to clusters consisting of strong neighborhoods intersecting with the set of remaining nodes. This aggregation process will create the corresponding next coarse level grid set $\Omega_1 = \{C_1^l, C_2^l, \cdots, C_{n_l}^l\}$. Each C_k^l is called a cluster. By repeating same process, $\Omega_2, \Omega_3, \cdots, \Omega_{l_{max}}$ are obtained.

As explained earlier, the next important task in AMG is to construct an interpolation operator I_{l+l}^l. In smoothed aggregation AMG [13, 14], we solve a local linear system to obtain an interpolation vector that interpolates a value for a coarse-grid cluster onto its neighborhood. Assume that a set of grid points at level l is $\Omega_l = \{C_1^l, C_2^l, \cdots, C_{n_l}^l\}$. For each cluster C_k^l, we define a neighborhood N_k^l as

$$N_k^l = \{j \notin C_k^l \mid \overline{\omega}_{ij} \geq \eta_2, i \in C_k^l\},$$

Performing the coarsening process explained earlier yields the next coarse-grid set, $\Omega_{l+1} = \{C_1^{l+1}, C_2^{l+1}, \cdots, C_{n_{l+1}}^{l+1}\}$. Suppose that C_k^{l+1} is obtained by aggregating $\{C_i^l, C_j^l\}$ from the grid set Ω_l. Furthermore, assume that clusters C_i^l, and C_j^l have the neighborhoods $N_i^l = \{C_s^l, C_t^l\}$ and $N_j^l = \{C_u^l, C_v^l\}$, respectively.

The corresponding interpolation operator from level $l+1$ to l, I_{l+1}^l, is computed column-by-column by the following procedure: The kth column $\mathbf{p}_k$ of the interpolation operator $I_{l+1}^l = [\mathbf{p}_1|\mathbf{p}_2|\cdots|\mathbf{p}_{n_{l+1}}]$ is obtained by solving the following small local system:

$$L_{C_k^l}\ \mathbf{p}_k = \mathbf{e}, \tag{3.7}$$

where $\mathbf{e}$ is the vector given by

$$\mathbf{e}_i = \begin{cases} 1 & \text{if } i \in C_k^l, \\ 0 & \text{otherwise,} \end{cases} \tag{3.8}$$

and the local matrix $L_{C_k^l}$ is given by

$$L_{C_k^l} = \begin{bmatrix} K_{C_k^l C_k^l} & K_{C_k^l N_k^l} \\ K_{N_k^l C_k^l} & K_{N_k^l N_k^l} \end{bmatrix}. \tag{3.9}$$

Note that $K_{IJ} = [\ k_{ij} \mid i \in I, j \in J\]$ indicates an $|I| \times |J|$ matrix where I and J are sets of nodes.

After completion of the AMG setup, either V-, or W-cycle is typically used for the preconditioning step. In the sequel, the AMG V-cycle algorithm 3.1 is utilized as a preconditioner to the iterative methods used.

Algorithm 3.1 AMG V-Cycle

$$\mathbf{u}_l \leftarrow \mathbf{AMGV}(K_l, \mathbf{u}_l, \mathbf{b}_l)$$

if $\Omega_l =$ *coarsest grid,* **then**

$\mathbf{u}_l \leftarrow$ *Solve* $(K_l\mathbf{u}_l = \mathbf{b}_l)$ *using Direct Method*

else

$\mathbf{x}_l \leftarrow$ *Relax* ν_1 *times on* $K_l\mathbf{u}_l = \mathbf{b}_l$ *on* Ω_l *with initial guess* $\mathbf{u}_l$
$\mathbf{b}_{l+1} \leftarrow I_l^{l+1}(\mathbf{b}_l - K_l\mathbf{u}_l)$
$\mathbf{u}_{l+1} \leftarrow 0,$
$\mathbf{u}_{l+1} \leftarrow \mathbf{AMGV}(K_{l+1}, \mathbf{u}_{l+1}, \mathbf{b}_{l+1}).$
Correct $\mathbf{u}_l = \mathbf{u}_l + I_{l+1}^l\mathbf{u}_{l+1}.$
$\mathbf{u}_l \leftarrow$ *Relax* ν_1 *times on* $K_l\mathbf{u}_l = \mathbf{b}_l$ *on* Ω_l *with* $\mathbf{u}_l$.

endif

Note that we employ Gauss-Seidel iterations for the relaxation scheme. More details on classical AMG can be found in [5], and [7].

Table 2: Convergence Results

n	ϵ	No preconditioner		AMG preconditioner			
		# of iter	total time	AMG setup	# of iter	iter time	total time
4,727	1.0e-03	452	2.86	0.40	24	2.40	2.80
18,563	1.0e-03	1554	40.89	2.56	44	17.65	20.21
73,569	1.0e-02	1777	187.42	17.86	38	61.34	79.20

Table 3: Thresholds

n	η_1	η_2
4,727	0.01	0.05
18,563	0.01	0.1
73,569	0.0001	0.1

4 Numerical Results

A two dimensional FEM algorithm [9] is used to solve the scattering problem described in section 2. In our numerical experiments the systems were solved via the BiConjugate Gradient Stabilized method (BiCGSTAB). The radius of the circular conducting cylinder, i.e., the inner circular boundary, is $\lambda/2$ whereas the radius of the ABC boundary, i.e., the outer cylinder boundary, is $3\lambda/2$. In other words, the first-order ABC is imposed at a distance of one wavelength away from the outer surface of the object. If the discretization size is $h = 0.4$ wavelengths, then the size of the corresponding coefficient matrix K is $n = 4727$ unknowns. Figure 3 shows the comparison between the total field calculated using the analytical solution and the field computed via the AMG preconditioned BiCGSTAB algorithm while figure 1 displays the sparsity pattern of the coefficient matrix K when the number of unknowns is $n = 4727$. Note that K is very sparse with density of only 1.4%. In addition, figure 2(b) shows that the number of iterations has been significantly reduced from 452 (without preconditioning) to 24(with AMG preconditioning).

Moreover, as h reduces to 0.2 and 0.1 wavelengths, the size of the coefficient matrix increases to $n = 18,563$ and $n = 73,569$ unknowns respectively. Figure 3(a) shows that for the case of $18,563$ unknowns, the number of iterations is reduced by 97%. Moreover, figure 3(b) shows that even when the number of unknowns is increased to 73,569, the number of iterations is still reduced by 97%. For a summary of the above results see table 2. The parameter ϵ indicates our stoping criterion. The reason why ϵ in the case of $n = 73,569$ is set to 1.0e-02 is that BiCGSTAB without preconditioning does not converge for a tolerance of 1.0e-03 even after 4,000 iterations. However, the BiCGSTAB with AMG preconditioning did converge in less than 100 iterations for the tolerance set above.

Table 4: Convergence Rates

n	No Preconditioner	AMG Preconditioner
4,727	6.22e-03	1.20e-01
18,563	1.79e-03	6.05e-02
73,569	9.97e-04	3.77e-02

In AMG, thresholds for the coarsening process, η_1 and for the interpolation process, η_2, are also needed. Table 4 shows the thresholds used in each case. Table 3 shows the size of the coarse-grid operators in each level. For instance, K_0 is the original matrix and K_3 is the coarsest level matrix in which the direct solver was used as indicated in the algorithm 3.1 for $n = 73,569$. It is also observed that the size of the coarsest level matrix, K_3 can't be reduced to even smaller size. This is due to the sparsity of the matrices on finer levels.

The computational complexity of the AMG preconditioner depends on the aggregations and the size of the neighborhoods of clusters at each level, which vary from problem to problem. In our experiments, it was observed that despite the fact that the AMG preconditioner requires some setup time, reduction to the overall CPU time was significant as shown in table 2. Hence the convergence rates have been improved with AMG preconditioning as shown in table 4. Note that the convergence rates are computed by the log(1/(convergence factor)) where the convergence factor is the geometric average of successive residual norms [13]. For our numerical experiments, Meschach [16] was used as a matrix library to speed software development.

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ON THE CONVERGENCE OF THE ISHIKAWA ITERATES TO A COMMON FIXED POINT OF $2k$ MAPPINGS

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Abstract. Let C be a convex subset of a complete convex metric space X, and $A_1, A_2, ..., A_{2k}$ be some self-mappings on C. In this paper, it is shown that if the sequence of Ishikawa iterations associated with $A_1, A_2, ..., A_{2k}$ converges, then its limit point is the common fixed point of $A_1, A_2, ..., A_{2k}$. This result extends and generalizes the corresponding results of Ćirić et. al. [Archivum Math., 39 (2003), 123–127.], Naimpally and Singh [J. Math. Anal. Appl. 96 (1983), 437–446.], Rhoades [Trans. Amer. Math. Soc. 196 (1974), 161–176.] and Hicks and Kubicek [J. Math. Anal. Appl. 59 (1977), 498–504.]

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1. Introduction

In the recent years several authors [1, 2, 4, 5] have studied the convergence of the sequence of the Mann iterates [3] of a mapping H to a fixed point of H, under various contractive conditions. The Ishikawa iteration scheme [2] was first used to establish the strong convergence for a pseudo contractive self-mapping of a convex compact subset of a Hilbert space. Very soon both iterative processes were used to establish the strong convergence of the respective iterates for some contractive type mappings in Hilbert spaces and then in more general normed linear spaces. Recently, Ćirić et. al. [1] showed that if the sequence of Ishikawa iterations associated with two mappings converges in the a convex metric space, then its limit point is the common fixed point of two mappings. In this paper, we prove that if the sequence of Ishikawa iterations associated with $2k$ mappings converges in the a convex metric space, then its limit point is the common fixed point of $2k$ mappings.

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2. Main Results

Definition 2.1. ([6]) Let X be a metric space and $I = [0, 1]$ the closed unit interval. A continuous mapping $W : X \times X \times I \longrightarrow X$ is said to be a convex structure on X if for all x, y in X, λ in I, $d[u, W(x, y, \lambda)] \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$ for all u in X. A space X together with a convex structure is called a convex metric space.

Theorem 2.2. *Let C be a nonempty closed convex subset of a convex metric space X and let $A, B, T, S : X \longrightarrow X$ be self-mappings satisfy the following condition:*

$$\begin{aligned} (2.1) \qquad & d(Ax, By) + d(Sx, Ty) \\ \leq\ & h[2d(x, y) + d(x, Ty) + d(y, Sx) + d(y, Ax) + d(x, By)] \end{aligned}$$

for all x, y in X where $0 < h < 1$. Suppose that $\{x_n\}$ is Ishikawa type iterative scheme with A, B, S and T, defined by

(1) $x_0 \in C$;

(2) $y_n = W(Ax_n, x_n, \beta_n) = W(Sx_n, x_n, \delta_n), \quad n \geq 0$;

(3) $x_{n+1} = W(By_n, x_n, \alpha_n) = W(Ty_n, x_n, \gamma_n), \quad n \geq 0$;

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy $0 \leq \alpha_n, \beta_n, \delta_n, \gamma_n \leq 1$ and $\{\alpha_n\}$ and $\{\gamma_n\}$ are bounded away from zero. If $\{x_n\}$ converges to some $p \in C$, then p is common fixed point of A, B, S and T.

Proof. It is clear that

$$d[x, W(x, y, \lambda)] = (1 - \lambda)d(x, y); \quad d[y, W(x, y, \lambda)] = \lambda d(x, y),$$

(see [1]).
From (3) it follows that

$$d(x_n, x_{n+1}) = d[x_n, W(By_n, x_n, \alpha_n)] = \alpha_n d(x_n, By_n)$$

and

$$d(x_n, x_{n+1}) = d[x_n, W(Ty_n, x_n, \alpha_n)] = \gamma_n d(x_n, Ty_n).$$

Since $x_n \longrightarrow p$, then $d(x_n, x_{n+1}) \longrightarrow 0$. Since $\{\alpha_n\}$ and $\{\gamma_n\}$ are bounded away from zero, it follows that

$$(2.2) \qquad \lim_{n \longrightarrow \infty} d(x_n, By_n) = d(x_n, Ty_n) = 0.$$

Using (2.1) we get:

$$\begin{aligned} & d(Ax_n, By_n) + d(Sx_n, Ty_n) \\ \leq\ & h[2d(x_n, y_n) + d(x_n, Ty_n) + d(y_n, Sx_n) + d(y_n, Ax_n) + d(x_n, By_n)]. \end{aligned}$$

From (2) and (3) we have that

$$\begin{aligned} d(x_n, y_n) &= d[x_n, W(Ax_n, x_n, \beta_n)] = \beta_n d(x_n, Ax_n), \\ d(x_n, y_n) &= d[x_n, W(Sx_n, x_n, \delta_n)] = \delta_n d(x_n, Sx_n), \\ d(Ax_n, y_n) &= d[Ax_n, W(Ax_n, x_n, \beta_n)] = (1 - \beta_n)d(x_n, Ax_n), \\ d(Sx_n, y_n) &= d[Sx_n, W(Sx_n, x_n, \delta_n)] = \delta_n d(x_n, Sx_n). \end{aligned}$$

Thus we have

$$\begin{aligned} & d(Ax_n, By_n) + d(Sx_n, Ty_n) \\ \leq\ & h[d(x_n, Ty_n) + d(x_n, Sx_n) + d(x_n, Ax_n) + d(x_n, By_n)]. \end{aligned}$$

Since

$$d(x_n, Ax_n) \leq d(Ax_n, By_n) + d(x_n, By_n)$$

and

$$d(x_n, Sx_n) \leq d(Sx_n, Ty_n) + d(x_n, Ty_n),$$

we get

$$d(Ax_n, By_n) + d(Sx_n, Ty_n) \leq h[2d(x_n, Ty_n) + d(Sx_n, Ty_n) + d(Ax_n, By_n) + 2d(x_n, By_n)].$$

Therefore

$$d(Ax_n, By_n) + d(Sx_n, Ty_n) \leq \frac{2h}{1-h}[d(x_n, Ty_n) + d(x_n, By_n)].$$

Taking the limit as $n \longrightarrow \infty$ we obtain, by (2.2),

$$\lim_{n\longrightarrow\infty} d(Ax_n, By_n) = \lim_{n\longrightarrow\infty} d(Sx_n, Ty_n) = 0.$$

Since $By_n \longrightarrow p$, it follows that $Ax_n \longrightarrow p$ also since Since $Ty_n \longrightarrow p$, it follows that $Sx_n \longrightarrow p$. Since $d(x_n, y_n) = \beta_n d(x_n, Ax_n)$, it follows that $y_n \longrightarrow p$.

From (2.1) again, we have that

$$\begin{aligned} & d(Ax_n, Bp) + d(Sx_n, Tp) \\ \leq\ & h[2d(x_n, p) + d(x_n, Tp) + d(p, Sx_n) + d(p, Ax_n) + d(x_n, Bp)] \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ we obtain

$$d(p, Bp) + d(p, Tp) \leq h[d(p, Bp) + d(p, Tp)].$$

Since $h \in (0, 1)$ then $d(p, Bp) + d(p, Tp) = 0$. Then $Bp = Tp = p$. Similarly, from (2.1),

$$\begin{aligned} & d(Ap, Bx_n) + d(Sp, Tx_n) \\ \leq\ & h[2d(p, x_n) + d(p, Tx_n) + d(x_n, Sp) + d(x_n, Ax_n) + d(p, Bx_n)] \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ we get

$$d(Ap, p) + d(Sp, p) \leq h[d(Ap, p) + d(Sp, p)].$$

Since $h \in (0, 1)$ then $d(Ap, p) + d(Sp, p) = 0$. Then $Ap = Sp = p$. Hence, $Ap = Bp = Sp = Tp = p$ and the proof is complete. □

Corollary 2.3. *Let C be a nonempty closed convex subset of a convex metric space X and let $A_1, A_2, ..., A_{2k} : X \longrightarrow X$ be self-mappings satisfy the following condition:*

$$\sum_{i=1}^{2k-1} d(A_i x, A_{i+1} y) \leq h[kd(x, y) + \sum_{i=1}^{k} d(y, A_{2i-1} y) + \sum_{i=1}^{k} d(x, A_{2i} y)]$$

for all x, y in X where $0 < h < 1$. Suppose that $\{x_n\}$ is Ishikawa type iterative scheme with $A_1, A_2, ..., A_{2k}$ defined by

(1) $x_0 \in C$;

(2) $y_n = W(A_1 x_n, x_n, \beta_{1,n}) = W(A_2 x_n, x_n, \beta_{2,n}) = \cdots = W(A_{2k-1} x_n, x_n, \beta_{k,n})$, $n \geq 0$;

(3) $x_{n+1} = W(A_2 y_n, x_n, \alpha_{1,n}) = W(A_4 y_n, x_n, \alpha_{2,n}) = \cdots = W(A_{2k} x_n, x_n, \alpha_{k,n})$, $n \geq 0$;

where $\{\alpha_{i,n}\}$ and $\{\beta_{i,n}\}$ for $i = 1, 2, ..., k$ satisfy $0 \leq \alpha_{i,n}, \beta_{i,n} \leq 1$ for $i = 1, 2, ..., k$ and $\{\alpha_{i,n}\}$ for $i = 1, 2, ..., k$ are bounded away from zero. If $\{x_n\}$ converges to some $p \in C$, then p is common fixed point of $A_1, A_2, ..., A_{2k}$.

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DOUBLE σ-CONVERGENCE LACUNARY STATISTICAL SEQUENCES

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ABSTRACT. In this paper we introduce two notions of σ-convergence for double sequences namely, σ-statistically P-convergence and lacunary σ-statistically P-convergence. These concepts are used to present multidimensional inclusion theorems.

1. INTRODUCTION

Let us consider the following: Let l_∞ and c be the Banach spaces of bounded and convergent sequences with the usual supremum norm. Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean if and only if

i $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all k,
ii $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and
iii $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in l_\infty$.

Throughout this paper we shall consider the mapping σ has having on finite orbits, that is $\sigma^m(k) \neq k$ for all nonnegative integers with $m \geq 1$, where $\sigma^m(k)$ is the m-th iterate of σ at k. Thus σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all off whose σ-mean are equal.

In the case when $\sigma(k) = k + 1$, the σ-mean is often called the Banach limit and V_σ is the set of almost convergent sequences, which was presented by Lorentz in [6]. It can be shown that

$$V_\sigma = \left\{ x \in l_\infty : \lim_m t_{m,n}(x) = s \text{ uniformly in } n,\ s = \sigma - \lim x \right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^m(n)}}{m+1},\ t_{-1,n}(x) = 0.$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent provided that $x \in V_\sigma$.

Definition 1.1 (Fast, Fridy; [5, 4]). *The sequence* $[x]$ *has* **statistic limit** L, *denoted by* $st - \lim x = L$ *provided that for every* $\epsilon > 0$,

$$\lim_n \frac{1}{n} \{ \textit{the number of } k \leq n : |x_k - L| \geq \epsilon \} = 0.$$

The idea of statistically convergence of sequence of real numbers was introduced by Fast in [4]. Schonberg in [13] studied statistically convergence as a summability method and listed some of the elementary properties of statistical convergence.

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Both of these authors noted that if a bounded sequence is statistically convergent to L, then it is Cesàro summable to L. Recently, Connor in [1] extended the statistical convergence definition to A-statistical convergence by using nonnegative regular matrices.

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\{\frac{K_{m,n}}{mn}\}_{m,n=1,1}^{\infty,\infty}$ has a limit then we say that K has a natural density and is defined

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero). Quite recently, Mursaleen and Edely [7], defined the statistical analogue for double sequences $x = (x_{k,l})$ as follows: A real double sequences $x = (x_{k,l})$ is said to be P-statistically convergent to L provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \{\text{number of } (j,k) : j < m \text{ and } k < n, |x_{j,k} - L| \geq \epsilon\} = 0.$$

In this case we write $st_2 - \lim_{m,n} x_{m,n} = L$ and we denote the set of all P-statistical convergent double sequences by st_2. By a bounded double sequence we shall mean a positive number M exists such that $|x_{j,k}| < M$ for all j and k, $||x||_{(\infty,2)} = \sup_{j,k} |x_{j,k}| < \infty$. We will denote the set of all bounded double sequences by $l_\infty^{''}$.

Definition 1.2. *The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called* **double lacunary** *if there exist two increasing of integers such that*

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Notations: $k_{r,s} = k_r l_s$, $h_{r,s} = h_r h_s$, $\theta_{r,s}$ is determine by $I_{r,s} = \{(i,j) : k_{r-1} < i \leq k_r \ \& \ l_{s-1} < j \leq l_s\}$, with $q_r = \frac{k_r}{k_{r-1}}$, $q_s = \frac{l_s}{l_{s-1}}$, and $q_{r,s} = q_r q_s$.

2. Main Result

We shall extend notion presented in the introduction to σ-statistically P-convergence.

Definition 2.1. *A double complex number sequence $x = x_{k,l}$ is said to be σ-**statistically P-convergent** to the number L if for every $\epsilon > 0$*

$$P - \lim_{p,q} \frac{1}{pq} \left\{ \text{the number of } k,l \leq p,q : \left| x_{\sigma^k(m),\sigma^l(n)} - L \right| \geq \epsilon \right\} = 0$$

uniformly in (m,n).

In this case we write
$S_{\sigma}^{''} - \lim x = L$ or $x_{k,l} \to L(S_{\sigma}^{''})$ where

$$S_{\sigma}^{''} = \{x = (x_{k,l}) : S_{\sigma}^{''} - \lim x = L, \text{ for some } L\}$$

Definition 2.2. *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary. The double real number sequence $x = (x_{k,l})$ is said to be $S_{\sigma,\theta}^{''}$-**convergent** to the number L provided that for every $\epsilon > 0$*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in I_{r,s} : \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \geq \epsilon\right\}\right| = 0$$

uniformly in (m,n). $S_{\sigma,\theta}^{''} - \lim x = L$ or $x_{k,l} \to L(S_{\sigma,\theta}^{''})$ The set such double sequences shall be denoted by

$$S_{\sigma,\theta}^{''} = \{x = (x_{k,l}) : S_{\sigma,\theta}^{''} - \lim x = L, \text{ for some } L\}.$$

Throughout this paper we will also use the following notations:

$$A_{\alpha,\beta}^{\gamma,\delta} = \left\{ (i,j) : \begin{array}{c} \alpha \leq i < \gamma - 1 \\ \text{or} \\ \beta \leq j < \delta - 1 \end{array} \right\}$$

and

$$B_{m,n} = \left\{ (i,j) : \begin{array}{c} 0 \leq i < m - 1 \\ \text{or} \\ 0 \leq j < n - 1 \end{array} \right\}.$$

Definition 2.3. *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary. The double real number sequence $x = (x_{k,l})$ is said to be $\bar{S}_{\sigma,\theta}^{''}$-**convergent** to the number L provided that for every $\epsilon > 0$*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \left|\left\{(k,l) \in B_{h_r,h_s} : \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \geq \epsilon\right\}\right| = 0$$

uniformly in (m,n). $\bar{S}_{\sigma,\theta}^{''} - \lim x = L$ or $x_{k,l} \to L(\bar{S}_{\sigma,\theta}^{''})$ The set such double sequences shall be denoted by

$$\bar{S}_{\sigma,\theta}^{''} = \{x = (x_{k,l}) : \bar{S}_{\sigma,\theta}^{''} - \lim x = L, \text{ for some } L\}.$$

We now introduce a new concept of strong double σ-convergence by combining σ-convergence with lacunary sequences, which yields multidimensional analog of Definition 2.1 in [11].

$$L_{\theta}^{''} = \{x = (x_{k,l}) : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| = 0 \text{ uniformly in } (m,n)\}.$$

We shall now prove some analogues for double sequences. For single sequences such results have been presented by Savaş and Nuray in [10].

Theorem 2.1. *Let $\theta_{r,s} = \{(k_r, l_s)\}$ be a double lacunary then*

1. *if $x \to L(L_{\theta}^{''})$ then $x \to L(S_{\sigma,\theta}^{''})$,*
2. *if $x \in l_{\infty}^{''}$ and $x \to L(S_{\sigma,\theta}^{''})$ then $x \to L(L_{\theta}^{''})$, and*
3. *$S_{\sigma,\theta}^{''} \cap l_{\infty}^{''} = L_{\theta}^{''}$.*

Proof. Part (1): If $\epsilon > 0$ and $x \to L(L_\theta^{''})$ then

$$\begin{aligned}\sum_{(k,l)\in I_{r,s}} \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| &\geq \sum_{(k,l)\in I_{r,s}\&\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|\geq\epsilon} \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \\ &\geq \epsilon\left|\left\{(k,l)\in I_{r,s} : \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \geq \epsilon\right\}\right|.\end{aligned}$$

Therefore $x \to L(S_{\sigma,\theta}^{''})$. Part (2): Suppose $[x]$ is in $l_\infty^{''}$ and $x \to L(S_{\sigma,\theta}^{''})$. Then we can assume that

$$\left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \leq M \text{ for all } k, l, m, \text{ and } n.$$

Given $\epsilon > 0$

$$\begin{aligned}\frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| &= \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}\&\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|\geq\epsilon} \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \\ &+ \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}\&\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|<\epsilon} \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \\ &\leq \frac{M}{h_{r,s}}\left|\left\{(k,l)\in I_{r,s} : \left|x_{\sigma^k(m),\sigma^l(n)} - L\right| \geq \epsilon\right\}\right| + \epsilon.\end{aligned}$$

Therefore $x \to L(L_\theta^{''})$. Part (3): follows from (1) and (2). □

We now present a lemma which shall be used in the following theorem.

Lemma 2.1. *Suppose for given $\epsilon_1 > 0$ and every $\epsilon > 0$ there exist m_0 and n_0 such that*

$$\frac{1}{mn}\left|\left\{(k,l)\in B_{m,n} : \left|x_{\sigma^k(r),\sigma^l(s)} - L\right| \geq \epsilon\right\}\right| < \epsilon_1,$$

for all $m \geq m_0$ $n \geq n_0$, $r \geq r_0$ and $s \geq s_0$ then $x \to L(S_{\sigma,\theta}^{''})$.

Proof. Let ϵ_1 be given. For every $\epsilon > 0$, choose n_0^1 m_0^1, r_0 and s_0 such that

$$\frac{1}{mn}\left|\left\{(k,l)\in B_{m,n} : \left|x_{\sigma^k(r),\sigma^l(s)} - L\right| \geq \epsilon\right\}\right| < \frac{\epsilon_1}{2}, \tag{2.1}$$

for all $m \geq m_0^1$, $n \geq n_0^1$, $r \geq r_0$ and $s \geq s_0$.

It is sufficient to prove that there exists $m_0^{1,1}$ and $n_0^{1,1}$ such that for $m \geq m_0^{1,1}$, $n \geq n_0^{1,1}$, $0 \leq r \leq r_0$, and $0 \leq s \leq s_0$

$$\frac{1}{mn}\left|\left\{(k,l)\in B_{m,n} : \left|x_{\sigma^k(r),\sigma^l(s)} - L\right| \geq \epsilon\right\}\right| < \epsilon_1. \tag{2.2}$$

If we let $m_0 = \max\{m_0^1, m_0^{1,1}\}$ and $n_0 = \max\{n_0^1, n_0^{1,1}\}$ equation (2.2) holds for $m > m_0$ and $n > n_0$ and for all r and s. Note once r_0 and s_0 are chosen where $0 \leq r \leq r_0$ and $0 \leq s \leq s_0$. Thus fixed (r_0, s_0). We can let

$$\left|\left\{(k,l)\in B_{r_0,s_0} : \left|x_{\sigma^k(r),\sigma^l(s)} - L\right| \geq \epsilon\right\}\right| = K.$$

Now equation (2.1) grants us the following:

$$\begin{aligned}\frac{1}{mn}\left|\left\{(k,l)\in B_{m,n}:\left|x_{\sigma^k(r),\sigma^l(s)}-L\right|\geq\epsilon\right\}\right| &\leq \frac{1}{mn}\left|\left\{(k,l)\in B_{r_0,s_0}:\left|x_{\sigma^k(r),\sigma^l(s)}-L\right|\geq\epsilon\right\}\right| \\ &+ \frac{1}{mn}\left|\left\{(k,l)\in A^{m,n}_{r_0,s_0}:\left|x_{\sigma^k(r),\sigma^l(s)}-L\right|\geq\epsilon\right\}\right| \\ &\leq \frac{K}{mn}+\frac{1}{mn}\left|\left\{(k,l)\in A^{m,n}_{r_0,s_0}:\left|x_{\sigma^k(r),\sigma^l(s)}-L\right|\geq\epsilon\right\}\right| \\ &\leq \frac{K}{mn}+\frac{\epsilon_1}{2}.\end{aligned}$$

Thus for m and n sufficiently large

$$\frac{1}{mn}\left|\left\{(k,l)\in B_{m,n}:\left|x_{\sigma^k(r),\sigma^l(s)}-L\right|\geq\epsilon\right\}\right|\leq\frac{M}{mn}+\frac{\epsilon_1}{2}<\epsilon_1.$$

Thus equation (2.2) holds. This completes the proof of the lemma. □

We now establish the next theorem.

Theorem 2.2. *$\bar{S}''_{\sigma,\theta}=S''_{\sigma}$ for every double lacunary θ.*

Proof. Let $x\in\bar{S}''_{\sigma,\theta}$, then Definition 2.2 assures us that, given $\epsilon_1>0$ there exist r_0, s_0, $\epsilon>0$, and L such that

$$\frac{1}{h_{r,s}}\left|\left\{(k,l)\in B_{h_r,h_s}:\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|\geq\epsilon\right\}\right|<\epsilon_1,$$

for $r\geq r_0$ and $s\geq s_0$ and $m=k_{r-1}+1+u$ and $n=l_{s-1}+1+v$ where $u\geq 0$ and $v\geq 0$. Let $p\geq h_r$ and $q\geq h_s$ and write $p=ih_r+\alpha$ and $q=ih_s+\beta$ where $0\leq\alpha\leq h_r$, and $0\leq\beta\leq h_s$, i is an integer. Since $p\geq h_r$, $q\geq h_s$, and $i\geq 1$ we obtain the following:

$$\begin{aligned}&\frac{1}{pq}\left|\left\{(k,l)\in B_{p,q}:\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|\geq\epsilon\right\}\right| \\ \leq\ &\frac{1}{pq}\left|\left\{(k,l)\in B_{(i+1)h_r,(i+1)h_s}:\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|\geq\epsilon\right\}\right| \\ =\ &\frac{1}{pq}\sum_{\gamma=0}^{i}\left|\left\{\gamma h_r\leq k\leq(\gamma+1)h_r-1\cup\gamma h_s\leq l\leq(\gamma+1)h_s-1:\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|\geq\epsilon\right\}\right| \\ \leq\ &\frac{(i+1)^2h_rh_s}{pq}\epsilon_1 o(1) \\ \leq\ &\frac{4i^2h_rh_s\epsilon_1}{pq}o(1);\ \text{for } i\geq 1.\end{aligned}$$

Since $\frac{h_r}{p}\leq 1$ and $\frac{h_s}{q}\leq 1$ it is clear that $\frac{ih_r}{p}\leq 1$ and $\frac{ih_s}{q}\leq 1$. Therefore

$$\frac{1}{pq}\left|\left\{(k,l)\in B_{p,q}:\left|x_{\sigma^k(m),\sigma^l(n)}-L\right|\geq\epsilon\right\}\right|\leq\epsilon_1.$$

Hence Lemma (2.1) implies $\bar{S}''_{\sigma,\theta}\subseteq S''_{\sigma}$. It is also clear that $S''_{\sigma}\subseteq\bar{S}''_{\sigma,\theta}$. This completes the proof. □

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COMPUTATIONALLY TESTABLE CONDITIONS FOR EXCITABILITY AND TRANSPARENCY OF A CLASS OF TIME-DELAY SYSTEMS WITH POINT DELAYS

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Abstract. This article is concerned with the excitability of positive linear time-invariant systems subject to internal point delays. It is proved that the excitability independent of delay is guaranteed if an auxiliary delay-free system is excitable. Necessary and sufficient conditions for excitability and transparency are formulated in terms of the parameterization of the dynamics and control matrices and, equivalently, in terms of strict positivity of a matrix of an associate system obtained from the influence graph of the original system. Such conditions are testable through simple algebraic tests involving moderate computational effort.

Keywords*:* excitable systems, positive systems, point delays, time-delay systems, transparency.

I. Introduction

The so-called positive systems are characterized by the fact that its relevant signals are nonnegative for all time. In particular, the property of external positivity implies that all the components of the output are non-negative for all time and all non-negative input and the property of internal positivity means that all the components of both the state and the output are non-negative for all time and for any non-negative input and initial conditions, [1-9]. Positive systems are very relevant in some continuous-time and discrete- time problems of the common life which cannot be described by negative signals, like, for instance, population dynamics evolutions, prey-predator problems, Ecological and Biological problems, like for instance, the population evolution in a certain habitat being modeled by the well-known Beverton- Holt equation,[27], the models for chemostat devices etc, [1-3, 5, 27]. Internally positive (usually referred to as positive) systems have non-negative control, output and input-output interconnection matrices (i.e. all the entries of those matrices are non-negative) and, furthermore, their matrix of dynamics is a so-called Metzler matrix (i.e. all its off-diagonal entries are non-negative), [1], [5]. Positive systems are also described in [24] for Leontieff models using in the economical production system,[24] and may be characterized in general arbitrary cones and also, in particular, for abstract delayed equations rather than in the first orthant (see, for instance, [24-25]). However, it is preferred in this manuscript to give a clear simpler characterization of positivity characterized in the first orthant which has a clear insight in related Engineering and Biology/ Ecology problems. Some major facts associated with internally positive systems which make very peculiar from the point of view of systems theory are the following ones:

1) Internal positivity of realizations in real canonical forms is always associated to real eigenvalues of the dynamics of the system. Otherwise, some state- solution trajectories possessing sub-trajectories inside the first quadrant of the appropriate two-dimensional phase plane of the phase space with conjugate complex eigenvalues would leave such a

quadrant locally around the zero equilibrium at certain time intervals. The reason is the presence of vortex or focus equilibrium points for the pair of coordinates defining such a phase plane. Under state/output feedback, this would result also in controls having negative values at certain time intervals, [1].This would result in the system not being internally positive.

2) Internal positivity is dependent on the chosen state-space realization in the sense that if a state-space realization is internally positive another one related to it via a non-singular transformation might not be internally positive. For instance, a canonical Jordan realization with real (simple or multiple) eigenvalues with all the components of the control and output matrices being non-negative is internally positive. However, if all the coefficients of its characteristic polynomial are positive, which is always the case if the system is stable, then its algebraically equivalent companion controllability canonical form is never internally positive since the entries of the last row of its dynamics matrix are always negative.

3) Stability and internal positivity might be properties in conflict as emphasized in the above discussion since a stable canonical controllability state-space realization cannot be internally positive although the algebraically equivalent stable canonical Jordan form leading to the same associate transfer matrix is internally positive.

On the other hand, many dynamic systems common in Nature have associated delays which may be internal (i.e. in the state) or external (i.e. in the input or output), [11-16]. Examples of systems subject to delays are abundant in the literature like, for instance, transportation problems, population growth problems, electric power transmission to large distances, peace-war models, chemical processes, heat exchangers and some biological problems like, for instance, the sunflower dynamics, [10]. A lot of scientific work is being devoted in this last years to the study of the basic properties of such systems like, for instance, observability, controllability, stabilizability, closed-loop stabilization and model-matching as well as the use of the associate formalism in practical control implementations [17-22]. An important effort has been recently devoted to new perspectives and applications in the field of time-delay systems. For instance, the robust decentralized closed-loop stabilization of interconnected systems with certain nonlinearities has been investigated in [28] by means of a fuzzy controller. On the other hand, the robust stabilization of cellular neural networks involving both discrete and distributed delays has been focused on in [29]. The state-feedback stabilization of systems possessing delays trough a memoryless controller and the optimal filtering under multiple state and observation delays have been investigated in [30] and [31], respectively. A major drawback to cope with time-delay systems from an analytical pointy of view is that internal delays make a dynamic system to become infinite-dimensional then possessing infinitely many associated modes [11-13]. *This paper is devoted to investigate the so-called excitability and transparency properties of time-invariant systems subject to constant point delays*. Those properties have been characterized for positive delay-free systems in [5] but the existing background results in the field of time-delays are rather scarce. The excitability is the system capability of exciting all its state variables (i.e. none of them remains identically zero for all finite time) while being initially at rest by application of some positive input. Transparency is the property of exciting all the output components of the unforced response by any given positive function of initial conditions. Excitability of the output components rather than all the state variables will be referred to as external excitability. The

transparency will be referred to as “weak transparency” if the excitation of all the output components is only achievable under positive point initial conditions. It will be referred to as “strong transparency” or, simply, “transparency” if any positive initial condition of a subinterval of nonzero measure of its definition domain excites all the output components. This subdivision is inherent to the fact that the system is subject to delays. A main consequence of this research is that the properties of excitability and transparency can be achieved by a combined effect of the delay- free and delayed dynamics. In this paper, both properties are characterized for systems under internal delays and related to parallel properties of two auxiliary systems defined for zero and infinity delay, respectively. In particular, related new results dependent on and independent of the delay size are obtained. The mechanism which allows obtaining those two different results is to split the characterization of the excitability/transparency properties into two vectors characterizing excitability, respectively, transparency, which are summed up in the same right-hand-side formula. If one of those vectors, which is independent of the delayed dynamics, is strictly positive, then the corresponding property is automatically guaranteed independently of the delay size. The extension of both characterizations to the multivariable case would consist of the fact that at least one entry per row of a certain matrix related to excitability/ transparency is positive. This implies that a corresponding nonzero component of the input excites the associate component of the state vector for the excitability property. The external excitability is characterized in a similar way as the (internal) excitability as the property implying that all the components of the output trajectory solution take strictly positive values in finite time. The transparency property is characterized mutatis- mutandis via the corresponding vector/ matrix for testing. If the property holds independent of the delay, then such a property is not dependent on the matrix which defines the delayed dynamics. Less stringent conditions are obtained if the whole matrix or vector characterizing the excitability / transparency has a positive entry per row by jointly considering both right-hand-side terms of the summed-up vector or matrix. If the excitability/transparency condition holds in this case but not in the case of just examining the first time then the property is characterized as being dependent of the delay size and it also depends on the delayed dynamics matrix as a result. In this context, a system which is not excitable independent of the delay might become excitable for a particular delay depending on the matrix which defines the delayed dynamics.

Notation Notes: Consider real matrices $V=\left(V_{ij}\right)\in \mathbf{R}^{n\times m}$ and vectors $v=\left(v_i\right)\in \mathbf{R}^{n}$. Then, the subsequent notation will be then useful: $v\geq 0 / V\geq 0$ (no vector component / matrix entry is negative), $v>0 / V>0$ (at least a component / entry is positive), $v>>0 / V>>0$ (all components/ entries are positive). The whole set of nonnegative real $n\times m$ matrices (n-vectors) including the positive and strictly positive ones is denoted by $V\in \mathbf{R}_{+}^{n\times m}$ ($v\in \mathbf{R}_{+}^{n}$). $\mathbf{R}_{0+}^{n}=\mathbf{R}_{+}^{n}\cup\left\{0\in \mathbf{R}^{n}\right\}$ for any $n\geq 1$ and a similar definition would apply to real matrices .

On the other hand, the notation “ iff “ is an abbreviation for “ if and only if “ as usual. A finite set with the first n natural numbers is defined as $\overline{n}=\left\{1,2,...,n\right\}$.

e_i denotes the i-th Euclidean unity vector of $\mathbf{R}^{n}$ for any $n\geq 2$ whose the only nonzero component is the i-th one.

II. The class of single point constant time-delay systems: State-trajectory solution

Consider the single-input single-output linear time-invariant system

$$S_h: \dot{x}(t) = A x(t) + A_0 x(t-h) + b u(t) \quad ; \qquad y(t) = c^T x(t) \tag{1}$$

where $x(t)$ is the state n-vector and u (t) and y (t) are, respectively, the scalar control input signal and $h \geq 0$ is the point delay. If h >0, then the above system possesses internal delayed dynamics. The real square n-matrices A and A_0 are associated with the delay-free and delayed dynamics, respectively and b and c are n dimensioned real control and output vectors, respectively. Particular delay- free systems which lie in the class (1) are:

$$S_0: \quad \dot{x}(t) = (A + A_0) x(t) + b u(t) \quad ; \qquad y(t) = c^T x(t) \tag{2.a}$$

$$S_\infty: \quad \dot{x}(t) = A x(t) + b u(t) \quad ; \qquad y(t) = c^T x(t) \tag{2.b}$$

Note that S_0 and S_∞ are obtained from (1) for h = 0 and $h \to \infty$, respectively. In the first case, the delayed dynamics becomes delay–free for zero delay since (2.a) equalizes (1) for h = 0. In the second one, an infinity delay makes the delayed dynamics to be trivially irrelevant since the initial conditions have a finite time interval definition domain. In this context, note that $\dot{x}(t) = A x(t)$ for any finite time as $h \to \infty$ and zero control so that the unforced state-trajectory solution is generated by the infinitesimal generator A for all finite time. A close description to the first equation (1) applies to linear time-invariant discrete-time single-input single-output systems of delayed discrete dynamics defined by $x_{k+1} = A x_k + A_0 x_{k-1} + b u_k$ with the replacements $t = kT \to k$ (integer index for running samples), $h = T = 1$(sampling period) $\dot{x}(t) \to x_{k+1}$, $x(t) \to x_k$, $u(t) \to u_k$. The auxiliary system (2.a) describes (1) for infinite delay. The following result holds:

Theorem 1: For any absolutely continuous function of initial conditions $\varphi: [-h, 0] \to \mathbf{R}^n$ with $\varphi(0) = x(0) = x_0$ and any piecewise- continuous control input, the state- trajectory solution of S_h is unique and given by any of the equivalent expressions below:

$$x(t, \varphi, u) = e^{At} x_0 + \int_{-h}^{0} e^{A(t-h-\tau)} A_0 \varphi(\tau) d\tau + \int_{0}^{t-h} e^{A(t-h-\tau)} A_0 \mathbf{1}(\tau - h) x(\tau) d\tau + \int_{0}^{t} e^{A(t-\tau)} b u(\tau) d\tau$$

$$= \Psi_h(t) x_0 + \int_{-h}^{0} \Psi_h(t-h-\tau) A_0 \varphi(\tau) d\tau + \int_{0}^{t} \Psi_h(t-\tau) b u(\tau) d\tau \tag{3}$$

where $\mathbf{1}(t)$ is the unit step (Heaviside) function with discontinuity at t = 0, e^{At} is a C_0- semigroup (popularly known as the state transition matrix) associated with the infinitesimal generator A of S_∞, which satisfies $d(e^{At})/dt = A e^{At}$, and $\Psi_h(t)$ is the evolution operator of S_h for any $h \geq 0$ which satisfies

$$\dot{\Psi}_h(t) = A\,\Psi_h(t) + A_0\,\Psi_h(t-h) \tag{4}$$

and it is explicitly defined by

$$\Psi_h(t) = e^{At}\left(I + \int_0^t e^{-A\tau} A_0\,\Psi_h(\tau-h)\mathbf{1}(\tau-h)\,d\tau\right) = \begin{cases} 0 & \text{for } t<0 \\ e^{At} & \text{for } t\in(0,h] \\ e^{At}\left(I + \int_0^t e^{-A\tau} A_0\,\Psi_h(\tau-h)\,d\tau\right) & \text{for } t\geq h \end{cases} \tag{5}$$

where I is the n-identity matrix. The output trajectory solution $y(t,\varphi,u) = c^T x(t,\varphi,u)$ follows directly from (3).

Outline of Proof: The first expression in (3) follows directly from (1) by considering $v(t,t-h,x,u) = A_0 x(t-h) + b\,u(t)$ as an extended forcing time at time t. Taking time-derivatives in (5) for $\Psi_h(t)$, Eq. 4 follows, provided again that (5) is true, so that (5) is the solution of (4) which is unique by using standard arguments from Picard –Lindeloff ´s theorem since the right- hand side of (1) is uniformly Lipschitz continuous. Then, by taking the first time-derivative of the second expression of (3) with the use of (4)-(5) in its right hand- side to separate x (t) and x (t-h) , Eq. 1 holds for $t \geq 0$.

III. Positivity and Excitability

In the following, admissible input functions $u:[0,\infty)\to \mathbf{R}_{0+}$ and admissible functions of initial conditions $\varphi:[-h,0]\to \mathbf{R}^n_{0+}$ are those which satisfy the respective constraints of Theorem 1 for the existence of unique trajectory solutions. The following precise definitions concerning the properties of positivity and excitability [1-2] for point time-delay systems are then used.

Definition 1: The system S_h is said to be internally positive (or simply positive) if all its state components and its output are nonnegative for all time for any admissible $u:[0,\infty)\to \mathbf{R}_{0+}$ which satisfies the constraints of Theorem 1 for the existence of solutions ($\mathbf{R}_{0+}$ being the set of nonnegative real numbers) and any admissible function of initial conditions $\varphi:[-h,0]\to \mathbf{R}^n_{0+}$, namely, fulfilling their respective constraints in Theorem 1 for existence of solution. □

Definition 2: The system S_h is said to be externally positive if its output is nonnegative for all time for any admissible control $u:[0,\infty)\to \mathbf{R}_{0+}$ and $\varphi \equiv 0$ on its definition domain [-h, 0]. □

Definition 3: The positive system S_h is said to be internally excitable (or simply excitable) if all its state components are positive (i.e. $x(t,0,u) >> 0$) for all time $t > t_0$, for some finite $t_0 \geq 0$ and some non

identically zero admissible $u:[0,\infty)\to \mathbf{R}_{0+}$ if the system is initially at rest, i.e. $\varphi \equiv 0$ on its definition domain [-h, 0]. □

Definition 4: The positive system S_h is said to be externally excitable if its output is positive (i.e. $y(t,0,u)>>0$) for all time $t>t_0$, for some finite $t_0 \geq 0$ and some non identically zero admissible $u:[0,\infty)\to \mathbf{R}_{0+}$ if the system being initially at rest. □

Note from the above definitions that positivity is related to the fact that the involved signals are nonnegative for all time for admissible nonnegative inputs while excitability stated below implies strict positivity of the involved signals after a finite interval of time for some admissible nonnegative and non identically zero inputs. More precisely, internal positivity is related to the property $x(t,\varphi,u)\geq 0$ and $y(t,\varphi,u)\geq 0$ for all $t>t_0$ for any admissible $u:[0,\infty)\to \mathbf{R}_{0+}$ and $\varphi:[-h,0]\to \mathbf{R}^{n}_{0+}$. (Internal) excitability of a positive system implies that there is some non identically zero input $u:[0,\infty)\to \mathbf{R}_{0+}$ such that $x(t,0,u)>>0$ for $t>t_0$, some finite $t_0 \geq 0$, if the system is initially at rest, i.e. $\varphi \equiv 0$ on its definition domain [-h, 0]; i.e. all the state components are strictly positive for $t>t_0$, some finite $t_0 \geq 0$. It has been proved in [2] that linear time-invariant delay-free systems are excitable iff $x(t,0,u)>>0$ for $t>0$. Similar characterizations apply for external excitability referred to as the strict positivity of the output under zero initial conditions for some non identically zero input $u:[0,\infty)\to \mathbf{R}_{0+}$.

Three assumptions follow to characterize positivity and external positivity of the system S_h:

Assumption 1: The matrix A is a Metzler matrix (namely, all its off-diagonal entries are nonnegative, [1-2]) $A_0 \geq 0$ and $b>0$.

Assumption 2: The matrix $(A+A_0)$ is a Metzler matrix and $b>0$.

Assumption 3: $c>0$ □

It is well-known that $e^{At}>0$, $\forall t\geq 0$ iff A is a Metzler matrix, [1]. Note also that *Assumption 1* implies *Assumption 2* but the converse is not true in general. The positivity of system (1) is characterized precisely in the following result.

Theorem 2. The subsequent properties follow:

(i) The system S_h is positive independent of the delay; i.e. for any $h\in[0,\infty)$, iff *Assumptions 1* and *3* hold. As a result, S_0 and S_∞ are both positive. Also, $\Psi_h(t)>0$ for any $h\in[0,\infty)$.

(ii) The system S_0 is positive iff *Assumptions 2-3* hold.

Proof: (i) "If part" Since A is a Metzler matrix, $e^{At} > 0$, $\forall t \geq 0$. Since $A_0 \geq 0$ and $b > 0$, $x(t) \geq 0$ for $t \in [0, h)$, $\forall h \in [0, \infty)$, from (3) for any $u: [0, \infty) \to \mathbf{R}_{0+}$ and $\varphi: [-h, 0] \to \mathbf{R}^n_{0+}$. Proceeding then constructively with (3), it follows that $x(t, \varphi, u) \geq 0$ and $y(t, \varphi, u) \geq 0$, since c > 0, for all $t \geq 0$ and S_h is positive for all $h \in [0, \infty)$ and as $h \to \infty$ so that S_0 and S_∞ are both positive as well. From (4) , the evolution operator $\Psi_h(t)$ evolves through time as the unforced (1) so that $\Psi_h(t) \geq 0$ for any $h \in [0, \infty)$ and all $t \geq 0$, since $\Psi_h(t)$ is an evolution operator of (1), it is nonsingular for all $t \geq 0$ so that $\Psi_h(t) > 0$.

"Only if part" It follows proceeding by contradiction for all possible cases. If b > 0 fails then there is some $b_i < 0$, some $i \in \bar{n} := \{1, 2, ..., n\}$. For $\varphi \equiv 0$, some sufficiently large real constant M > 0, and any input fulfilling $u(t) > M$, $\forall t \geq 0$, it follows $x_i(t, 0, u) < 0$ ($i \in \bar{n}$), some $t \geq 0$, and S_h is not internally positive for any $h \in [0, \infty)$. If $A_0 \geq 0$ fails (i.e. $-(A_0) \geq 0$) then, there is some $(i, j) \in \bar{n} \times \bar{n}$ such that $A_{0ij} < 0$. Assume $u \equiv 0$ and initial conditions $\varphi(t) = (0, ..., \varphi_i(t) = k_i > 0, ..., 0)^T$, k_i being a constant, $\forall t \in [-h, 0]$, some $i \in \bar{n}$. Then, $x_i(0^+, \varphi, 0) < 0$ so that S_h is not positive for any delay $h \in [0, \infty)$. If c > 0 fails then there is $j \in \bar{n}$ such that $c_j < 0$. From (1) and (3), $y(0, x_0, 0) < 0$, and then S_h is not positive, for zero input provided that $\varphi(t) = 0$, $\forall t \in [-h, 0)$ and $\varphi(0) = x(0) = x_0 > 0$ satisfies the constraint $x_{0j} := x_j(0, x_0, 0) > \dfrac{\sum_{j \neq i = 1}^{n} c_i x_{0i}}{|c_j|}$. Finally, if A is not a Metzler matrix then $x(t) \geq 0$ fails for $u \equiv 0$, $\varphi(\tau) \equiv 0$, $\forall \tau \in [-h, 0)$, and $\varphi(0) = x_0 > 0$.

(ii) $(A + A_0)$ is a Metzler matrix iff $e^{(A + A_0)t} > 0$ for all $t \geq 0$, [1], which together with $b > 0$, implies and it is implied by the delay-free system S_0 being positive. □

Note that if the system S_h is positive then it is also externally positive but the converse is not true in general. For instance, if the system S_h is positive with b> 0 and c > 0 and all the signs of all nonzero components of b and c are changed simultaneously, then the modified S_h still remains positive but it is not internally positive The external positivity of the system S_h is now formally characterized.

Theorem 3: The subsequent properties follow:

(i) The system S_h is externally positive independent of the delay iff its impulse response $g_h(t) = L^{-1}\left(c^T \left(sI - A - A_0 e^{-hs}\right) b\right)$ fulfills $g_h: [0, \infty) \to \mathbf{R}_{0+}$, where L, L^{-1} denote the Laplace transform operator and its inverse, respectively. As a result, S_0 and S_∞ are both externally positive.

(ii) The systems S_0 and S_∞ are externally positive iff $g_0:[0,\infty)\to\mathbf{R}_{0+}$ and $g_\infty:[0,\infty)\to\mathbf{R}_{0+}$, respectively.

Proof: (i) Note that for zero initial conditions and an admissible input $u:[0,\infty)\to\mathbf{R}_{+0}$:

$$y(t)=\int_{P_u(t-\tau)} g_h(t-\tau)u(\tau)d\tau-\left|\int_{N_u(t-\tau)} g_h(t-\tau)u(\tau)d\tau\right|$$

where $P_u(t)$ and $N_u(t)$ are disjoint and complementary real subintervals of [0, t] , one of them being possibly empty, defined as $P_u(t)=\{\tau\in[0,t]:u(t)\geq 0\}$ and $N_u(t)=[0,t]/P_u(t)$. If $N_u(t)=\varnothing$ (the empty set) for all $t\geq 0$ then $y(t)\geq 0$, $\forall t\in\mathbf{R}_{+0}$ and the sufficiency part has been proved. To prove the necessary condition, proceed by contradiction. Assume a time instant $t>0$ and an admissible $u:[0,\infty)\to\mathbf{R}_{+0}$ which is nonzero only on $N_u(t)$, i.e. $P_u(t)=\{\tau\in[0,t]:u(t)\geq 0\}=\{\tau\in[0,t]:u(t)=0\}$. Since the impulse response is a real continuous function $N_u(t)=[0,t]$ has a finite measure since it is nonempty. Then $y(t)<0$ and $\varphi\equiv 0$ on its definition domain [-h, 0] so that S_h is not externally positive. Thus, Property (i) has been proved.

(ii) The proof is similar to that of Property (i).

Excitability conditions: Note that the basic difference between positivity (respectively, external positivity) and excitability of positive systems (respectively, external excitability of externally positive systems) is that in the first case all the state (respectively, output) components are nonnegative for all time while in the second one they are strictly positive on some time interval. The following key technical result is then used to consider only constant control signals to characterize excitability.

Lemma 1: The positive system S_h is excitable independent of the delay iff there is a constant input $u(t)=k_u$ such that $x(t)>>0$ for the system being initially at rest.

Proof: *"If part"* If the system is excitable then there is an admissible bounded nonzero constant input $u:[0,\infty)\to\mathbf{R}_{0+}$ when applied to S_h results in

$$k_u\left(\int_0^t \Psi_h(t-\tau)b\,d\tau\right)\geq x(t,0,u(t))=\int_0^t \Psi_h(t-\tau)b\,u(\tau)d\tau>>0$$

if S_h is initially at rest, from the second expression in (3) for some $t>0$, provided that $\sup_{t\geq 0}(u(t))\leq k_u$ if S_h is positive (so that $\Psi_h(t)>0, b>0$) and initially at rest. Thus, the constant input $u(t)=k_u$ when injected to S_h initially at rest makes $x(t,0,k_u)=k_u\left(\int_0^t \Psi_h(t-\tau)b\,d\tau\right)>>0$.

"Only if part" Assume that there is no constant input k_u implying $x(t,0,k_u)=k_u\left(\int_0^t \Psi_h(t-\tau)b\,d\tau\right)>>0$, for some $t>0$. Then, for some $i\in\bar{n}$ and any arbitrary positive real constant k_u, $x_i(t,0,k_u)=k_u\left(\int_0^t e_i^T\Psi_h(t-\tau)b\,d\tau\right)=0\geq x_i(t,0,u)=\int_0^t e_i^T\Psi_h(t-\tau)b\,u(\tau)d\tau$ for all $t\geq 0$ and all admissible u(t) satisfying $u(t)\geq k_u>0$ for all $t\geq 0$, if the internally positive system S_h is initially at rest where e_i is the i-th unity Euclidean vector in $\mathbf{R}^n$. Since $k_u>0$ is arbitrary there is no bounded nonnegative input which makes all the state variables to be positive for all t > 0.

Remark 1: *Lemma 1* implies that it is necessary and sufficient to consider constant unity inputs to characterize the internal excitability property since $k_u\left(\int_0^t \Psi_h(t-\tau)b\,d\tau\right)>>0$ for $k_u>0 \Leftrightarrow \left(\int_0^t \Psi_h(t-\tau)b\,d\tau\right)>>0$.

The influence graph G of a single-input single-output dynamic system [1-2] of n-th order has (n+2) nodes associated with its the n state variables, input and output and all paths in-between those nodes provided they are linked through nonzero entries of its parameterization. An associate system S_h^G to S_h is defined from its influence graph from (1) as :

$$S_h^G:\ \dot{x}^{(G)}(t)=A^{(G)}x^{(G)}(t)+A_0^{(G)}x^{(G)}(t-h)+b^{(G)}u(t)\ ;\ y^{(G)}(t)=c^{(G)^T}x^{(G)}(t) \qquad (6)$$

where all the components and entries of the various matrix and vectors of parameters are defined from their counterparts of S_h as unity entries if their corresponding ones are nonzero and zero otherwise. Note that the associate system S_h^G to S_h is always positive by construction even if S_h is not positive.

The following result will then allow the formulation of the main result of the paper. The concept of support of the input is invoked. Note that the support of a real function is the proper or improper subset of its definition domain where the function range takes nonzero values. The support is said to be compact if it is a compact set.

Proposition 1: Assume that A is a Metzler matrix and $A_0=0$. Then, there exists an input $u:[0,t]\to\mathbf{R}_{0+}^n$ of compact support (i.e. cl (X_t), the closure of $X_t:=\{\tau\in[0,t]:u(\tau)>0\}$) being of nonzero measure for all time interval $[0,t]$ of nonzero measure leading to $x_i(t)\neq 0$, $\forall i\in\bar{n}\ \forall t>0$ under identically zero initial conditions iff there is an integer $k=k(i)\in\{0\}\cup\overline{n-1}$, dependent in general on i, such that the condition $e_i^T A^k b\neq 0$ holds for each $i\in\bar{n}$.

- The above condition is also:

(1) A sufficient condition of positivity (provided that, in addition, c > 0) and excitability independent of delay of S_h for any $A_0\geq 0$; and

(2) A necessary and a sufficient condition of positivity (provided that, in addition, c > 0) and excitability of S_∞.

Proof: From *Lemma* 1, it suffices to consider constant inputs $u(t)=k_u\ (t\geq 0)$ to characterize excitability. For zero initial conditions, constant unity input and $A_0=0$, the state–trajectory solution is given by $x(t,0,u)=\sum_{k=0}^{n-1}\beta_k^{(u)}(t)A^k b$ from (3) where $\beta_k^{(u)}(t):=k_u\left(\int_0^t \alpha_k(t-\tau)d\tau\right)$ and $\alpha_k(\tau)$ are a set of n unique continuously differentiable and linearly independent real functions on any real interval $(0,t)$ such that $e^{At}=\sum_{k=0}^{n-1}\alpha_k(t)A^k$ (see [4]).

Note from the constraint $\alpha_0(0)I+\sum_{k=1}^{n-1}\alpha_k(0)A^k=\left[e^{At}\right]_{t=0}=I$, $\alpha_0(0)=1$ and $\alpha_k(0)=0$ for $k\in\overline{n-1}$.

It is now proved that the set of functions $\beta_k^{(u)}(\tau)$ are also linearly independent on any interval [0, t] of nonzero measure. Proceed by contradiction. Assume that they are linearly dependent. Thus, $\sum_{k=0}^{n-1}\lambda_k\beta_k^{(u)}(\tau)\equiv 0$ on $\tau\in[0,t]$ for some non identically zero set of real scalar constants λ_k; k = 0,.

Assume with no loss of generality that $\lambda_1\neq 0$. Note that $\sum_{k=0}^{n-1}\lambda_k\beta_k^{(u)}(\tau)\equiv 0\Rightarrow\sum_{k=0}^{n-1}\lambda_k\dot{\beta}_k^{(u)}(\tau)\equiv 0$ on$(0,t)$. From the definition of the functions $\beta_k^{(u)}$, it follows that $\dot{\beta}_k^{(u)}(\tau)=2\alpha_k(0)-\alpha_k(\tau)$; $k=0,1,...,n-1$ which leads to $\dot{\beta}_0^{(u)}(\tau)=1+k_u-\alpha_0(\tau)$ and $\dot{\beta}_k^{(u)}(\tau)=-\alpha_k(\tau)$ for $k\in\overline{n-1}$ on $(0,t)$ since $\alpha_0(0)=1$ and $\alpha_k(0)=0$ for $k\in\overline{n-1}$. Replacing these expressions into $\sum_{k=0}^{n-1}\lambda_k\dot{\beta}_k^{(u)}(\tau)\equiv 0$, it follows that $\delta(\tau):=\sum_{k=0}^{n-1}\lambda_k\alpha_k(\tau)=(1+k_u)\lambda_0$. The following situations may occur:

a) $\lambda_0=0\Rightarrow\delta(\tau)\equiv 0$ on $(0,t)$ with $\lambda_1\neq 0$. Thus, the functions $\alpha_k(\tau)$ are not linearly independent on $(0,t)$, so that they are not linearly independent on $[0,t]$, what leads to a contradiction.

b) $\lambda_0\neq 0\Rightarrow\delta(\tau)-(1+k_u)\lambda_0\equiv 0$ on $(0,t)$ with $\lambda_1\neq 0$. Then the set of functions $\alpha_0'(t):=\alpha_0(t)-(1+k_u)\lambda_0$, $\alpha_k(t)$, $k\in\overline{n-1}$ are linearly dependent on $(0,t)$. Since the set $\alpha_k(t);k=0,1,...,n-1$ are independent of k_u, but only dependent on A, that means that for almost all values of k_u, the set $\alpha_k(t);k=0,1,...,n-1$ is not linearly independent on $(0,t)$ and then on $[0,t]$.

As a result, for some nonnegative input of compact support on any interval $[0,t]$ of nonzero measure (and , in particular, for almost any constant positive input on $[0,t]$) one has

$$x_{S_{\infty_i}}(\tau,0,u)=\left[e_i^T b, e_i^T Ab, \ldots, e_i^T A^{n-1} b\right]\left[\beta_0^{(u)}(\tau), \beta_1^{(u)}(\tau), \ldots, \beta_{n-1}^{(u)}(\tau)\right]^T \neq 0 \qquad (7)$$

i.e. the i-th state component of the state trajectory solution of the system S_∞ on $(0,t]$ for all $i\in\bar{n}$ iff $e_i^T A^k b \neq 0$ for some $k=k(i)\in\{0\}\cup\overline{n-1}$ since $\beta_k^{(u)}(\tau)$ are linearly independent on $(0,t]$ for almost all nonnegative input of compact support, and thus non identically zero, on $[0,t]$. Thus, $x_{S_\infty}(\tau,0,u)>>0$ on $(0,t]$ for some admissible input of the given class. Since A is a Metzler matrix, S_∞ is positive so that $x_{S_\infty}(\tau,0,u)\geq 0$ on $[0,t]$ for all $t\geq 0$. For zero initial conditions and any $h\in[0,\infty)$, $x_{S_h}(\tau,0,u)\geq x_{S_\infty}(\tau,0,u)\geq 0$ on $[0,t]$ for all $t\geq 0$ for any admissible input and $x_{S_h}(\tau,0,u)\geq x_{S_\infty}(\tau,0,u)>>0$ on $[0,t]$ for all $t\geq 0$, $\forall h\in[0,\infty)$ or some admissible input of the given class. Since A is Metzler and $A_0\geq 0$, the above constraint still holds for any admissible function of initial conditions. Then, the system is positive and excitable independent of the delay. The condition is also a necessary and sufficient condition for positivity and excitability of S_∞; i.e. for the case $A_0=0$.

Proposition 2: (i) S_h is positive for a given delay $h\geq 0$ iff $c > 0$, $A_0\geq 0$, $b > 0$ and $\left[\int_0^t e^{A(t-\tau)}\left(I_n+\int_0^{\tau-h} A_0\Psi_h(\tau-h-\varsigma)\,d\varsigma\right)d\tau\right]b>0$ for all $t > 0$. External positivity is characterized by changing the last expression by its pre-multiplication by c^T.

(ii) S_h is positive independent of the delay for any delay $h\geq 0$ if the last of the above conditions is replaced with $e^{At}b>0$, $\forall t\in[t_0, t_0+T)$ $t\geq t_0$, some finite $t_0\geq 0$, some finite $T > 0$ and any given $h\geq 0$. As a result, S_0 and S_∞ are also positive. External positivity independent of the delay is characterized by changing the condition $e^{At}b>0$ by $c^T e^{At}b>0$.

(iii) S_h being positive (respectively, externally positive) is excitable (respectively, externally excitable) for a given delay $h\geq 0$ iff $\left[\int_0^t e^{A(t-\tau)}\left(I_n+\int_0^{\tau-h} A_0\Psi_h(\tau-h-\varsigma)\,d\varsigma\right)d\tau\right]b>>0$. External positivity is characterized by changing the last expression by its pre-multiplication by c^T.

(iv) S_h being positive (respectively, externally positive) is excitable (respectively, externally excitable) independent of the delay for any delay $h\geq 0$ iff $\left(e^{At}b\right)_i>0$ $\left(\text{respectively}, \left(c^T e^{At}b\right)_i>0\right)$ for all $i\in\bar{n}$, $\forall t\in[t_0, t_0+T)$, some finite $t_0\geq 0$ and some finite $T > 0$. As a result, S_0 and S_∞ are excitable (respectively, externally excitable).

(v) S_0 is positive iff $(A + A_0)$ is a Metzler matrix , b > 0 and c > 0 (i.e. if *Assumptions* 1-3 hold) what is guaranteed by S_h being positive independent of the delay. In addition, it is excitable (respectively, externally excitable) iff $\left(e^{At} b\right)_i > 0$ $\left(\text{respectively}, \left(c^T e^{At} b\right)_i > 0\right)$ for all $i \in \bar{n}$, $\forall t \in [t_0, t_0 + T)$ $t \geq t_0$, some finite $t_0 \geq 0$ and some finite T > 0. S_0 being excitable (respectively, externally excitable) is guaranteed by S_h being excitable independent of the delay.

(vi) S_∞ is positive iff A is a Metzler matrix , b > 0 and c > 0 (i.e. if *Assumptions 1-3* hold with $A_0 = 0$) what is guaranteed by S_h being positive independent of the delay. In addition, it is excitable (respectively, externally excitable) iff $\left(e^{(A+A_0)t} b\right)_i > 0$ $\left(\text{respectively}, \left(c^T e^{(A+A_0)t} b\right)_i > 0\right)$ for all $i \in \bar{n}$, $\forall t \in [t_0, t_0 + T)$ $t \geq t_0$, some finite $t_0 \geq 0$ and some finite T > 0. S_∞ being excitable (respectively, externally excitable) is guaranteed by S_h being excitable independent of the delay.

Proof: First note from *Theorem 1* that for a unity step control the forced response of S_h, $x(t) = \left[\int_0^t e^{A(t-\tau)}\left(I_n + \int_0^{\tau-h} A_0 \Psi_h(\tau - h - \varsigma)\, d\varsigma\right) d\tau\right] b$. Then, $x(t) > 0$ for any set of nonnegative initial conditions, any nonnegative control and any t > 0. This follows since $b > 0$, $c > 0$ and since A is a Metzler matrix what implying $e^{At} x_0 \geq 0$ and $\int_0^{\tau-h} e^{A(t-\tau)} A_0 x(\tau - h) d\tau \geq 0$. Thus, the system is positive independent of the delay iff $\left[\int_0^t e^{A(t-\tau)}\left(I_n + \int_0^{\tau-h} A_0 \Psi_h(\tau - h - \varsigma)\, d\varsigma\right) d\tau\right] b > 0$ for all t > 0. If the above expression is, furthermore, >> 0 for some time interval then. all the state components are positive on such an interval and the system is excitable independent of the delay since positivity, respectively, excitability hold for any nonnegative control, respectively, any control being positive over some finite interval of time. Properties (i)-(iii) have been proved. Properties (ii) – (iv) follow correspondingly by noting the following: A being a Metzler matrix is equivalent to $e^{At} > 0$ for all $t \geq 0$. Since e^{At} is a fundamental matrix of the unforced system S_∞, it is nonsingular for all $t \geq 0$ then its kernel is zero so that $e^{At} b = 0$ iff b = 0. Since b > 0 then $\left(e^{At} b + e^{A\tau} A_0 x(\tau - h)\right) > 0$ and $\left(e^{At} b + e^{A\tau} A_0 x(\tau - h)\right) >> 0$ are related to positivity and excitability, respectively, is equivalent to A being a Metzler matrix together with b > 0 then the system is positive and either $x(0^+) > 0$ or $x(0^+) >> 0$, respectively, for any positive constant control and then for any nonnegative control being positive on some interval of nonzero measure for zero initial conditions. Then, the conditions given are equivalent to the joint *Assumptions* 1 -3 and S_h is positive independent of the delay and the first part has been proved. Also, note that for identically zero initial conditions $\left(e^{At} b\right)_i > 0$ for all $i \in \bar{n}$, $\forall t \in [t_0, t_0 + T)$, some finite $t_0 \geq 0$ and some finite T > 0. for all $i \in \bar{n} \Leftrightarrow x_i(t, 0, u) > 0$ for all $i \in \bar{n}$, $\forall t \in [t_0, t_0 + T)$, some finite $t_0 \geq 0$ and some finite T > 0.so that S_h is excitable independent of the delay. Properties (v)-(vi) apply for particular systems (2) with delay zero

or infinity. The proofs of positivity of the delay-free auxiliary systems S_0 and S_∞ are direct from (2) and the above result for positivity independent of the delay .The proof related to external excitability follows in a similar way concerning $y(t,0,u)>0$ for all $i\in\bar{n}$, $\forall t\in[t_0, t_0+T)$, some finite $t_0\geq 0$ and some finite T > 0. External positivity and external excitability follow by pre-multiplying directly the relevant conditions for positivity/ excitability by c^T . □

Remark 2: Note that the condition $\left(e^{At}b\right)_i>0$ invoked in *Proposition 2* for all $i\in\bar{n}$ and is equivalent (although more difficult to test) to the previous excitability condition $e_i^T A^k b\neq 0$ for some integer $k=k(i)\in\{0\}\cup\overline{n-1}$, dependent in general on i, for each $i\in\bar{n}$. Note also that, since, the auxiliary system S_∞ is delay-free and time-invariant if excitability holds then $x\left(0^+,0,u(t)\right)>>0$ for some admissible input so that $t_0=0$ with no loss in generality, [1-2]. □

Proposition 3: There exists an integer $k=k(i)\in\{0\}\cup\overline{n-1}$, dependent in general on i, for all $i\in\bar{n}$, such that $e_i^T A_{(G)}^k b_{(G)}>0$ for each $i\in\bar{n}$ iff $\sum_{k=0}^{n-1} A_{(G)}^k b>>0$.

Proof : Since $A_{(G)}\geq 0$ and $b_{(G)}\geq 0$ by construction, $e_i^T A_{(G)}^k b_{(G)}\geq 0$, $\forall(i,k)\in\bar{n}\times\left(\{0\}\cup\overline{n-1}\right)$. Then, $e_i^T A_{(G)}^k b_{(G)}>0, \forall i\in\bar{n}$ and some k = k(i) depending in general on each $i\in\bar{n}$

$$\Rightarrow\left(A_{(G)}^k b_{(G)}\right)_i>0 \text{ for some } k=k(i)\in\{0\}\cup\overline{n-1}, \forall i\in\bar{n}$$

$$\Rightarrow\left(\sum_{k=0}^{n-1} A_{(G)}^k b_{(G)}\right)_i>0\Rightarrow\sum_{k=0}^{n-1} A_{(G)}^k b>>0.$$

Conversely, $\sum_{k=0}^{n-1} A_{(G)}^k b>>0\Rightarrow\left(\sum_{k=0}^{n-1} A_{(G)}^k b_{(G)}\right)_i>0\Rightarrow\left(A_{(G)}^k b_{(G)}\right)_i>0$

for some $k=k(i)\in\{0\}\cup\overline{n-1}, \forall i\in\bar{n}$. □

Proposition 4: Assume that S_h is positive independent of the delay. Then, it is excitable independent of the delay iff the associate system S_h^G is excitable independent of the delay, in particular, if any of the conditions below below, which mutually imply and are implied by all the remaining ones, hold:

. $e_i^T A^k b\neq 0$ for some $k=k(i)\in\{0\}\cup\overline{n-1}$, depending in general on i, for each $i\in\bar{n}$.

. $e_i^T A_{(G)}^k b_{(G)}>0$ for some $k=k(i)\in\{0\}\cup\overline{n-1}$, depending in general on i, for each $i\in\bar{n}$.

. For each $i\in\bar{n}$, $e_i^T A_{(G)}^k b_{(G)}$ is not identically zero for all $k\in\{0\}\cup\overline{n-1}$.

. $\sum_{k=0}^{n-1} A^{(G)\,k} b >> 0$

Any of the above equivalent conditions are necessary and sufficient for both S_∞ and S^{G}_{∞} to be excitable.

Also, S_h and S^{G}_{h} are both output excitable independent of the delay if any of the equivalent constraints below, which mutually imply and are implied by all the remaining ones, hold:

. $c^T A^k b \neq 0$ for some $k \in \{0\} \cup \overline{n-1}$.

. $c^T A^{(G)\,k} b^{(G)} > 0$ for some $k \in \{0\} \cup \overline{n-1}$.

. For each $i \in \overline{n}$, $c^T A^{(G)\,k} b^{(G)}$ is not identically zero for all $k \in \{0\} \cup \overline{n-1}$.

. $\sum_{k=0}^{n-1} c^T A^{(G)\,k} b >> 0$

. There exist $i \in \overline{n}$; $k \in \{0\} \cup \overline{n-1}$, $j \in \overline{m}$ such that $c_i^T A^k b_j \neq 0$ or, equivalently, $c_i^{(G)T} A^{(G)\,k} b_j^{(G)} > 0$

Any of the above equivalent conditions are necessary and sufficient for both S_∞ and S^{G}_{∞} to be output excitable.

Proof : Note that S^{G}_{h} is positive by construction. Note also that b and $b^{(G)}$ have exactly the same zero components by construction of $b^{(G)}$ from b and that $\left(e^{At}\right)_{ij} = 0 \Leftrightarrow \left(e^{A^{(G)}t}\right)_{ij} = 0$ for any (off – diagonal) entry $(i, j \neq i) \in \overline{n} \times \overline{n}$ such that $A_{ij} = A_{ij}^{(G)} = 0$. As a result, $\left(e^{At} b\right)_i = 0 \Leftrightarrow \left(e^{A^{(G)}t} b^{(G)}\right)_i = 0$ for any $i \in \overline{n} \Leftrightarrow \left(e^{At} b\right)_i \neq 0 \Leftrightarrow \left(e^{A^{(G)}t} b^{(G)}\right)_i > 0$ for any $i \in \overline{n} \Leftrightarrow S_h$, if positive , is excitable iff S^{G}_{h} (being always positive by construction) is excitable. The remaining part of the proof follows directly from *Propositions 1- 3*. The conditions for (internal) excitability have been proven. Those for external excitability are close by replacing the Euclidean canonical unity vector by the output vector c. □

Proposition 5: Assume that S_h is externally positive independent of the delay. Then, S_h is externally excitable independent of the delay iff S^{G}_{h} is externally excitable independent of the delay. Also, both S_h and S^{G}_{h} are both output excitable independent of the delay if any of the equivalent constraints below, which mutually imply and are implied by each other, hold

. $c^T A^k b \neq 0$ for some $k \in \{0\} \cup \overline{n-1}$.

. $c^{(G)\,T}A^{(G)\,k}b^{(G)}>0$ for some $k\in\{0\}\cup\overline{n-1}$.

. $c^{(G)T}A^{(G)\,k}b^{(G)}$ is not identically zero for all $k\in\{0\}\cup\overline{n-1}$.

. $\sum_{k=0}^{n-1}c^{(G)T}A^{(G)\,k}b>>0$.

Any of the above equivalent conditions are necessary and sufficient for both S_∞ and S_∞^G to be output excitable.

Proof : It is similar to the proof of *Proposition 3* . The proof outline is to extend the proof of *Proposition 1* by replacing (7) with

$$y_{S_{\infty i}}(\tau,0,u)=\left[c^T b, c^T Ab,\ldots,c^T A^{n-1}b\right]\left[\beta_0^{(u)}(\tau),\beta_1^{(u)}(\tau),\ldots,\beta_{n-1}^{(u)}(\tau)\right]^T\neq 0 \qquad (8) \qquad \square$$

Proposition 6: Assume that S_0 is positive. Then, it is excitable iff the associate system S_0^G is excitable. Both S_0 and S_0^G are excitable iff any of the equivalent constraint below, which mutually imply and are implied by all the remaining ones, hold:

. $e_i^T\left(A+A_0\right)^k b\neq 0$ for some $k=k(i)\in\{0\}\cup\overline{n-1}$, depending in general on i, for each $i\in\bar{n}$.

. $e_i^T\left(A^{(G)}+A_0^{(G)}\right)^k b^{(G)}>0$ for some $k=k(i)\in\{0\}\cup\overline{n-1}$, depending in general on i, for each $i\in\bar{n}$.

. For each $i\in\bar{n}$, $e_i^T\left(A^{(G)}+A_0^{(G)}\right)^k b^{(G)}$ is not identically zero for all $k\in\{0\}\cup\overline{n-1}$.

. $\sum_{k=0}^{n-1}e_i^T\left(A^{(G)}+A_0^{(G)}\right)^k b>>0$.

Now, assume that S_0 is externally positive. Then, it is externally excitable iff the associate system S_0^G is externally excitable. Both S_0 and S_0^G are externally excitable iff any of the equivalent constraint below, which mutually imply and are implied by all the remaining ones, hold:

. $c^T\left(A+A_0\right)^k b\neq 0$ for some $k\in\{0\}\cup\overline{n-1}$.

. $c^{(G)\,T}\left(A^{(G)}+A_0^{(G)}\right)^k b^{(G)}>0$ for some $k\in\{0\}\cup\overline{n-1}$.

. $c^{(G)T}\left(A^{(G)}+A_0^{(G)}\right)^k b^{(G)}$ is not identically zero for all $k\in\{0\}\cup\overline{n-1}$.

. $\sum_{k=0}^{n-1}c^{(G)T}\left(A^{(G)}+A_0^{(G)}\right)^k b>0$.

Excitability conditions dependent on the delay size are obtained below for the case when A and A_0 commute. The basis of the proof is that in such a case, a finite set of (delay-dependent) more general real scalar linearly independent functions that the set $\alpha_k(t)$; k = 0, 1, …, n-1 used in the expansion $e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k$ are obtained to expand the evolution operator $\Psi_h(t)$ in the matrix products $A_0^i A^j$.

Proposition 7: Assume that A and A_0 commute and that that S_h is positive. Then, S_h is excitable iff

$\left|\left(A^k b\right)_\ell\right| + \left|\left(e^{-Ah} A_0^i A^j b\right)_\ell\right| \neq 0$ for some $(i,j,k) \in \left(\{0\} \cup \overline{n-1}\right) \times \left(\{0\} \cup \overline{n-1}\right) \times \left(\{0\} \cup \overline{n-1}\right)$ depending on ℓ for each $\ell \in \{0\} \cup \overline{n-1}$, or, equivalently, iff S_h^G is excitable, that is iff

$\left(A^{(G)\,k} b^{(G)}\right)_\ell + \left(e^{-Ah} A_0^{(G)i} A^{(G)\,j} b^{(G)}\right)_\ell \neq 0$ for some $(i,j,k) \in \left(\{0\} \cup \overline{n-1}\right) \times \left(\{0\} \cup \overline{n-1}\right) \times \left(\{0\} \cup \overline{n-1}\right)$

depending on ℓ, for each $\ell \in \{0\} \cup \overline{n-1}$, or, equivalently, iff $A^{(G)\,k}\left(I + e^{-A^{(G)}h} \sum_{i=0}^{n-1} A_0^{(G)i}\right) b^{(G)} >> 0$

Proof : First , postulate that for the case when A and A_0 commute,

$$\Psi_h(t) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \gamma_{ij}^{(h)}(t) A_0^i A^j = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \gamma_{ij}^{(h)}(t) A^j A_0^i \qquad (9.a)$$

$$= \sum_{k=0}^{n-1} \alpha_k(t) A^k + \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} \int_0^t \alpha_j(t-\tau)\gamma_{ik}^{(h)}(\tau-h)\mathbf{1}(\tau-h) A^{k+j} A_0^{i+1}\, d\tau \qquad (9.b)$$

by using (5), since A and A_0 commute, where $\gamma_{ij}^{(h)}: [0,\infty) \to \mathbf{R}_+$, $(i,j) \in \left(\{0\} \cup \overline{n-1}\right) \times \left(\{0\} \cup \overline{n-1}\right)$ and $e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k$. If (9.a) holds then (9.b) holds as well after substituting (9.a) into he right-hand-side of (5) under the integral symbol. From Cayley- Hamilton theorem, there are real scalars δ_i, σ_i (i = 0,1,…, n-1) such that:

$$A^n = \sum_{i=0}^{n-1} \delta_i A^i \; ; \quad A_0^n = \sum_{i=0}^{n-1} \sigma_i A_0^i \qquad (10)$$

Taking time-derivatives in (9.a) and using (10), one gets

$$\dot{\Psi}_h(t) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \dot{\gamma}_{i,j-1}^{(h)}(t) A_0^i A^j$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \gamma_{ij}^{(h)}\, A_0^i\, A^{j+1} + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \gamma_{ij}^{(h)}(t-h)\delta_j A_0^{i+1} A^j$$

$$= \sum_{i=0}^{n-1}\sum_{j=1}^{n-1} \gamma_{i,j-1}^{(h)} A_0^i\, A^j + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \gamma_{i,n-1}^{(h)}(t-h)\delta_j A_0^i\, A^j$$

$$+ \sum_{i=1}^{n-1}\sum_{j=0}^{n-1} \gamma_{i-1,j}^{(h)} A_0^i\, A^j + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \gamma_{n-1,j}^{(h)}(t-h)\sigma_i A_0^i\, A^j \qquad (11)$$

By equalizing the first and last identities in (11), one gets for $(i,j)\in\left(\{0\}\cup\overline{n-1}\right)\times\left(\{0\}\cup\overline{n-1}\right)$

$$\left[\dot{\gamma}_{ij}^{(h)}(t) - \gamma_{i,j-1}^{(h)}(t) - \gamma_{i,n-1}^{(h)}(t)\delta_j - \gamma_{i-1,j}^{(h)}(t-h) - \gamma_{n-1,j}^{i-1,j}(t-h)\sigma_i\right] A_0^i A^j = 0 \qquad (12)$$

so that the scalar real functions in the expansion (9.a) exist , are linearly independent and subject to the differential constraints :

$$\dot{\gamma}_{ij}^{(h)}(t) = \gamma_{i,j-1}^{(h)}(t) + \gamma_{i,n-1}^{(h)}(t)\delta_j + \gamma_{i-1,j}^{(h)}(t-h) + \gamma_{n-1,j}^{i-1,j}(t-h)\sigma_i \qquad (13)$$

for $(i,j)\in\left(\{0\}\cup\overline{n-1}\right)\times\left(\{0\}\cup\overline{n-1}\right)$ subject to initial conditions $\gamma_{00}^{(h)}(0)=1$, $\gamma_{ij}^{(h)}(0)=0$ for $(i,j)\in\left(\left(\{0\}\cup\overline{n-1}\right)\times\overline{n-1}\right)\cup\left(\left(\overline{n-1}\right)\times\left(\{0\}\cup\overline{n-1}\right)\right)$ implying in (9) that $\Psi_h(0)=I$. As a result the expansion postulated in (9.a) holds with unique scalar functions $\gamma_{ij}^{(h)}(t)$ satisfying (12). Now, from (10) into (9.b) the state-trajectory solution (3) under zero initial conditions becomes for a unity step input (see *Lemma 1* and *Remark 1*) :

$$x(t,0,u) = \left(\int_0^t \Psi_h(t-\tau)\mathbf{1}(\tau)\,d\tau\right)b = \sum_{k=0}^{n-1}\beta_k(t)A^k b + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\omega_{ij}^{(h)}(t)\,e^{-Ah}A_0^i A^j b \qquad (14.a)$$

$$= \left[\beta_0(t),....,\beta_{n-1}(t),\omega_{00}(t)....,\omega_{n-1,n-1}(t)\right]\left[b,...,A^{n-1}b,\ e^{-Ah}A_0 Ab,...,e^{-Ah}A_0^{n-1}A^{n-1}b\right]^T \qquad (14.b)$$

where $\beta_k(t) := \int_0^t \alpha_k(t-\tau)A^k\,d\tau$; k = 0,1, …, n-1. Since the functions in the first brackets of the right-hand-side of (14.b) are linearly independent and the system is positive, $x(t,0,u) >> 0$ for $t>0$ (i.e. it is not zero) iff for some $(i,j,k)\in\left(\{0\}\cup\overline{n-1}\right)\times\left(\{0\}\cup\overline{n-1}\right)\times\left(\{0\}\cup\overline{n-1}\right)$ depending on ℓ for each $\ell\in\{0\}\cup\overline{n-1}$. The remaining of the proof follows in the same way by using the associated system. □

Remark 3: The assumption that A and A_0 commute is a key one in postulating and then proving that the expansion of the evolution operator into a finite number of powers of both matrices is given by a set of linearly independent functions. Other alternative expansions into powers of matrix A or, even, arbitrary

matrices may be found but the linearly independent real functions become replaced by matrix functions so that *Proposition 7* cannot be proved. Note that the conditions of excitability involve two terms. If the first one is not zero then the excitability holds independent of the delay and it only depends on the matrix A and the vector b. All the state components become positive at time $t = 0^+$. If the second condition holds then that property is achievable at time $t = h^+$ since S_h is a positive system so that A is a Metzler matrix , $A_0 \geq 0$ and $b > 0$. Thus, the general condition of excitability also includes a stronger one of excitability independent of the delay. □

Proposition 8: Assume that A and A_0 commute and that that S_h is externally positive. Then, S_h is externally excitable iff $\left| c^T A^k b \right| + \left| c^T e^{-Ah} A_0^i A^j b \right| \neq 0$

or, equivalently, iff S_h^G is externally excitable, that is iff $c^{(G)T} A^{(G)k} b^{(G)} + c^{(G)T} e^{-Ah} A_0^{(G)i} A^{(G)j} b^{(G)} \neq 0$ or, equivalently, iff $c^{(G)T} A^{(G)k} \left(I + e^{-A^{(G)}h} \sum_{i=0}^{n-1} A_0^{(G)i} \right) b^{(G)} > 0$

Proof: The proof is very similar to that of *Proposition 7* and, hence, is omitted. □

Similar considerations as those given in the last part of *Remark 3* for the (state) excitability either for the current delay or independent of the delay apply. In particular, if $c^T A^k b$ is not zero then external excitability independent of the delay is guaranteed. If $c^T A^k b$ is zero then external excitability is still guaranteed in finite time for the system S_h under the current delay h and all $t \geq h^+$. It is immediate to give a direct extension of excitability from *Propositions 2-* 3 for a positive multi-input system of input components $u_i(t)$; $\forall i \in \overline{m}$ and control matrix $B = (b_1, b_2, ..., b_m) \in \mathbf{R}_+^{n \times m}$ with B > 0 as follows:

Theorem 4. Assume that A is a Metzler matrix and B > 0. The following properties hold:

(i) S_h is excitable if for some finite $t_0 \geq 0$ and $\forall t \in [t_0, t_0 + T)$ for some finite T > 0, $e^{At} \left(\sum_{j \in S\bar{n}}^{m} b_j \right)_i > 0$ and $e^{A(t-\tau)} \left(\sum_{j \notin S\bar{n}}^{m} A_{0jk} x_k(\tau - h) \right)_i > 0$, some $\tau \in [t-h, t)$, $\forall i \in S\bar{n} \subset \vec{n}$; $\forall j \in \bar{n} \setminus S\bar{n}$.

(ii) S_h is excitable if for some finite $t_0 \geq 0$ and some finite T > 0, $e^{A^{(G)}t} \left(\sum_{j \in S\bar{n}}^{m} b_j^{(G)} \right)_i > 0$, $\forall t \in [t_0, t_0 + T)$ and $e^{A^{(G)}(t-\tau)} \left(\sum_{j \notin S\bar{n}}^{m} A_{0jk}^{(G)} x_k^{(G)}(\tau - h) \right)_i > 0$, some $\tau \in [t-h, t)$; $\forall i \in S\bar{n} \subset \vec{n}$; $\forall j \in \bar{n} \setminus S\bar{n}$.

(iii) S_h is excitable if $\sum_{k=0}^{n-1} A^{(G)^k}\left(\left(\sum_{j\in\bar{n}}^{m} b_j^{(G)}\right)+\sum_{j\in\bar{n}}^{m} A_{0jk}^{(G)}\right)_i > 0;\ \forall i \in \bar{n}$.

(iv) S_h is excitable if $\sum_{k=0}^{n-1} A^{(G)^k}\left(\left(\sum_{j\in\bar{n}}^{m} b_j^{(G)}\right)+\sum_{j\in\bar{n}}^{m} A_{0jk}^{(G)}\right) >> 0$ □

Remark 4 : The proof follows directly from the fact that S_h is excitable if and only if $S_h^{(G)}$ is excitable and that excitability is achievable by exciting each state component either from an input component through the corresponding component of the associate control vector or through excitation of some other state component which is coupled with it through the delayed dynamics. Also, *Theorem 4* may be reformulated in terms of the associate system defined form its influence graph by substituting all matrices by the corresponding ones of the associate system leading to *Theorem 4 (ii)*. Finally, since if S_h is positive then $A^{(G)}$ is both Metzler and positive then another alternative equivalent of formulating *Theorem 4* is $\sum_{k=0}^{n-1} A^{(G)^k}\left(\left(\sum_{j\in\bar{n}}^{m} b_j^{(G)}\right)+\sum_{j\in\bar{n}}^{m} A_{0jk}^{(G)}\right)_i > 0;\ \forall i \in \bar{n}$ [*Theorem 4 (iii)*] or its equivalent form of *Theorem 4* (iv). *Theorem 4* extends directly to output excitability by pre-multiplying all the given excitability conditions by the output vector $c^T > 0$.

IV. Transparency of Positive Systems

To deal with the property of transparency, consider a p- dimensional output vector $y(t)=Cx(t)$ replacing the output equation of (1) with $C=(c_1, c_2, ..., c_p)^T \in \mathbf{R}_+^{p\times n}$ of columns $c_i \in \mathbf{R}_+^n$ with C > 0: $\forall i\in\bar{p}$. The delayed system and delay-free dynamic systems (2) together with this output are positive if A is a Metzler n-matrix, $A_0 \in \mathbf{R}_+^{n\times n}$, $C=(c_1, c_2, ..., c_p)^T \in \mathbf{R}_+^{p\times n}$, $b\in\mathbf{R}_+^n$ (or each column of B in the multi-input case of Theorem 4 is nonnegative) . The property of transparency is associated with the unforced system so that it is independent of the positivity of the control vector o matrix.

Definition 5: A positive system S_h is said to be weakly transparent if and only if each unforced output response can be made positive in some finite time for any given $x_0 (>0)\in\mathbf{R}_+^n$, $\forall t\in\mathbf{R}_+$ nonnegative input to the system at rest on $t\in[-h, 0)$; i.e. $\varphi(t)=0$, $\forall t\in[-h, 0)$. S_h is excitable at time $T(>0)\in\mathbf{R}_+$ if and only if $x(T)>>0$ under the above conditions.

Definition 6: A positive system S_h is said to be strongly transparent (or simply transparent) if and only if it is weakly transparent and, furthermore, each free output response can be made positive in some finite time for initial conditions $x_0=\varphi(0)=0$, $\varphi(t)(>0)\in\mathbf{R}_+^n$, for all t on some subinterval of finite measure of $[-h, 0)$. □

Note that weak transparency only depend on point initial conditions while strong transparency applies to arbitrary interval initial conditions since strong transparency requires weak transparency the whole function of initial conditions considered might be positive on its definition domain. The following result related to transparency of S_h is given:

Theorem 5: Assume that S_h is positive with $A \in \mathbf{R}_+^{n\times n}$. Then, the following items hold:

(i) S_h is weakly transparent if $\sum_{k=0}^{n-1} C_j^T \Psi_h^{(k)}(0)\, e_i = \sum_{k=0}^{n-1} C_j^T A^k e_i \neq 0$, $\forall i \in \bar{n}$, $\forall j \in \bar{p}$ where C_j^T is the j-th row of C, $\forall j \in \bar{p}$. If, furthermore, $C_j^T \left(\int_0^h \Psi_h(\tau) d\tau \right) A_0 e_i \neq 0$ [what holds if $\sum_{k=0}^{n-1} C_j^T A^k A_0 e_i \neq 0$] $\forall i \in \bar{n}$, $\forall j \in \bar{p}$ then S_h is transparent. The condition is equivalent to $\sum_{j=1}^{p} C_j^T \left(\int_0^h \Psi_h(\tau) d\tau \right) A_0 >> 0$.

(ii) S_h is weakly transparent if

$\sum_{k=0}^{n-1} C_j^T \Psi_h^{(k)}(h^+)\, e_i = \sum_{k=0}^{n-1} C_j^T \left(A^k e^{Ah} + \sum_{\ell=0}^{k-1} A^\ell A_0 A^{k-1-\ell} \right) e_i \neq 0, \quad \forall i \in \bar{n}, \quad \forall j \in \bar{p}.$ The condition is equivalent to $\sum_{j=1}^{p} \sum_{k=0}^{n-1} C_j^T \Psi_h^{(k)}(h^+)\, e_i = \sum_{k=0}^{n-1} C_j^T \left(A^k e^{Ah} + \sum_{\ell=0}^{k-1} A^\ell A_0 A^{k-1-\ell} \right) >> 0$

If, furthermore, $C_j^T \left(\int_h^{2h} \Psi_h(\tau) d\tau \right) A_0 e_i \neq 0$ [what holds if $\sum_{k=0}^{n-1} C_j^T \left(A^i e^{Ah} + \sum_{j=0}^{i-1} A^j A_0^{i-j} \right) A_0 e_i \neq 0$] $\forall i \in \bar{n}$, $\forall j \in \bar{p}$ then S_h is transparent. Both conditions are , respectively, equivalent to $\sum_{j=1}^{p} C_j^T \left(\int_h^{2h} \Psi_h(\tau) d\tau \right) A_0 >> 0$ and to $\sum_{j=1}^{p} \sum_{k=0}^{n-1} C_j^T \left(A^i e^{Ah} + \sum_{j=0}^{i-1} A^j A_0^{i-j} \right) A_0 >> 0$.

(iii) S_h is weakly transparent if and only if

$$\sum_{k=0}^{n-1} C_j^T \Psi_h^{(k)}(2h^+)\, e_i = \left[\sum_{k=0}^{n-1} C_j^T \left(A^k e^{2Ah} \left[I_n + \int_h^{2h} e^{-A\tau} A_0 e^{A(\tau-h)} d\tau \right] + \sum_{\ell=0}^{k-1} A^j A_0 A^{k-1-\ell} e^{Ah} \right) A_0 \right] e_i \neq 0, \forall i \in \bar{n},$$

$\forall j \in \bar{p}$ or, equivalently,

$$\sum_{j=1}^{p} \sum_{k=0}^{n-1} C_j^T \left(A^k e^{2Ah} \left[I_n + \int_h^{2h} e^{-A\tau} A_0 e^{A(\tau-h)} d\tau \right] + \sum_{\ell=0}^{k-1} A^j A_0 A^{k-1-\ell} e^{Ah} \right) A_0 >> 0.$$ S_h is (strongly) transparent if and only if, furthermore, $C_j^T \left(\int_{2h}^{3h} \Psi_h(\tau) d\tau \right) A_0 e_i \neq 0$; what holds if for all $i \in \bar{n}$ and $j \in \bar{p}$:

$$\sum_{k=0}^{n-1} C_j^T \left(\sum_{i=0}^{n-1} \left(A^i e^{2Ah} \left[I_n + \int_h^{2h} e^{-A\tau} A_0 e^{A(\tau-h)} d\tau \right] + \sum_{j=0}^{i-1} A^{i-j} A_0^{i-1-j} e^{Ah} \right) \right) A_0 e_i \neq 0$$

or, equivalently,

$$\sum_{j=1}^{p}\sum_{k=0}^{n-1}C_j^T\left(\sum_{i=0}^{n-1}\left(A^i e^{2Ah}\left[I_n+\int_h^{2h}e^{-A\tau}A_0e^{A(\tau-h)}d\tau\right]+\sum_{j=0}^{i-1}A^{i-j}A_0^{i-1-j}e^{Ah}\right)\right)A_0>>0$$

(iv) Assume that S_h is positive (without requiring $A\in\mathbf{R}_+^{n\times n}$ but only $A\in M_E^n$). Thus, Properties (i)-(iii) hold if all the matrices of the parameterization of S_h are replaced by the corresponding ones of the system S_h^G associated with its influence graph in each one of the given conditions.

Proof: *(i)* Since S_h is positive $\sum_{k=0}^{n-1}C_j^TA^ke_i\neq 0\Rightarrow\sum_{k=0}^{n-1}C_j^TA^ke_i>0\Rightarrow y_j(0^+)>0$ for any $x_0>0$ and any $\varphi(t)\in\mathbf{R}_+^n$ for $t\in[-h,0)$ [even if $\varphi(t)=0$ for $t\in[-h,0)$] for any $j\in\bar{p}$ and then S_h is weakly transparent. If, furthermore, $C_j^T\left(\int_0^h\Psi_h(\tau)d\tau\right)A_0e_i\neq 0\ \forall i\in\bar{n}$, $\forall j\in\bar{p}$ then $y_j(t)>0$ for some $t\in[0,h)$ any $\mathbf{R}_n^+\ni\varphi(t)=\gamma_\varphi\neq 0$ (and constant with no loss in generality) even if $x_0=0$. Such a condition is guaranteed from (3)-(5) if $\sum_{k=0}^{n-1}C_j^T\Psi_h^{(k)}(0)A_0e_i=\sum_{k=0}^{n-1}C_j^TA^kA_0e_i\neq 0;\ \forall j\in\bar{p}$. The weak transparency of Property *(ii)* follows in the same way by noting that $\sum_{k=0}^{n-1}C_j^T\Psi_h^{(k)}(h^+)e_i=\sum_{k=0}^{n-1}C_j^T\left(A^ke^{Ah}+\sum_{\ell=0}^{k-1}A^\ell A_0A^{k-1-\ell}\right)e_i\neq 0$, $\forall i\in\bar{n}$, $\forall j\in\bar{p}$ implies that $y(h^+)>>0$ for any $x_0>0$ and any $\varphi(t)\in\mathbf{R}_+^n$ for $t\in[-h,0)$. If, furthermore, $C_j^T\left(\int_h^{2h}\Psi_h(\tau)d\tau\right)A_0e_i\neq 0$, guaranteed if $\sum_{k=0}^{n-1}C_j^T\Psi_h^{(k)}(h^+)A_0=e_i\sum_{k=0}^{n-1}C_j^T\left(A^ie^{Ah}A_0+\sum_{j=0}^{i-1}A^jA_0^{i-j}\right)e_i\neq 0$; $\forall i\in\bar{n},\forall j\in\bar{p}$ via (28). Finally, the "sufficiency part" of weak transparency of Property (iii) holds since $\sum_{k=0}^{n-1}C_j^T\Psi_h^{(k)}(2h^+)e_i>>0\Rightarrow y(2h^+)>>0$ for any $x_0>0$ and any $\varphi(t)\in\mathbf{R}_+^n$ for $t\in[-h,0)$. The additional sufficient condition for transparency is $\sum_{k=0}^{n-1}C_j^T\Psi_h^{(k)}(2h^+)A_0e_i>>0$ which holds if

$$\sum_{k=0}^{n-1}C_j^T\left(\sum_{i=0}^{n-1}\left(A^ie^{2Ah}\left[I_n+\int_h^{2h}e^{-A\tau}A_0e^{A(\tau-h)}d\tau\right]+\sum_{j=0}^{i-1}A^{i-j}A_0^{i-1-j}e^{Ah}\right)\right)A_0e_i>>0;\ \forall i\in\bar{n},\ \forall j\in\bar{p},$$

from (29), what implies that $y(2h^+)>>0$ for any $x_0>0$ and any function $\varphi(t)\in\mathbf{R}_+^n$ for $t\in[-h,0)$. The "necessity parts "of (iii) for weak transparency and transparency follow directly by contradiction. If $\sum_{k=0}^{n-1}C_j^T\Psi_h^{(k)}(2h^+)e_i=0$ (respectively, $\sum_{k=0}^{n-1}C_j^T\Psi_h^{(k)}(2h^+)A_0e_i=0$) for some $i\in\bar{n}$, $j\in\bar{p}$ then any $y_j(t)=0$ for all $t\geq 2h$ if $x_i(0)=\gamma_{\varphi_i}>0$ is the only nonzero component of $x(0)=x_0$ and $\varphi(t)=0;\ \forall t\in[-h,0)$ (respectively, if $x(0)=x_0=0$ and $\varphi_i(t)=\gamma_{\varphi_i}>0\ \forall t\in[-h,0)$ is the only nonzero component of the function of

initial conditions) then S_h is not weakly transparent (respectively transparent). Property (iv) is a direct consequence of Properties (i)-(iii) since $A^{\Gamma} \in \mathbf{R}_{+}^{n \times n} \cap M_E^n$ and S_h^{Γ} is positive if and only if S_h is positive. □

Note that the conditions for transparency for a single input- single output time – delay system (1) follow directly, as a particular case, from the given one in this section by replacing the control and output matrices B and C by the corresponding control and output vectors b and c^T. Note from Theorem 5 that the transparency property is achievable through the combined delay- free and delayed dynamics of the unforced system. It is interesting to characterize a property which relates the properties of external excitability and strong transparency in the specific way that the output is positive in finite time either by the action of initial conditions or by that of the input.

Definition 7: S_h is said to be almost externally excitable if the output vector is strictly positive in finite time for some nonzero pair $(\varphi^T, u^T): [-h, 0] \times \mathbf{R}_+ \to \mathbf{R}_+^{n \times m}$ of initial conditions and input. □

Definition 7 and Theorems 4- 5 with Remark 4, together with the superposition property of the unforced and forced output trajectories to conform the whole output trajectory, yield directly the following (non-necessary) sufficiency-type condition for almost external excitability:

Theorem 6: If S_h is either externally excitable or strongly transparent then it is almost externally excitable. □

Explicit conditions for almost external excitability follow by summing up the corresponding left-hand- sides of the external excitability conditions (obtained by extending Theorem 4 with Remark 4) with those obtained for strong transparency from Theorem 5.

Example 1: Consider the positive second-order system:

$\dot{x}_1(t) = x_1(t) + u(t)$; $\dot{x}_2(t) = \delta x_1(t-h) + x_2(t-h)$

initially at rest on $[-h, 0]$ with $u(t) = \gamma > 0$ for all $t \in \mathbf{R}_+$ and $\delta \geq 0$. The state trajectory is:

$$x_1(t) = \left(\int_0^t e^{(t-\tau)} d\tau \right) \gamma = \gamma \left(e^t - 1 \right) > 0 \text{ for } t > 0 \text{ ; and}$$

$$x_2(t) = \int_{-h}^{t-h} \left(\delta x_1(\tau) + x_2(\tau) \right) d\tau = \gamma \, \delta \left(e^{t-h} - 1 + h - t \right) U(t-h) > 0 \text{ for } t > h \text{ if } \delta \neq 0$$

and identically zero for $\forall t \in \mathbf{R}_+$ if $\delta = 0$ and the system is initially at rest. As a result, excitability does not hold for any delay if $\delta = 0$ since the second state component initially at rest never takes strictly positive values. excitability is achieved after the first delay interval. If h = 0 then $x_1(0^+) > 0$; $x_2(0^+) > 0$. For u (t) being the Heaviside function, it is obvious that $x_1(t) > 0$ for t > 0 and any delay $h \geq 0$. It is also seen that if the output vector is e_1 (i.e. the output equalizes the first state component, then the system is externally excitable (equivalent to the first state component being excitable) since $c^T (b + Ab) > 0$. However, since b+ Ab > 0 , but it is not >>0, the system is not excitable for infinite delay then S_∞ is not excitable. S_0 is

excitable since $b + Ab + A_0 b >> 0$. S_h is not excitable independent of the delay size since S_∞ is not excitable. These results follow directly from *Propositions 2* and *4*, However, the system is excitable for any h > 0 if $\delta > 0$.

Now, consider the modified system:

$$\dot{x}_1(t) = x_1(t) + u(t); \quad \dot{x}_2(t) = x_2(t) + \delta_1 x_1(t) + \delta_2 x_1(t-h)$$

initially at rest with $\delta_{1,2} \geq 0$. The first state variable is immediately excited (i.e. at $t = 0^+$) by any constant positive control while the second one is excited immediately if $\delta_1 > 0$, it is excited at $t = h^+$ if $\delta_1 = 0$ and $\delta_2 > 0$ and it is never excited if $\delta_{1,2} = 0$. In that case, the system is not excitable. Note that the first differential equation is unstable but if the first right-hand-side sign is changed to negative (i.e. , $\dot{x}_1(t) = -x_1(t) + u(t)$) so that the equation becomes asymptotically stable then the system is still positive and excitable except if $\delta_{1,2} = 0$. If the first right-hand side term of the second equation is also negative then the whole system is still excitable although the system is not already positive.

It is obvious that the system is also transparent if $c >> 0$ and $\delta_i \geq 0 \left(i \in \overline{2}\right)$ and if $c > 0$ with $c_1 > 0$, $\delta_i \geq 0 \left(i \in \overline{2}\right)$ and $\delta_{1,2} \neq 0$ if $\delta_{2,1} = 0$ since the output becomes strictly positive under zero input . On the other hand, it is obvious that the system is also transparent by applying Theorem 5 iff $c >> 0$ since, for zero input, the output reaches positive values in finite time $\delta_i \geq 0 \left(i \in \overline{2}\right)$ and if $c > 0$ with $c_1 > 0$, $\delta_i \geq 0 \left(i \in \overline{2}\right)$ and $\delta_{1,2} \neq 0$ if:

1) $c_1 > 0 \vee c_2 > 0 \vee c_2 \delta_i > 0$ (for some $i \in \overline{2}$ what requires $c_2 > 0$) when $x_{i0} > 0$ for $i \in \overline{2}$
2) $c_2 > 0$ for $x_{10} = 0$ and $x_{20} > 0$
3) $c_1 > 0 \vee c_2 \delta_i > 0$ (for some $i \in \overline{2}$ what requires $c_2 > 0$) for $x_{10} > 0$ and $x_{20} = 0$

The three conditions hold simultaneously iff c >> 0 since transparency is defined for any positive point initial condition not necessarily being strictly positive. □

V. CONCLUSIONS

This paper focused on the study of the fundamental properties of positivity (i.e. all the component of the state are nonnegative for all time for nonnegative controls and initial conditions), excitability (i.e. all the components of the state of a positive system are strictly positive for some finite time for some admissible input under identically zero initial conditions) and transparency, namely, all the output components are positive at some finite time for any nonzero admissible initial conditions and zero controls of linear time-invariant dynamic systems under delayed dynamics. Excitability/external excitability have been also discussed as related properties obtained separately from each input component. Results depending on and independent of the delay size have been obtained in the manuscript. Although only one single delay has been considered, the extensions to any finite set of point commensurate or incommensurate delays are direct by the only application of superposition related techniques.

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ALPHA-STABLE PARADIGM IN FINANCIAL MARKETS

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Abstract

Statistical models of financial data series and algorithms of forecasting and investment are the topic of this research. The objects of research are the historical data of financial securities, statistical models of stock returns, parameter estimation methods, effects of self-similarity and multifractality, and algorithms of financial portfolio selection. The numerical methods (MLE and robust) for parameter estimation of stable models have been created and their efficiency were compared. Complex analysis methods of testing stability hypotheses have been created and special software was developed (nonparametric distribution fitting tests were performed and homogeneity of aggregated and original series was tested; theoretical and practical analysis of self-similarity and multifractality was made). The passivity problem in emerging markets is solved by introducing the mixed-stable model. This model generalizes the stable financial series modeling. 99% of the Baltic States series satisfy the mixed stable model proposed. Analysis of stagnation periods in data series was made. It has been shown that lengths of stagnation periods may be modeled by the Hurwitz zeta law (insteed of geometrical). Since series of the lengths of each run are not geometrically distributed, the state series must have some internal dependence (Wald-Wolfowitz runs test corroborates this assumption). The inner series dependence was tested by the Hoel criterion on the order of the Markov chain. It has been concluded that there are no

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zero order Markov chain series or Bernoulli scheme series. A new mixed-stable model with dependent states has been proposed and the formulas for probabilities of calculating states (zeros and units) have been obtained. Methods of statistical relationship measures (covariation and codifference) between shares returns were studied and algorithms of significance were introduced.

Keywords: codifference, covariation; mixed-stable model; portfolio selection; stable law; pasivity and stagnation phenomenon; Hurwitz zeta distribution; financial modeling; self-similarity; multifractal; infinite variance; Hurst exponent; Anderson–Darling, Kolmogorov–Smirnov criteria

1 Introduction

Modeling and analysis of financial processes is an important and fast developing branch of computer science, applied mathematics, statistics, and economy. Probabilistic-statistical models are widely applied in the analysis of investment strategies. Adequate distributional fitting of empirical financial series has a great influence on forecast and investment decisions. Real financial data are often characterized by skewness, kurtosis, heavy tails, self-similarity and multifractality. Stable models are proposed (in scientific literature, [7, 24, 28, 29, 30]) to model such behavior.

Since the middle of the last century, financial engineering has become very popular among mathematicians and analysts. Stochastic methods were widely applied in financial engineering. Gaussian models were the first to be applied, but it has been noticed that they inadequately describe the behavior of financial series. Since the classical Gaussian models were taken with more and more criticism and eventually have lost their positions, new models were proposed. Stable models attracted special attention; however their adequacy in the real market should be verified. Nowadays, they have become an extremely powerful and versatile tool in financial modeling [28, 29]. There are two essential reasons why the models with a stable paradigm (max-stable, geometric stable, α-stable, symmetric stable and other) are applied to model financial processes. The first one is that stable random variables (r.vs) justify the generalized central limit theorem (CLT), which states that stable distributions are the only asymptotic distributions for adequately scaled and centered sums of independent identically distributed random variables (i.i.d.r.vs). The second one is that they are leptokurtotic and asymmetric. This property is illustrated in Figure 1, where (a) and (c) are graphs of stable probability density functions (with additional parameters) and (b) is the graph of the Gaussian probability density function, which is also a special case of stable law.

Following to S.Z. Rachev [29], "the α-stable distribution offers a reasonable improvement if not the best choice among the alternative distributions that have been proposed in the literature over the past four decades".

Each stable distribution $S_\alpha(\sigma, \beta, \mu)$ has the stability index α that can be treated as the main parameter, when we make an investment decision, β is the

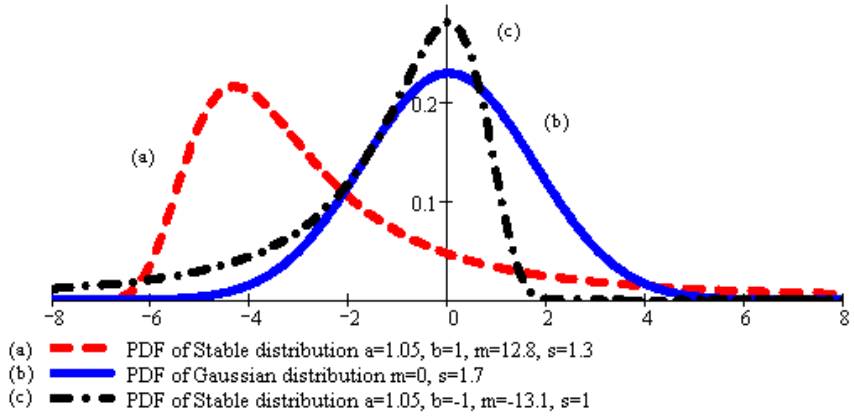


Figure 1: Stable distributions are leptokurtotic and asymmetric (here ais a stability parameter, b - asymmetry parameter, m – location parameter and s is a scale parameter)

parameter of asymmetry, σ is that of scale, and μ is the parameter of position. In models that use financial data, it is generally assumed that $\alpha \in (1; 2]$. Stable distributions only in few special cases have analytical distribution and density functions. That is why they are often described by characteristic functions (CF). Several statistical and robust procedures are examined in creating the system for stock portfolio simulation and optimization. The problem of estimating the parameters of stable distribution is usually severely hampered by the lack of known closed form density functions for almost all stable distributions. Most of the methods in mathematical statistics cannot be used in this case, since these methods depend on an explicit form of the PDF. However, there are numerical methods [26] that have been found useful in practice and are described below in this paper.

Since fat tails and asymmetry are typical of stable random variables, they better (than Gaussian) fit the empirical data distribution. Long ago in empirical studies [23, 24] it was noted that returns of stocks (indexes, funds) were badly fitted by the Gaussian law, while stable laws were one of the solutions in creating mathematical models of stock returns. There arises a question, why stable laws, but not any others are chosen in financial models. The answer is: because the sum of n independent stable random variables has a stable and only stable distribution, which is similar to the CLT for distributions with a finite second moment (Gaussian). If we are speaking about hyperbolic distributions, so, in general, the Generalized Hyperbolic distribution does not have this property, whereas the Normal-inverse Gaussian (NIG) has it [1]. In particular, if Y_1 and Y_2 are independent normal inverse Gaussian random variables with common parameters α and β but having different scale and location parameters $\delta_{1,2}$ and $\mu_{1,2}$, respectively, then $Y = Y_1 + Y_2$ is NIG$(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2)$. So

NIG fails against a stable random variable, because, in the stable case, only the stability parameter α must be fixed and the others may be different, i.e., stable parameters are more flexible for portfolio construction of different asymmetry. Another reason why stable distributions are selected from the list of other laws is that they have heavier tails than the NIG and other distributions from the generalized hyperbolic family (its tail behavior is often classified as "semi-heavy").

Foreign financial markets and their challenges were always of top interest for stock brokers. The new investment opportunities emerged after expansion of the European Union in 2004. Undiscovered markets of the Baltic States and other countries of Central and Eastern Europe became very attractive for investors. Unbelievable growth of the gross domestic product (GDP) 3-8% (the average of the EU is 1.5–1.8%) and high profitability overcame the risk. But a deep analysis has not yet been made in those markets. For a long time it has been known that financial series are the source of self-similar and multifractal phenomena and numerous empirical studies support that [3, 7]. In this research, the analysis of daily stock returns of the Baltic States and some world wide known indexes is made. Financial series in the Baltic States bear two very important features (compared with the markets of the USA and EU):

1. Series are rather short: 10–12 years (not exceeding 2000 data points), but only recent 1000–1500 data points are relevant for the analysis;

2. A stagnation phenomenon is observed in empirical data (1993–2005). Stagnation effects are characterized by an extremely strong passivity: at some time periods stock prices do not change because there are no transactions at all.

To avoid the short series problem, the bootstrap method was used [14]. The bootstrap is a method for estimating the distribution of an estimator or test statistic by treating the data as if they were the population of interest. In a word, the bootstrap method allows us to "make" long enough series required in multifractality and self-similarity analysis, from the short ones.

The second problem, called a "daily zero return" problem, is more serious than it may seem. The Baltic States and other Central and Eastern Europe countries have "young" financial markets and they are still developing (small emerging markets), financial instruments are hardly realizable and therefore they are often non-stationary, and any assumptions or conclusions may be inadequate when speaking about longtime series. Stagnation effects are often observed in young markets [2, 4]. In such a case, the number of daily zero returns can reach 89% . A new kind of model should be developed and analyzed, i.e., we have to include one more additional condition into the model – the daily stock return is equal to zero with a certain (rather high) probability p. Anyway, this problem may be solved by extending a continuous model to the mixed one, where daily returns equal to zero are excluded from the series when estimating the stability parameters. The series of non-zero returns are fitted to the stable distribution. Stable parameters are estimated by the maximal likelihood method. Goodness of fit is verified by the Anderson-Darling distributional adequacy test. The stability is also tested by the homogeneity test, based on the fundamental property of stable laws. Unfortunately, because of strong

passivity, continuous distribution fitting tests (Anderson-Darling, Kolmogorov-Smirnov, etc) are hardly applicable. An improvement based on mixed distributions is proposed and its adequacy in the Baltic States market is tested. In this dissertation the Koutrouvelis goodness-of-fit, test based on the empirical characteristic function and modified χ^2 (Romanovski) test, was used.

When constructing a financial portfolio, it is essential to determine relationships between different stock returns. In the classical economics and statistics (the data have finite first and second moments), the relationship between random variables (returns) is characterized by covariance or correlation. However under the assumption of stability (sets of stock returns are modeled by stable laws), covariance and correlation (Pearson correlation coefficient) cannot be applied, since the variance (if the index of stability $\alpha < 2$) and the mean (if the index of stability $\alpha < 1$) do not exist. In this case, we can apply rank correlation coefficients (ex. Spearman or Kendall [17, 18]) or the contingency coefficient. Under the assumption of stability, it is reasonable to apply generalized covariance coefficients – covariation or codifference. Therefore the generalized Markowitz problem is solved taking the generalized relationship measures (covariation, codifference [30]). It has been showed that the implementation of codifference between different stocks greatly simplifies the construction of the portfolio.

Typical characteristics of the passivity phenomenon are constancy periods of stock prices. The dissertation deals with the distributional analysis of constancy period lengths. Empirical study of 69 data series from the Baltic states market and modeling experiments have showed that constancy period lengths are distributed by the Hurwitz zeta distribution instead of geometrical distribution. An improved mixed stable model with dependent states of stock price returns is proposed.

2 The object of research

The objects of this research are the historical data of financial securities (stock, equity, currency exchange rates, financial indices, etc.), statistical models of stock returns, parameter estimation methods, effects of self-similarity and multifractality, and algorithms of financial portfolio selection.

In this paper, data series of the developed and emerging financial markets are used as an example. The studied series represent a wide spectrum of stock market. Information that is typically (finance.yahoo.com, www.omxgroup.com, etc.) included into a financial database is [34]:

- Unique trade session number and date of trade;
- Stock issuer;
- Par value;
- Stock price of last trade;
- Opening price;
- High - low price of trade;
- Average price;

- Closure price;
- Price change % ;
- Supply – Demand;
- Number of Central Market (CM) transactions;
- Volume;
- Maximal – Minimal price in 4 weeks;
- Maximal – Minimal price in 52 weeks;
- Other related market information.

We use here only the closure price, because we will not analyze data as a time series and its dependence. We analyzed the following r.vs

$$X_i = \frac{P_{i+1} - P_i}{P_i}$$

where P is a set of stock prices. While calculating such a variable, we transform data (Figure 2) from price to return.

The length of series is very different starting from 1566 (6 years, NASDAQ) to 29296 (107 years, DJTA). Also very different industries are chosen, to represent the whole market. The Baltic States (64 companies) series studied represent a wide spectrum of the stock market (the whole Baltic Main list and Baltic I-list). The length of series is very different, starting from 407 to 1544. The average of data points is 1402. The number of zero daily stock returns differs from 12% to 89% , on the average 52% .

Almost all the data series are strongly asymmetric ($\hat{\gamma}_1$), and the empirical kurtosis ($\hat{\gamma}_2$) shows that density functions of the series are more peaked than that of Gaussian. That is why we make an assumption that Gaussian models are not applicable to these financial series.

3 The stable distributions and an overview of their properties

We say [29, 30] that a r.v. X is distributed by the stable law and denote

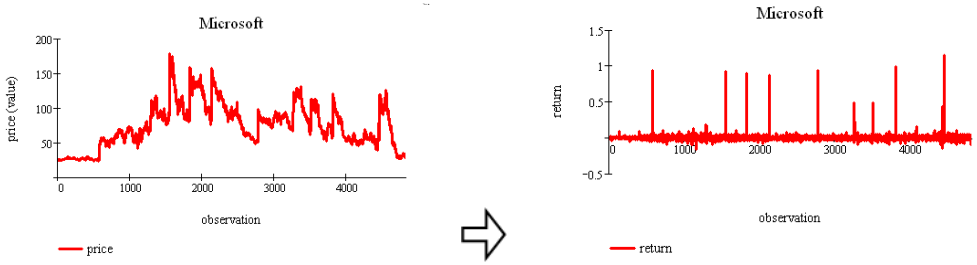


Figure 2: Data transformation

$$X \stackrel{d}{=} S_\alpha(\sigma, \beta, \mu) \quad ,$$

where S_α is the probability density function, if a r.v. has the characteristic function:

$$\phi(t) = \begin{cases} \exp\left\{-\sigma^\alpha \cdot |t|^\alpha \cdot \left(1 - i\beta sgn(t) \tan(\frac{\pi\alpha}{2})\right) + i\mu t\right\}, \text{if } \alpha \neq 1 \\ \exp\left\{-\sigma \cdot |t| \cdot \left(1 + i\beta sgn(t) \frac{2}{\pi} \cdot \log|t|\right) + i\mu t\right\}, \text{if } \alpha = 1 \end{cases} .$$

Each stable distribution is described by 4 parameters: the first one and most important is the stability index $\alpha \in (0;2]$, which is essential when characterizing financial data. The others, respectively are: skewness $\beta \in [-1, 1]$, a position $\mu \in R$, the parameter of scale $\sigma > 0$.

The probability density function is

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(t) \cdot \exp(-ixt) dt.$$

In the general case, this function cannot be expressed as elementary functions. The infinite polynomial expressions of the density function are well known, but it is not very useful for Maximal Likelihood estimation because of infinite summation of its members, for error estimation in the tails, and so on. We use an integral expression of the PDF in standard parameterization

$$p(x, \alpha, \beta, \mu, \sigma) = \frac{1}{\pi\sigma} \int_0^\infty e^{-t^\alpha} \cdot \cos\left(t \cdot \left(\frac{x-\mu}{\sigma}\right) - \beta t^\alpha \tan\left(\frac{\pi\alpha}{2}\right)\right) dt.$$

It is important to note that Fourier integrals are not always convenient to calculate PDF because the integrated function oscillates. That is why a new Zolotarev formula is proposed which does not have this problem:

$$p(x, \alpha, \beta, \mu, \sigma) = \begin{cases} \frac{\alpha \left|\frac{x-\mu}{\sigma}\right|^{\frac{1}{\alpha-1}}}{2\sigma \cdot |\alpha - 1|} \int_{-\theta}^{1} U_\alpha(\varphi, \theta) \exp\left\{-\left|\frac{x-\mu}{\sigma}\right|^{\frac{a}{\alpha-1}} U_\alpha(\varphi, \theta)\right\} d\varphi, \text{ if } x \neq \mu \\ \frac{1}{\pi\sigma} \cdot \Gamma\left(1 + \frac{1}{\alpha}\right) \cdot \cos\left(\frac{1}{\alpha} \arctan\left(\beta \cdot \tan\left(\frac{\pi\alpha}{2}\right)\right)\right), \text{ if } x = \mu \end{cases}$$

$$U_\alpha(\varphi, \vartheta) = \left(\frac{\sin\left(\frac{\pi}{2}\alpha(\varphi + \vartheta)\right)}{\cos\left(\frac{\pi\varphi}{2}\right)}\right)^{\frac{\alpha}{1-\alpha}} \cdot \left(\frac{\cos\left(\frac{\pi}{2}\left((\alpha - 1)\varphi + \alpha\vartheta\right)\right)}{\cos\left(\frac{\pi\varphi}{2}\right)}\right),$$

where $\theta = \arctan\left(\beta \tan\frac{\pi\alpha}{2}\right) \frac{2}{\alpha\pi} \cdot sgn(x - \mu)$.

If μ=0 and σ=1, then $p(x, \alpha, \beta) = p(-x, \alpha, -\beta)$.

A stable r.v. has a property, tat may be expressed in two equivalent forms:

If $X_1, X_2, \ldots, X_n$ are independent r.vs. distributed by $S_\alpha(\sigma, \beta, \mu)$, then $\sum_{i=1}^{n} X_i$ will be distributed by $S_\alpha(\sigma \cdot n^{1/a}, \beta, \mu \cdot n)$.

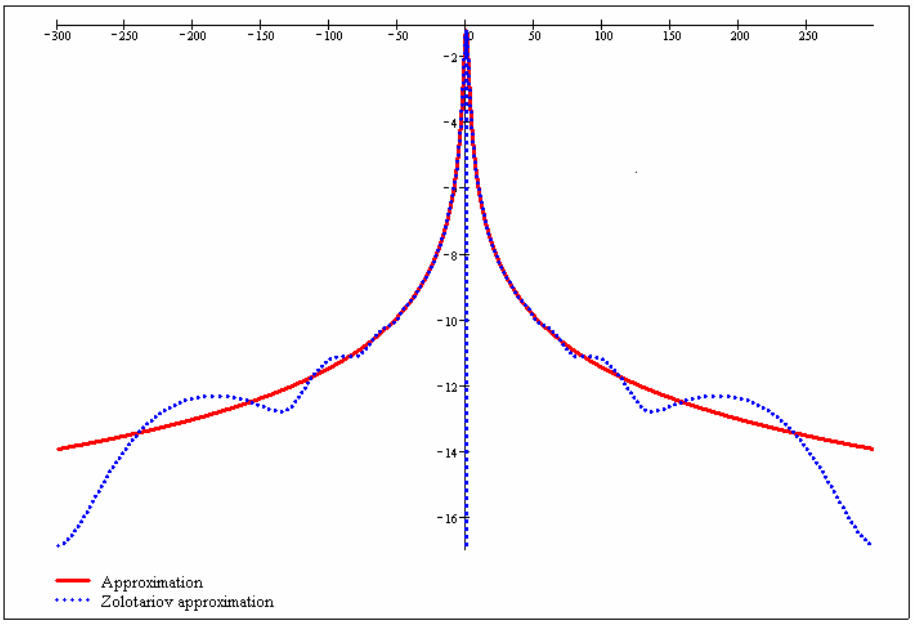


Figure 3: Logarithm of the probability density function $S_{1.5}(1,0,0)$

If $X_1, X_2, \ldots, X_n$ are independent r.vs. distributed by $S_\alpha(\sigma,\beta,\mu)$, then

$$\sum_{i=1}^{n} X_i \stackrel{d}{=} \begin{cases} n^{1/\alpha}\cdot X_1 + \mu\cdot(n-n^{1/\alpha}), \text{if} \alpha \neq 1 \\ n\cdot X_1 + \frac{2}{\pi}\cdot\sigma\cdot\beta\cdot n\ln n, \text{if} \alpha = 1 \end{cases}.$$

One of the most fundamental stable law statements is as follows.

Let X_1, X_2,...,X_n be independent identically distributed random variables and

$$\eta_n = \frac{1}{B_n}\sum_{k=1}^{n} X_k + A_n,$$

where B_n¿0 and A_n are constants of scaling and centering. If $F_n(x)$ is a cumulative distribution function of r.v. η_n, then the asymptotic distribution of functions $F_n(x)$, as $n \to \infty$, may be stable and only stable. And vice versa: for any stable distribution $F(x)$, there exists a series of random variables, such that $F_n(x)$ converges to $F(x)$, as $n \to \infty$.

The pth moment $E|X|^p = \int_0^\infty P(|X|^p > y)dy$ of the random variable X exists and is finite only if $0 < p < \alpha$. Otherwise, it does not exist.

3.1 Stable processes

A stochastic process $\{X(t), t \in T\}$ is stable if all its finite dimensional distributions are stable [30].

Let $\{X(t), t \in T\}$ be a stochastic process. $\{X(t), t \in T\}$ is α-stable if and only if all linear combinations $\sum_{k=1}^{d} b_k X(t_k)$ (here $d \geq 1$ $t_1, t_2, \ldots, t_d \in T$, $b_1, b_2, \ldots, b_d$ – real) are α-stable. A stochastic process $\{X(t), t \in T\}$ is called the (standard) α-stable Levy motion if:

1. $X(0) = 0$ (almost surely);
2. $\{X(t)$: $t \geq 0\}$ has independent increments;
3. $X(i) - X(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$, for any $0 < s < t < \infty$ and $0 < \alpha \leq 2$, $-1 < \beta < 1$.

Note that the α-stable Levy motion has stationary increments. As $\alpha = 2$, we have the Brownian motion.

3.2 Parameter Estimation Methods

The problem of estimating the parameters of stable distribution is usually severely hampered by the lack of known closed form density functions for almost all stable distributions [3]. Most of the methods in mathematical statistics cannot be used in this case, since these methods depend on an explicit form of the PDF. However, there are numerical methods that have been found useful in practice and are described below. Given a sample $x_1, \ldots, x_n$ from the stable law, we will provide estimates $\hat{\alpha}$, $\hat{\beta}$,$\hat{\mu}$, and $\hat{\sigma}$of α, β, μ, and σ. Also, some empirical methods were used:

- Method of Moments (empirical CF);
- Regression method.

3.2.1 Comparison of estimation methods

We simulated a sample of 10 thousand members with the parameters $\alpha = 1.75$, $\beta = 0.5$, $\mu = 0$ and $\sigma = 1$. Afterwards we estimated the parameters of a stable random variable with different estimators. All the methods are decent, but the maximal likelihood estimator yields the best results. From the practical point-of-view, MLM is the worst method, because it is very time-consuming. For large sets ($\sim$10.000 and more) we suggest using the regression (or moments) method to estimate α, β and σ, then estimate μ by MLM (optimization only by μ). As a starting point you should choose α, β, σ and sample mean, if $\alpha > 1$ and a median, otherwise, for μ. For short sets, use MLM with any starting points (optimization by all 4 parameters).

3.3 A mixed stable distribution model

Let $Y \sim B(1,p)$ and $X \sim S_a$ [2]. Let a mixed stable r. v. Z take the value 0 with probability 1 if $Y = 0$, else $Y = 1$ and $Z = X$. Then we can write the distribution function of the mixed stable distribution as

$$\begin{aligned} P(Z<z) &= P(Y=0)\cdot P(Z<z|Y=0) + P(Y=1)\cdot P(Z<z|Y=1) \\ &= p\cdot\varepsilon(z) + (1-p)\cdot S_\alpha(z) \end{aligned} \tag{1}$$

where $\varepsilon(\mathrm{x}) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$, is the cumulative distribution function (CDF) of the degenerate distribution. The PDF of the mixed-stable distribution is

$$f(x) = p\cdot\delta(x) + (1-p)\cdot p_\alpha(x),$$

where $\delta(x)$ is the Dirac delta function.

3.3.1 Cumulative density, probability density and characteristic functions of mixed distribution

For a given set of returns $\{x_1, x_2, \cdots, x_n\}$, let us construct a set of nonzero values $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{n-k}\}$. The equity ZMP1L (Žemaitijos pienas), from Vilnius stock exchange is given as an example (p=0,568). Then the likelihood function is given by

$$L(\bar{x},\theta,p) \sim (1-p)^k p^{n-k} \prod_{i=1}^{n-k} p_\alpha(\bar{x}_i,\theta) \tag{2}$$

where θ is the vector of parameters (in the stable case, $\theta = (\alpha,\beta,\mu,\sigma)$). The function $(1-p)^k p^{n-k}$is easily optimized: $p_{\max} = \frac{n-k}{n}$. So we can write the optimal CDF as

$$F(z) = \frac{n-k}{n} S_\alpha(z,\theta_{\max}) + \frac{k}{n}\varepsilon(z), \tag{3}$$

where the vector θ_{max} of parameters is estimated with nonzero returns.

The probability density function

$$p(z) = \frac{n-k}{n} p_\alpha(z,\theta_{\max}) + \frac{k}{n}\delta(z). \tag{4}$$

Finally we can write down and plot (Figure 6) the characteristic function (CF) of the mixed distribution.

$$\phi_{mix}(t) = \frac{n-k}{n}\cdot\phi(t) + \frac{k}{n}$$

The empirical characteristic function $\hat{\phi}(t,X) = \frac{1}{n}\sum_{j=1}^{n} e^{itX_j}$.

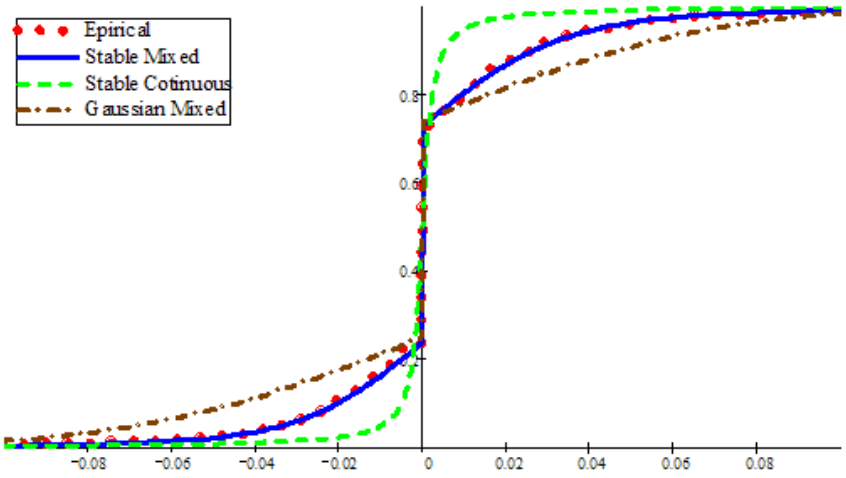


Figure 4: CDF of ZMP1L

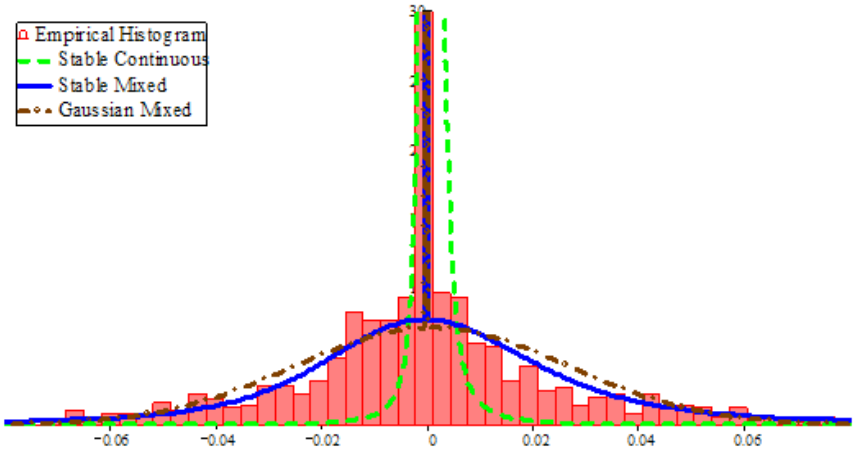


Figure 5: PDF and a histogram of ZMP1L

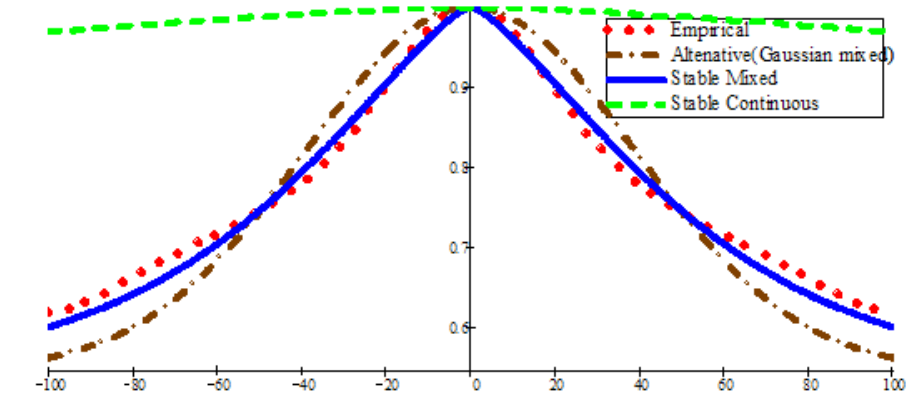


Figure 6: Empirical, Gaussian, Stable mixed, and Stable continuous CF of ZMP1L

Table 1: Results of goodness-of-fit tests (accepted/rejected cases)

Fit Method	Gaussian	mixed Gaussian	stable	mixed stable
Modified χ^2	0/64	7/64	0/64	52/64
Empirical CF	0/64	0/64	12/64	52/64
Anderson – Darling	0/64	-	0/64	-

3.3.2 Mixed model adequacy

There arises a problem when we are trying to test the adequacy hypothesis for these models. Since we have a discontinuous distribution function, the classic methods (Kolmogorov–Smirnov, Anderson-Darling) do not work for the continuous distribution, and we have to choose a goodness–of–fit test based on the empirical characteristic function [20, 21], or to trust a modified χ^2 (Romanovski) method [18]. The results (see Table 1) of both methods are similar (match in 48 cases).

The CF-based test of Brown and Saliu [6] is not so good (89% of all cases were rejected, since they are developed for symmetric distributions). A new stability test for asymmetric (skewed) alpha-stable distribution functions, based on the characteristic function, should be developed, since the existing tests are not reliable. Detailed results of stable-mixed model fitting are given in Table 2.

One can see that when the number of "zeros" increases, the mixed model fits the empirical data better.

A mixed-stable model of returns distribution was proposed. Our results show that this kind of distribution fits the empirical data better than any other. The implementation of this model is hampered by the lack of goodness-of-fit tests for discontinuous distributions. Since adequacy tests for continuous distribution

Table 2: Mixed model fit dependence on the number of zeros in series

Number of "zeros"	Number of such series	Fits mixed model (χ^2, %)	Fits mixed model (Empirical CF, %)
0,1-0,2	2	100	100
0,2-0,3	2	100	100
0,3-0,4	8	25,00	25,00
0,4-0,5	17	64,71	94,12
0,5-0,6	14	71,43	100
0,6-0,7	15	86,67	100
0,7-0,8	4	100	100
0,8-0,9	2	100	100

functions cannot be implemented, the tests, based on the empirical characteristic function and a modified χ^2 test, are used.

3.4 Modeling of stagnation intervals in emerging stock markets

We analyzed the following r.vs $X_i = 0$, if $P_{i+1} = P_i$ and $X_i = 1$, if $P_{i+1} \neq P_i$, where $\{P_i\}$ is a set of stock prices and $\{X_i\}$ is set of discrete states, representing behavior of stock price (change=1 or not=0).

3.4.1 Empirical study of lengths distribution of zero state runs

Theoretically if states are independent (Bernoulli scheme), then the series of lengths of zero state runs should be distributed by geometrical law. However, the results of empirical tests do not corroborate this assumption. We have fitted the series distribution of lengths of zero state runs by discrete laws (generalized logarithmic, generalized Poisson, Hurwitz zeta, generalized Hurwitz zeta, discrete stable). The probability mass function of Hurwitz zeta law is

$$P\left(\xi = k\right) = \nu_{s,q}\left(k+q\right)^{-s} \quad ,$$

where $\nu_{s,q} = \left(\sum_{i=0}^{\infty}\left(i+q\right)^{-s}\right)^{-1}$, $k \in N$,q¿0, s¿1. The parameters of all discrete distributions were estimated by the maximal likelihood method.

3.4.2 Transformation and distribution fitting

First of all, we will show how financial data from the Baltic States market are transformed to subsets length of zero state series and then we will fit each of the discrete distributions mentioned in above section. Carvalho, Angeja and Navarro have showed that data in network engineering fit the discrete logarithmic distribution better than the geometrical law. So we intend to test whether such a property is valid for financial data from the Baltic States market.

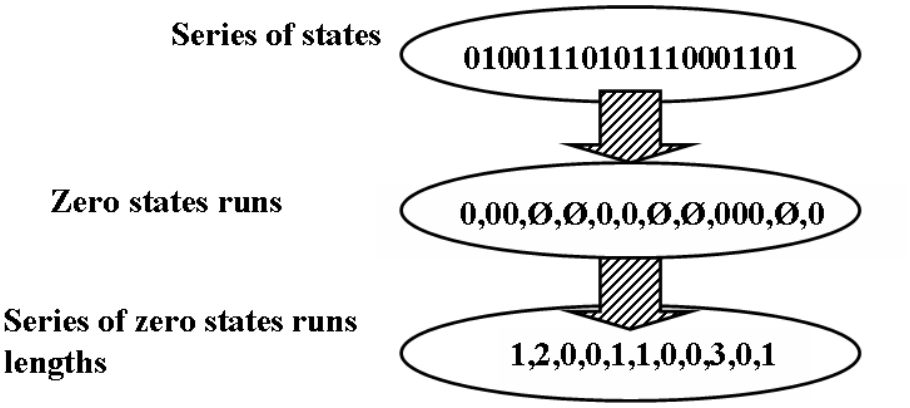


Figure 7: Data transformation

Table 3: Distribution of zero state series

Significance level	Hurwitz zeta	Generalized Hurwitz zeta	Generalized logarithmic	Discrete stable	Poissson	Generalized Poissson	Geometrical
0.01	94.74%	96.49%	63.16%	26.32%	0.00%	1.75%	1.75%
0.025	91.23%	91.23%	50.88%	22.81%	0.00%	1.75%	1.75%
0.05	87.72%	84.21%	42.11%	17.54%	0.00%	1.75%	1.75%
0.1	80.70%	78.95%	31.58%	12.28%	0.00%	1.75%	1.75%

A set of zeros between two units is called a run. The first run is a set of zeros before the first unit and the last one after the last unit. The length of the run is equal to the number of zeros between two units. If there are no zeros between two units, then such an empty set has zero length (Fig. 7).

To transform our data (from the state series, e.g., 010011101011100110) the two following steps should be taken: (a) extract the zero state runs (e.g., 0,00,0,0,000,0) from the states series; (b) calculate the length of each run (1,2,0,0, 1,1,0,0,3,0,1,). After the transformation, we estimated the parameters of each discrete distribution mentioned above and tested the nonparametric χ^2 distribution fitting hypothesis.

As mentioned above, theoretically this series should be distributed by geometrical law, however, from Table 3 we can see that other laws fit our data (57 series) much better. It means that zero state series from the Baltic States market are better described by the Hurwitz zeta distribution.

This result allows us to assume that zero-unit states are not purely independent. The Wald–Wolfowitz runs test [22] corroborates this assumption for almost all series from the Baltic States market. The inner series dependence was tested by the Hoel [15] criterion on the order of the Markov chain. It has been concluded that there are no zero order series or Bernoulli scheme series. 95% of given series are 4th-order Markov chains with ϕ=0.1% significance level.

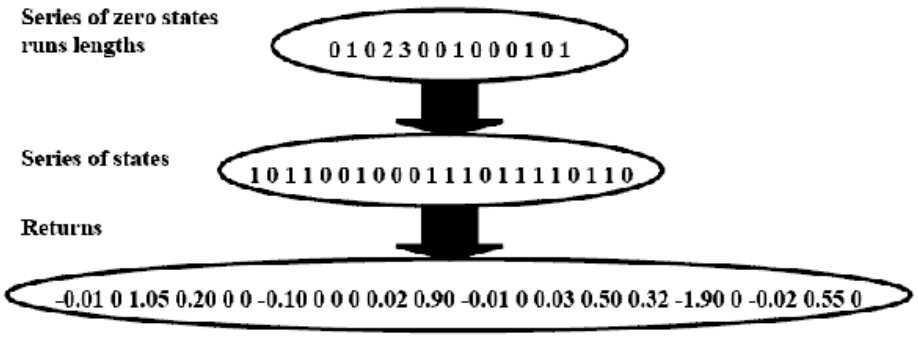


Figure 8: Simulation of passive stable series

3.4.3 The mixed stable model with dependent states

Since the runs test rejects the randomness hypothesis of the sequence of states, the probability of states (zeros and ones) depends on the position in the sequence. If the lengths of states sequences are distributed by Hurwitz zeta law, then the probabilities of states are

$$P(X_n = 1|..., X_{n-k-1} = 1, \underbrace{X_{n-k} = 0, ..., X_{n-1} = 0}_{k}) = \frac{p_k}{1 - \sum_{j=0}^{k-1} p_j}, n \in N, k \in Z_0,$$

where p_k are probabilities of Hurwitz zeta law; $P(X_0{=}1) = p_0$. It should be noted that $P(X_n = 0|...) = 1 - P(X_n = 1|...), n, k \in Z_0$.

With the probabilities of states and distribution of nonzero returns we can generate sequences of stock returns (interchanging in the state sequence units with a stable r.v.) see Fig. 8.

So, the mixed-stable modeling with dependent states is more advanced than that with independent (Bernoulli) states, it requires parameter estimation by both the stable (α, β, μ, σ) and Hurwitz zeta (q, s) law.

4 Analysis of stability

Examples of stability analysis can be found in the works of Rachev [5, 16, 31] and Weron [35]. In the latter paper, Weron analyzed the DJIA index (from 1985-01-02 to 1992-11-30, 2000 data points in all). The stability analysis was based on the Anderson–Darling criterion and by the weighted Kolmogorov criterion (D'Agostino), the parameters of stable distribution were estimated by the regression method proposed by Koutrouvelis [19]. The author states that DJIA characteristics perfectly correspond to stable distribution.

Almost all data series are strongly asymmetric ($\hat{\gamma}_1$), and the empirical kurtosis ($\hat{\gamma}_2$) shows that density functions of series are more peaked than Gaussian.

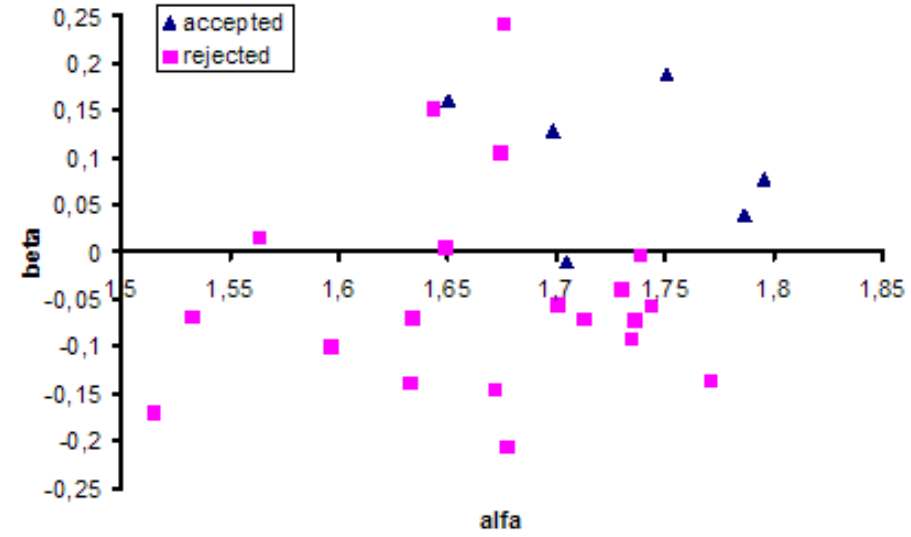


Figure 9: Distribution of α and β (*developed markets*)

That is why we make an assumption that Gaussian models are not applicable to these financial series. The distribution (Figure 9) of α and β estimates shows that usually α is over 1.5 and for sure less than 2 (this case 1.8) for financial data.

Now we will verify two hypotheses: the first one – H_0^1 is our sample (with empirical mean $\hat{\mu}$ and empirical variance $\hat{\sigma}$) distributed by the Gaussian distribution. The second – H_0^2 is our sample (with parameters α, β,μ and σ) distributed by the stable distribution. Both hypotheses are examined by two criteria: the Anderson–Darling (A-D) method and Kolmogorov–Smirnov (K-S) method. The first criterion is more sensitive to the difference between empirical and theoretical distribution functions in far quantiles (tails), in contrast to the K-S criterion that is more sensitive to the difference in the central part of distribution.

The A-D criterion rejects the hypothesis of Gaussianity in all cases with the confidence level of 5% . Hypotheses of stability fitting were rejected only in 15 cases out of 27, but the values of criteria, even in the rejected cases, are better than that of the Gaussian distributions.

To prove the stability hypothesis, other researchers [13, 25] applied the method of infinite variance, because non–Gaussian stable r.vs has infinite variance. The set of empirical variances S_n^2 of the random variable X with infinite variance diverges.

Let $x_1,\dots x_n$ be a series of i.i.d.r.vs X. Let $n \leq N < \infty$ and $\bar{x}_n$ be the mean of the first n observations, $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2$, $1 \leq n \leq N$. If a

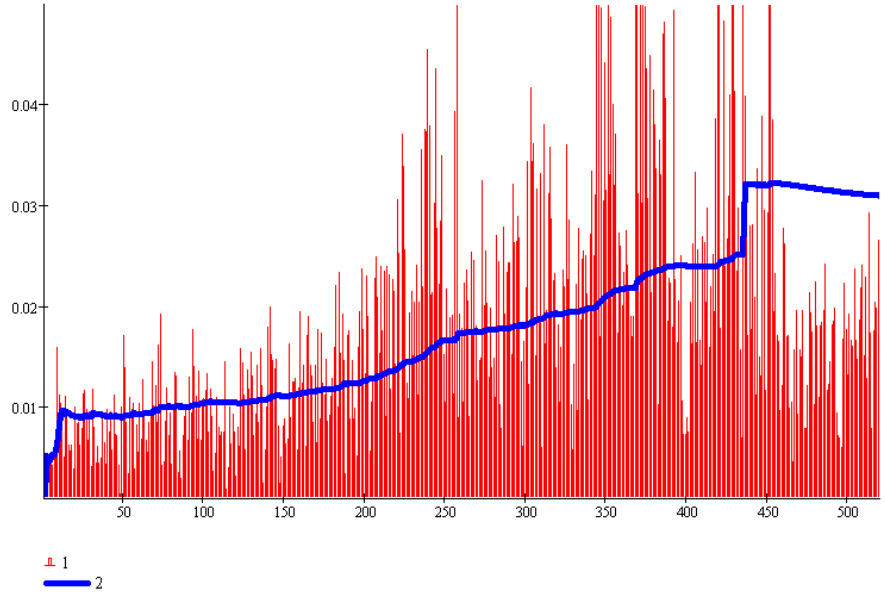


Figure 10: Series of empirical variance of the MICROSOFT company (13-03-86 – 27-05-05)

distribution has finite variance, then there exists a finite constant $c¡\infty$ such that $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 \to c$ (almost surely), as $n \to \infty$. And vice versa, if the series is simulated by the non–Gaussian stable law, then the series S_n^2 diverges. Fofack [11] has applied this assumption to a series with finite variance (standard normal, Gamma) and with infinite variance (Cauchy and totally skewed stable). In the first case, the series of variances converged very fast and, in the second case, the series of variances oscillated with a high frequency, as $n \to \infty$. Fofack and Nolan [12] applied this method in the analysis of distribution of Kenyan shilling and Morocco dirham exchange rates in the black market. Their results allow us to affirm that the exchange rates of those currencies in the black market change with infinite variance, and even worse – the authors state that distributions of parallel exchange rates of some other countries do not have the mean ($\alpha¡1$ in the stable case). We present, as an example, a graphical analysis of the variance process of Microsoft corporation stock prices returns (Figure 10).

The columns in this graph show the variance at different time intervals, the solid line shows the series of variances S_n^2. One can see that, as n increases, i.e. $n \to \infty$, the series of empirical variance S_n^2 not only diverges, but also oscillates with a high frequency. The same situation is for mostly all our data sets presented.

4.1 Stability by homogeneity of the data series and aggregated series

The third method to verify the stability hypothesis is based on the fundamental statement. Suppose we have an original financial series (returns or subtraction of logarithms of stock prices) X_1, X_2,...,X_n. Let us calculate the partial sums Y_1,Y_2,...,$Y_{[n/d]}$, where $Y_k = \sum_{i=(k-1)\cdot d+1}^{k\cdot d} X_i$, k=1...$[n/d]$, and d is the number of sum components (freely chosen). The fundamental statement implies that original and derivative series must be homogeneous. Homogeneity of original and derivative (aggregated) sums was tested by the Smirnov and Anderson criteria (ω^2).

The accuracy of both methods was tested with generated sets, that were distributed by the uniform R(−1,1), Gaussian N(0,1/$\sqrt{3}$), Cauchy C(0,1) and stable $S_{1.75}(1, 0.25, 0)$ distributions. Partial sums were scaled, respectively, by $\sqrt{d}$, $\sqrt{d}$, d, $d^{1/1.75}$. The test was repeated for a 100 times. The results of this modeling show that the Anderson criterion (with confidence levels 0.01, 0.05 and 0.1) is more precise than that of Smirnov with the additional confidence level.

It should be noted that these criteria require large samples (of size no less than 200), that is why the original sample must be large enough. The best choice would be if one could satisfy the condition $n/d > 200$.

The same test was performed with real data of the developed and emerging markets, but homogeneity was tested only by the Anderson criterion. Partial series were calculated by summing $d = 10$ and 15 elements and scaling with $d^{1/\alpha}$.

One may draw a conclusion from the fundamental statement that for international indexes ISPIX, AMEX, BP, FCHI, COCA, GDAXI, DJC, DJ, DJTA, GE, GM, IBM, LMT, MCD, MER, MSFT, NIKE, PHILE, S&P and SONY the hypothesis on stability is acceptable.

4.2 Self–similarity and multifractality

As mentioned before, for a long time it has been known that financial series are not properly described by normal models [31, 32]. Due to that, there arises a hypothesis of fractionallity or self–similarity. The Hurst indicator (or exponent) is used to characterize fractionallity. The process with the Hurst index $H = 1/2$ corresponds to the Brownian motion, when variance increases at the rate of $\sqrt{t}$, where t is the amount of time. Indeed, in real data this growth rate (Hurst exponent) is higher. As $0.5 < H \leq 1$, the Hurst exponent implies a persistent time series characterized by long memory effects, and when $0 \leq H < 0.5$, it implies an anti-persistent time series that covers less distance than a random process. Such behavior is observed in mean – reverting processes[32].

There are a number of different, in equivalent definitions of self-similarity [33]. The standard one states that a continuous time process $Y = \{Y(t),\ t \in T\}$

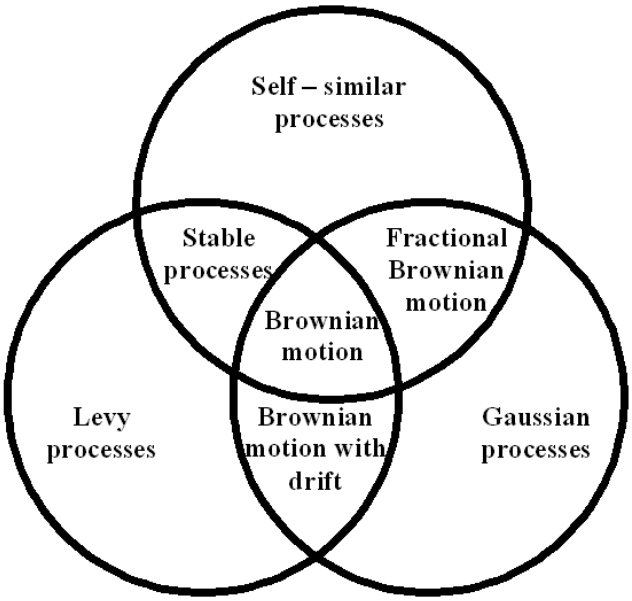


Figure 11: Self-similar processes and their relation to Levy and Gaussian processes

is self-similar, with the self-similarity parameter H(Hurst index), if it satisfies the condition:

$$Y(t) \stackrel{d}{=} a^{-H} Y(at), \quad \forall t \in T,\ \forall a > 0,\ 0 \leq H < 1, \tag{5}$$

where the equality is in the sense of finite-dimensional distributions. The canonical example of such a process is Fractional Brownian Motion ($H = 1/2$). Since the process Y satisfying (5) can never be stationary, it is typically assumed to have stationary increments [8].

Figure 11 shows that stable processes are the product of a class of self-similar processes and that of Levy processes [9]. Suppose a Levy process $X = \{X(t), t \geq 0\}$. Then X is self-similar if and only if each $X(t)$ is strictly stable. The index α of stability and the exponent H of self-similarity satisfy $\alpha = 1/H$.

Consider the aggregated series $X^{(m)}$, obtained by dividing a given series of length N into blocks of length m and averaging the series over each block.

$$X^{(m)}(k) = \frac{1}{m} \sum_{i=(k-1)m+1}^{km} X_i, \text{here } k = 1, 2 \ldots [N/m].$$

Self-similarity is often investigated not through the equality of finite-dimensional distributions, but through the behavior of the absolute moments. Thus, consider

$$AM^{(m)}(q) = E\left|\frac{1}{m}\sum_{i=1}^{m} X(i)\right|^q = \frac{1}{m}\sum_{k=1}^{m}\left|X^{(m)}(k) - \bar{X}\right|^q$$

If X is self-similar, then $AM^{(m)}(q)$ is proportional to $m^{\beta(q)}$, which means that $\ln AM^{(m)}(q)$ is linear in $\ln m$ for a fixed q:

$$\ln AM^{(m)}(q) = \beta(q)\ln m + C(q). \tag{6}$$

In addition, the exponent $\beta(q)$ is linear with respect to q. In fact, since $X^{(m)}(i) \stackrel{d}{=} m^{1-H}X(i)$, we have

$$\beta(q) = q(H-1) \tag{7}$$

Thus, the definition of self-similarity is simply that the moments must be proportional as in (6) and that $\beta(q)$ satisfies (7).

This definition of a self-similar process given above can be generalized to that of multifractal processes. A non-negative process $X(t)$ is called multifractal if the logarithms of the absolute moments scale linearly with the logarithm of the aggregation level m. Multifractals are commonly constructed through multiplicative cascades [10]. If a multifractal can take positive and negative values, then it is referred to as a signed multifractal (the term "multiaffine" is sometimes used instead of "signed multifractal"). The key point is that, unlike self-similar processes, the scaling exponent $\beta(q)$ in (6) is not required to be linear in q. Thus, signed multifractal processes are a generalization of self-similar processes. To discover whether a process is (signed) multifractal or self-similar, it is not enough to examine the second moment properties. One must analyze higher moments as well.

However this method is only graphical and linearity is only visual.

Finally, only 9 indices are self-similar: ISPX, AMEX, FCHI, GDAXI, DJC, DJ, DJTA, NIKKEI, S&P.

Hurst exponent estimation. There are many methods to evaluate this index, but in literature the following are usually used [33]:

- Time-domain estimators,
- Frequency-domain/wavelet-domain estimators,

The methods: absolute value method (absolute moments), variance method (aggregate variance), R/S method and variance of residuals are known as time domain estimators. Estimators of this type are based on investigating the power law relationship between a specific statistic of the series and the so-called aggregation block of size m.

The following three methods and their modifications are usually presented as time-domain estimators:

Periodogram method;

Whittle;
Abry-Veitch (AV).

The methods of this type are based on the frequency properties of wavelets.

All Hurst exponent estimates were calculated using SELFIS software, which is freeware and can be found on the web page http://www.cs.ucr.edu/∼tkarag.

4.3 Multifractality and self-similarity in the financial markets

In the case of The Baltic States and other Central and Eastern Europe financial markets, the number of daily zero returns can reach 89% . Anyway, this problem may be solved by extending a continuous model to the mixed one, where daily returns equal to zero are excluded from the series when estimating the stability parameters. The series of non-zero returns are fitted to the stable distribution. Stable parameters are estimated by the maximal likelihood method. Goodness-of-fit is verified by the Anderson-Darling distributional adequacy test. The stability is also tested by the homogeneity test, based on the fundamental property of stable laws. The summation scheme is based on the bootstrap method in order to get larger series. Multifractality and self-similarity are investigated through the behavior of the absolute moments. The Hurst analysis has been made by the R-S method.

We have investigated 26 international financial series focusing on the issues of stability, multifractality, and self-similarity. It has been established that the hypothesis of stability was ultimately rejected in 14.81% cases, definitely stable in 22.22% , and the rest are doubtful. It is important to note that, even in the case of rejection, the value of the A-D criterion was much better for stability testing than for the test of Gaussian distribution. No series was found distributed by the Gaussian law.

The stable model parameters were estimated by the maximal likelihood method. The stability indexes of stable series are concentrated between 1.65 and 1.8, which confirms the results of other authors that the stability parameter of financial data is over 1.5. Asymmetry parameters are scattered in the area between -0.017 and 0.2.

The investigation of self-similarity has concluded that only 66.67% of the series are multifractal and the other 33.33% concurrently are self-similar.

The Hurst analysis has showed that the methods of R/S and Variance of Residuals are significant in the stability analysis. Following these two methods, Hurst exponent estimates are in the interval $H \in (0.5; 0.7)$, which means that the stability index $\alpha \in (1.42; 2)$. If the Hurst exponent is calculated by the R/S method, $H \in (0.5; 0.6)$, then $\alpha \in (1.666; 2)$.

The stable models are suitable for financial engineering; however the analysis has shown that not all (only 22% in our case) the series are stable, so the model adequacy and other stability tests are necessary before model application. The studied series represent a wide spectrum of stock market, however it should be stressed that the research requires a further continuation: to extend the models.

The analysis of stability in the Baltic States market has showed that 49 series of 64 are multifractal and 8 of them are also self-similar. If we removed zero returns from the series, there would be 27 multifractal series, concurrently 9 of them are self-similar.

5 Relationship measures

In constructing a financial portfolio it is essential to determine relationships between different stock returns [28]. However, under the assumption of stability (sets of stock returns are modeled by stable laws), the classical relationship measures (covariance, correlation) cannot be applied. Therefore the generalized Markowitz problem is solved by generalized relationship measures (covariation, codifference). We show that implementation of the codifference between different stocks greatly simplifies the construction of the portfolio. We have constructed optimal portfolios of ten Baltic States stocks.

In the classical economic statistics (when the distributional law has two first moments, i.e., mean and variance), relations between two random variables (returns) are described by covariance or correlation. But if we assume that financial data follow the stable law (empirical studies corroborate this assumption), covariance and especially correlation (Pearson) cannot be calculated. In case when the first ($\alpha < 1$) and the second ($\alpha < 2$) moments do not exist, other correlation (rank, e.g., Spearmen, Kendall, etc. [17]) and contingency coefficients are proposed. However, in the portfolio selection problem Samorodnitsky and Taqqu suggest better alternatives, even when mean and variance do not exist. They have proposed alternative relation measures: covariation and codifference.

If X_1 and X_2 are two symmetric i.d. [30] (with $\alpha_1 = \alpha_2 = \alpha$) stable random variables, then the covariation is equal to

$$[X_1, X_2]_\alpha = \int_{S_2} s_1 s_2^{\langle \alpha - 1 \rangle} \Gamma(ds),$$

where α¿1 , $y^{\langle \alpha \rangle} = |y|^\alpha \text{sign}(\alpha)$ and Γ is a spectral measure of (X_1, X_2).

In such a parameterization, the scale parameter $\sigma_{X_1}^\alpha$ of symmetric stable r.v. can be calculated from $[X_1, X_1]_\alpha = \sigma_{X_1}^\alpha$. If $\alpha = 2$ (Gaussian distribution), the covariation is equal to half of the covariance $[X_1, X_2]_2 = \frac{1}{2}\text{Cov}(X_1, X_2)$ and $[X_1, X_1]_2 = \sigma_{X_1}^2$ becomes equal to the variance of X_1. However, the covariation norm of $X \in S_\alpha$ (α¿1) can be calculated as $\|X\| = ([X, X]_\alpha)^{1/\alpha}$. If $X \sim S_\alpha(\sigma, 0, 0)$ (SαS case), then the norm is equivalent to the scale parameter of the stable distribution $\|X\|_\alpha = \sigma$.

In general case [27] the codifference is defined through characteristic functions

$$\begin{aligned} cod_{X,Y} &= \ln(E\exp\{i(X-Y))\}) - \ln(E\exp\{iX\}) - \ln(E\exp\{-iY\}) \\ &= \ln\left(\frac{E\exp\{i(X-Y)\}}{E\exp\{iX\}\cdot E\exp\{-iY\}}\right) = \ln\left(\frac{\phi_{X-Y}}{\phi_X \cdot \phi_{-Y}}\right), \end{aligned}$$

or empirical characteristic functions

$$cod_{X,Y} = \ln \left(\frac{n \cdot \sum_{j=1}^{n} e^{i(X_j - Y_j)}}{\sum_{j=1}^{n} e^{iX_j} \cdot \sum_{j=1}^{n} e^{-iY_j}} \right)$$

The codifference of two symmetric ($S\alpha S$) r.vs X and Y ($0 < \alpha \leq 2$) can be expressed through the scale parameters

$$cod_{X,Y} = \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha - \|X - Y\|_\alpha^\alpha$$

If $\alpha = 2$, then $cod_{X,Y} = \mathrm{Cov}(X, Y)$.

Samorodnitsky and Taqqu have showed that

$$\left(1 - 2^{\alpha-1}\right) \left(\|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha\right) \leq cod_{X,Y} \leq \|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha,$$

here $1 \leq \alpha \leq 2$, and, if we normalize (divide by $\|X\|_\alpha^\alpha + \|Y\|_\alpha^\alpha$), we will get a generalized correlation coefficient.

In the general case [27], the following inequalities

$$\begin{array}{lll} \left(1 - 2^{\alpha-1}\right) \ln \left(\frac{1}{E \exp\{iX\} \cdot E \exp\{-iY\}} \right) & & \\ \leq & cod_{X,Y} & \\ = & \ln \left(\dfrac{E \exp\{i(X - Y)\}}{E \exp\{iX\} \cdot E \exp\{-iY\}} \right) & \\ \leq & \ln \left(\dfrac{1}{E \exp\{iX\} \cdot E \exp\{-iY\}} \right) & \end{array}$$

are proper, and if we divide both sides by $\ln\left(E \exp\{iX\} \cdot E \exp\{-iY\}\right)$, we will get the following system of inequalities for the correlation coefficient

$$\left(1 - 2^{\alpha-1}\right) \leq corr_{X,Y} = \frac{\ln \left(\frac{E \exp\{iX\} \cdot E \exp\{-iY\}}{E \exp\{i(X-Y)\}} \right)}{-\ln\left(E \exp\{iX\} \cdot E \exp\{-iY\}\right)} \leq 1$$

If $0 < \alpha \leq 1$ this correlation coefficient is only non-negative, and if $\alpha = 2$, $\beta = 0$, then $-1 \leq corr_{X,Y} = \rho_{X,Y} \leq 1$ is equivalent to the Pearson correlation coefficient.

5.1 Significance of codifference

The significance of the Pearson correlation coefficient is tested using Fisher statistics, and that of Spearmen and Kendall coefficients, respectively, are tested using Student and Gaussian distributions. But likely that there are no codifference significance tests created. In such a case, we use the bootstrap method (one of Monte-Carlo style methods). The following algorithm to test the codifference significance is proposed:

1. Estimate stable parameters (α, β, σ and μ) and stagnation probability p of all equity returns series;
2. Estimate relation matrix of measure ρ (covariation or codifference) for every pair of equities series;
3. Test the significance of each ρ_{ij} by the bootstrap method:
 i. generate a pair of two ith and jth mixed-stable (with estimated parameters) series, and proceed to the next step;
 ii. calculate the kth relation measure ρ_{ij}^k, between the ith and jth series;
 iii. repeat (i) and (ii) steps for $k = 1, ..., N$ (for example, 10000) times;
 iv. construct ordered series of estimates $\rho_{ij}^{(k)}$;
 v. if $\rho_{ij}^{([N\cdot 0.025])} \leq \rho_{ij} \leq \rho_{ij}^{([N\cdot 0.975])}$, then the significance of ρ_{ij} is rejected with the confidence level 0.05, i.e., it is assumed that $\rho_{ij} = 0$.
 vi. repeat 3i–3v steps for each pair of equities i and j.

Covariation and codifference are calculated for ten equities with the longest series (MNF1L, LDJ1L, VNF1R, NRM1T, MKO1T, GZE1R, ETLAT, VNG1L, SNG1L, TEO1L). The correlation tables are presented for the series of equalized length 1427.

However, in portfolio the theory covariance (or equivalent measure) is more useful, since in that case, there is no need to know the variance. The generalized covariance tables are calculated for previously mentioned series.

6 Conclusions

Parameter estimation methods and software has been developed for models with asymmetric stable distributions. The efficiency of estimation methods was tested by simulating the series. Empirical methods are more effective in time, but the maximal likelihood method (MLM) is more effective (for real data) in the sense of accuracy (Anderson-Darling goodness-of-fit test corroborate that). It should be noted that MLM is more sensitive to changes of the parameters α and σ.

Empirical parameters of the Baltic States series and developed market series (respectively 64 and 27 series) have been estimated. Most of the series are very asymmetric ($0.1 < |\gamma_1| < 30$), and the empirical skewness ($\gamma_2 \neq 0$) suggests that the probability density function of the series is more peaked and exhibits fatter tails than the Gaussian one. The normality hypothesis is rejected by the Anderson-Darling and Kolmogorov-Smirnov goodness-of-fit tests.

Distribution of the stability parameter α and asymmetry parameter β in the series of developed markets shows, that usually $1.5 < \alpha < 2$ and the parameter β is small. Distribution of the stability parameter in the series of the Baltic States market (full series) shows that usually α is lower than 1.5 and close to 1.

But if we remove the zero returns from the series, the parameter α is scattered near 1.5, while the parameter β is small usually, but positive.

An experimental test of the series homogeneity shows that for the stable series with asymmetry, the Anderson test is more powerful than the Smirnov one. The Anderson test for 27 series from the developed markets shows that 21 series are homogeneous with their aggregated series and only 2 series (64 at all) from the Baltic States market (and only when the zero returns are removed) are homogeneous with their aggregate series (they do not obey the fundamental stable theorem).

The analysis of self-similarity and multifractality, by the absolute moments method, indicates that all 27 series (from the developed markets) are multifractal and concurrently 9 of them are self-similar. On the other hand, 49 series (from the Baltic States market) are multifractal and 8 of them are also self-similar, but if we remove the zero returns from the series, then remain only 27 multifractal and 9 self-similar series. This is because the series becomes too short for multifractality analysis.

A mixed stable model of returns distribution in emerging markets has been proposed. We introduced the probability density, cumulative density, and the characteristic functions. Empirical results show that this kind of distribution fits the empirical data better than any other. The Baltic States equity lists are given as an example.

The implementation of the mixed-stable model is hampered by the lack of goodness-of-fit tests for discontinuous distributions. Since adequacy tests for continuous distribution functions cannot be implemented, the tests based on the empirical characteristic function (Koutrouvelis) as well as modified χ^2, are used. The experimental tests have showed that, if the stability parameter α and the number of zero returns are increasing, then the validity of the tests is also increasing. 99% of the Baltic States series satisfy the mixed stable model proposed (by the Koutrouvelis test).

The statistical analysis of the Baltic States equity stagnation intervals has been made. Empirical studies showed that the length series of the state runs of financial data in emerging markets are better described by the Hurwitz zeta distribution, rather than by geometrical. Since series of the lengths of each run are not geometrically distributed, the state series must have some internal dependence (Wald-Wolfowitz runs test corroborates this assumption). A new mixed-stable model with dependent states has been proposed and the formulas for probabilities of calculating states (zeros and units) have been obtained. Adequacy tests of this model are hampered by inner series dependence.

The inner series dependence was tested by the Hoel [15] criterion on the order of the Markov chain. It has been concluded that there are no zero order series or Bernoulli scheme series. 95% of given series are 4th-order Markov chains with $\phi = 0.1\%$ significance level.

When constructing an optimal portfolio, it is essential to determine possible relationships between different stock returns. However, under the assumption of stability (stock returns are modeled by mixed-stable laws) traditional relationship measures (covariance, correlation) cannot be applied, since

$(1,27 < \alpha < 1,78)$. In such a case, covariation (for asymmetric r.v.) and codifference are offered. The significance of these measures can be tested by the bootstrap method.

A wide spectrum of financial portfolio construction methods is known, but in the case of series stability it is suggested to use a generalized Markowitz model. The problem is solved by the generalized relationship measures (covariation, codifference). Portfolio construction strategies with and without the codifference coefficient matrix are presented. It has been shown that the codifference application considerably simplifies the construction of the optimal portfolio. Optimal stock portfolios (with 10 most realizable Baltic States stocks) with and without the codifference coefficient matrix are constructed.

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The numerical solution of partial differential equations with local polynomial regression(LPR)

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Abstract

In this paper, we extended the LPR method to solve the partial differential equations. Numerical experiments are presented to demonstrate the utility and the efficiency of the proposed computational procedure.

Keywords: Local polynomial regression(LPR); Convection diffusion;Heat equation.

1. Introduction

Consider the first problem is convection diffusion equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = \beta \frac{\partial^2 u}{\partial x^2} \ , \ 0 \leq x \leq 1, t \geq 0 \ , \tag{1}$$

and the second problem is heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \nu(x) \ , \ 0 \leq x \leq 1, t \geq 0 \ , \tag{2}$$

To Eq.(1) and Eq.(2) we attach the initial conditions and boundary condition

$$u(x,0) = f(x), \qquad 0 \leq x \leq 1 \ , \tag{3}$$

$$u(0,t) = g_0(t), \qquad t \geq 0 \ , \tag{4}$$

$$u(1,t) = g_1(t), \qquad t \geq 0. \tag{5}$$

In our previous work[1,2], solution of fifth order boundary value problems and integral equations by using local polynomial regression. In the present paper the local polynomial regression(LPR) is used to developed a technique for solving partial differential equations. We show that the method to achive the desired accuracy.

2. Local polynomial regression

Suppose that the $(p+1)$th derivative of $y(x)$ at point x_0 exists. We approximate the unknown regression function $y(x)$ locally at x_0 by a polynomial of order p. The theoretical justification is that we can approximate, in a neighborhood of x_0 , $y(x)$using a Taylor expansion

$$y(x) \approx \sum_{k=0}^{p} \beta_k (x_i - x_0)^k \tag{6}$$

where

$$\beta_k = \frac{y^{(k)}(x_0)}{k!} \tag{7}$$

This polynomial, used to approximate the unknown function locally at x_0, is obtained by solving a locally weighted least squares regression problem, i.e. by minimizing

$$\sum_{i=1}^{n} \left\{ Y_i - \sum_{k=0}^{p} \beta_k (x_i - x_0)^k \right\}^2 K(\tfrac{x_i - x_0}{h}) \tag{8}$$

where h is a parameter called bandwidth (also called a smoothing parameter), K is a weighting function called the kernel function. Let $\beta_k, k = 0, 1, ...p$ be the solution of the minimizing problem. From Eqs.(4), it is clear that j!β_j is an estimator for the derivatives $y^{(j)}(x_0), j = 0, 1, ...p$. Thus, the estimation obtained, of both the regression function and its derivatives, is local, and therefore, the process must be repeated at all points where an estimation is of interest. Let us see the analytical expression of the solution $\beta_k, k = 0, 1, ...p$ of the locally weighted least squares regression problem. Let X be the nx$(p+1)$ matrix

$$\mathrm{X}=\begin{pmatrix} 1 & (x_1-x_0) & ... & (x_1-x_0)^p \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & (x_n-x_0) & ... & (x_n-x_0)^p \end{pmatrix} \tag{9}$$

and the vectors $Y=(Y_1,Y_2,...,Y_n)^{'}$ and $\beta=(\beta_0,\beta_1,...\beta_p)^{'}$. Finally, let W denote the nxn diagonal matrix of weights $W=diag\{K_h(x_i-x_0)\}$. Then, the solution is

$$\beta=(X^TWX)^{-1}X^TWY. \tag{10}$$

The selection of K does not influence the results much. We selected the quartic kernel as follows

$$K(u)=\begin{Bmatrix} \frac{15}{16}(1-u^2)^2 & if|u|\leq 1 \\ 0 & otherwise \end{Bmatrix} \tag{11}$$

The fundemental idea of this methodology appears in [3].

3. LPR solutions for PDE

Difference schemes for the first problem considered as following:

$$\frac{u_{i+1}-u_i}{\Delta t}+\alpha_2\frac{\partial u}{\partial x}=\alpha_1\frac{\partial^2 u}{\partial x^2} \tag{12}$$

where $\Delta t=k$

$$-k\alpha_1u_{i+1}^{''}+k\alpha_2u_{i+1}^{'}+u_{i+1}=u_i \tag{13}$$

and the initial conditions are given in (3)-(4)

$$u(x,0)=f(x)=u_0, \tag{14}$$

Subsituting (14) in (13) then is obtained as follows

$$t = 0 + \Delta t \qquad -k\alpha_1 u_1'' + k\alpha_2 u_1' + u_1 = u_0 \tag{15}$$

$$t = 0 + 2\Delta t \qquad -k\alpha_1 u_2'' + k\alpha_2 u_2' + u_2 = u_1 \tag{16}$$

$$\begin{array}{cc} . & . \\ . & . \\ . & . \end{array}$$

$$t = 0 + n\Delta t \qquad -k\alpha_1 u_n'' + k\alpha_2 u_n' + u_n = u_{n-1} \tag{17}$$

In this section, the LPR method for solving Eqs.(1) is outlined. Let Eqs.(6) be an approximate solution of Eqs.(1).

$$y(t) = \sum_{j=0}^{p} \beta_j (x_i - x_0)^j \tag{18}$$

where

$x_1 = a, x_2, \ldots, x_n = b$

and it is required that the approximate solution(18) satisfies the PDEs at the points$x = x_i$. Putting (18) in (15), it follows that

$$-k\alpha_1 \left(\sum_{j=0}^{p} \beta_j (x - x_0)^j\right)'' + k\alpha_2 \left(\sum_{j=0}^{p} \beta_j (x - x_0)^j\right)' +$$

$$\left(\sum_{j=0}^{p} \beta_j (x - x_0)^j\right) = u_0 \qquad a \le x \le b \tag{19}$$

This leads to the system

$$i = 1, \qquad a_{1,j} = \beta_j (x_1 - x_0)^j, \quad j = 0, m \qquad y(i) = g_0(k) \tag{20a}$$

$$i = 2, n-1, \qquad b_{i,j} = -k\alpha_1 j(j-1)\beta_j (x_1 - x_0)^{j-2}, \quad j = 2, m \tag{20b}$$

$$i = 2, n-1, \qquad c_{i,j} = k\alpha_2 j \beta_j (x_1 - x_0)^{j-1}, \quad j = 1, m \tag{20c}$$

$$i = 2, n-1, \qquad d_{i,j} = \beta_j (x_1 - x_0)^j, \quad j = 0, m \tag{20d}$$

$$y(i) = f(x_i) \qquad (20e)$$

$$i = n, \qquad a_{n,j} = \beta_j (x_n - x_0)^j, \qquad j = 0, m \qquad y(i) = g_1(k) \qquad (20f)$$

Then, the matrix form(9) can be written as follows by using (20a-20f).

$$\text{X}= \begin{bmatrix} a_{1,0} & a_{1,1} & \cdots & \cdots & \cdots & a_{1,m} \\ d_{2,0} & d_{2,1}+c_{2,1} & d_{2,2}+c_{2,2}+b_{2,2} & \cdots & \cdots & d_{2,m}+c_{2,m}+b_{2,m} \\ d_{3,0} & d_{3,1}+c_{3,1} & d_{3,2}+c_{3,2}+b_{3,2} & \cdots & \cdots & d_{3,m}+c_{3,m}+b_{3,m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{n-1,0} & d_{n-1,1}+c_{n-1,1} & d_{n-1,2}+c_{n-1,2}+b_{n-1,2} & \cdots & \cdots & d_{n-1,m}+c_{n-1,m}+b_{n-1,m} \\ a_{n,0} & a_{n,1} & \cdots & \cdots & \cdots & a_{n,m} \end{bmatrix} \qquad (21)$$

$$\text{Y}= \begin{bmatrix} y(1) \\ \cdot \\ \cdot \\ \cdot \\ y(n) \end{bmatrix}$$

Putting (21) in (10), then estimated set of coefficients β_i are obtained by solving matrix system. Therefore, approximate solution (18) is obtained. Same procedure can be used for Eq(2).(See example2)

4. Numerical results

In this section, the method discussed in section 2 and 3 is tested on the following problems from the literature[4,5], and absolute error in the analytical solutions are calculated. All computations were carried out using MATLAB 6.5.

Example 1.

Consider the following convection-diffusion equation with the initial condition

$$\frac{\partial u}{\partial t} + 0.1\frac{\partial u}{\partial x} = 0.02\frac{\partial^2 u}{\partial x^2} \ , \ 0 \le x \le 1, t \ge 0 \ , \qquad (22)$$

$$u(x,0) = e^{1.17712434446770x}, \qquad 0 \le x \le 1 \ , \qquad (23)$$

and boundary conditions

$$u(0,t) = e^{-0.09t}, \qquad t \geq 0 \ , \tag{24}$$

$$u(1,t) = e^{1.17712434446770-0.09t}, \qquad t \geq 0, \tag{25}$$

The exact solution of this problem is $u(t,x) = e^{1.17712434446770x-0.09t}$. The observed maximum absolute errors for various values of k and for a fixed value of n=21 are given in Table 1. The numerical results are illustrated in Figure 1.

Example 2.

Consider the following problem,

$$u_t = u_{xx} + 4\pi^2 sin(2\pi x) \ , \ 0 < x < 1, 0 < t \leq 1 \ , \tag{26}$$

$$u(0,t) = u(1,t) = 1 \ , \ t \geq 0 \ , \tag{27}$$

$$u(x,0) = 1 \ . \tag{28}$$

The exact solution of this problem is $u(t,x) = 1 + (1 - e^{-4\pi^2 t}) sin(2\pi x)$. The observed maximum absolute errors for various values of k and for a fixed value of n=21 are given in Table 2. The numerical results are illustrated in Figure 2.

Table 1: The maximum absolute errors for problem 1

n,m,h	$k = 0.1$	$k = 0.01$	$k = 0.001$	$k = 0.0001$
21,7,1/10	0.0011383	1.1504286e-004	1.1829922e-005	9.0270108e-006

Table 2: The maximum absolute errors for problem 2

n,m,h	$k = 0.1$	$k = 0.01$	$k = 0.001$	$k = 0.0001$
21,9,1/10	0.1825205	0.0623807	0.0069566	6.9589135e-004

5. Conclusions

In this study, we have introduced a new method to solve the partial differential equations. The LPR has been tested on examples and have tabulated the numerical results. We have shown that the method is very fast convergent for solving partial differential equations. The numerical results showed that the present method approximates the exact solution very well. The method can be also extended to the different bandwidth and kernel functions.

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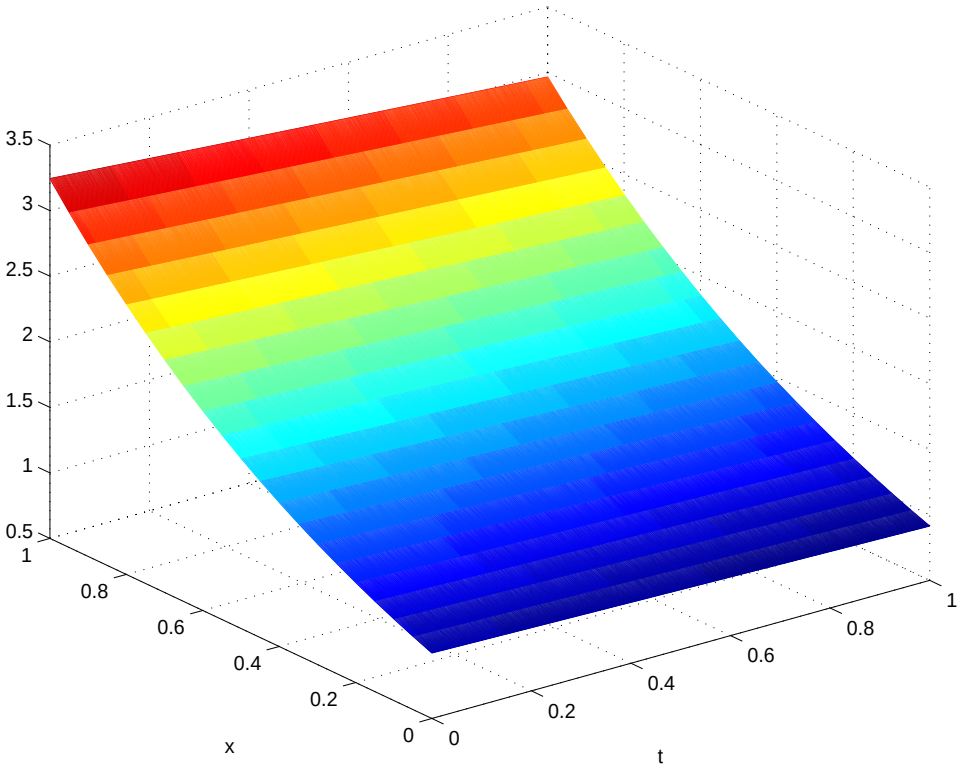


Figure 1: Results for $n = 21, m = 7, k = 0.0001$

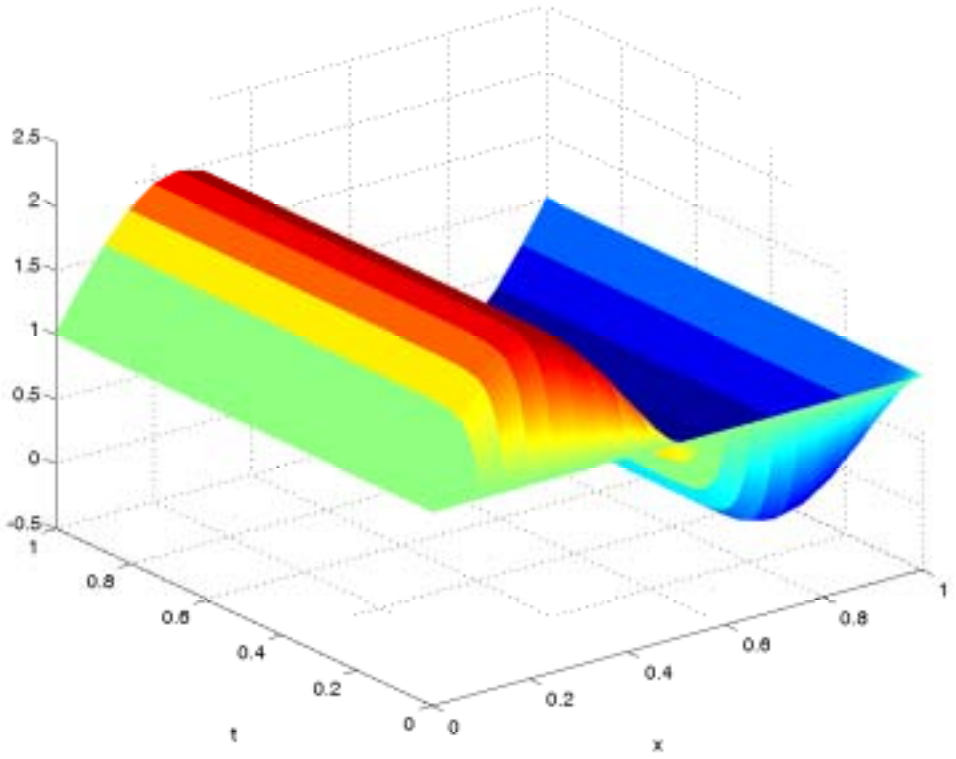


Figure 2: Results for $n = 21, m = 9, k = 0.0001$

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Stabilization of the wave equation with Neumann boundary condition and localized nonlinear damping

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Abstract

We show that the solutions of the wave equation with potential, Neumann boundary conditions and a locally distributed nonlinear damping, decay to zero, with an algebraic rate, that is, the total energy $E(t)$ satisfies for $t \geq 0$: $E(t) \leq C(1+t)^{-\gamma}$, where C is a positive constant depending on E(0) and $\gamma > 0$ is a constant. We assume geometrical conditions as in P. Martinez [7]. In the one/two-dimensional cases, we obtain exponential decay rate when the nonlinear dissipation behaves linearly close to the origin. The same result holds in higher dimension if the dissipative localized term behaves linearly.

Keywords: wave equation, Neumann boundary condition, non-linear damping, localized damping, decay rate.
2000Mathematics Subject Classification: 35B35, 35B40, 35LT0

1 Introduction

We study the stabilization of the total energy for the initial boundary value problem associated to the wave equation with potential, a nonlinear dissipative term and Neumann boundary condition given by

$$\begin{cases} u_{tt} - \Delta u + a(x)q(x)u + \rho(x,u_t) = 0, \ \ x \in \Omega, \ t > 0 \\ u(x,0) = u_0(x) \ , \ \ x \in \Omega \\ u_t(x,0) = u_1(x) \ , \ \ x \in \Omega \\ \frac{\partial u}{\partial \eta}(x,t) = 0 \ , \ \ x \in \partial\Omega, \ t \geq 0 \end{cases} \tag{1}$$

where $\eta = \eta(x)$ denotes the unit exterior normal vector at $x \in \Gamma = \partial\Omega$ and Ω is an open bounded set on $\mathbb{R}^N$ with smooth boundary. The functions u_0 and u_1 are the initial conditions and the dissipation is distributed by the function $\rho = \rho(x,s) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ which is localized on the domain by a continuous function

$$a(x) : \bar{\Omega} \to \mathbb{R}^+ \ , \ a \ \in L^\infty(\Omega)$$

with $a(x) \geq a_0 > 0$ on ω, $\omega \subset \bar{\Omega}$, a neighborhood of part of the boundary of Ω. The function $q(x) : \bar{\Omega} \to \mathbb{R}^+$ is continuous. We assume that $a(x)q(x)$ is not identically zero. Moreover, we consider the following hypotheses on the dissipative function $\rho(x,s)$

I) $\rho(x,s)s \geq 0$, $s \in \mathbb{R}$, $x \in \Omega$;

II) ρ and $\frac{\partial \rho}{\partial s}$ are continuous functions in $\overline{\Omega} \times \mathbb{R}$;

III) There exist constants $K_1, K_2, K_3, K_4 > 0$ and numbers:
$r, p \in \mathbb{R}$, $-1 < r < \infty$ and $-1 < p \leq 2/(N-2)$ if $N \geq 3$ or $-1 < p < \infty$ if $N = 1$ or $N = 2$
such that:

$$K_1\, a(x)\, |s|^{r+1} \ \leq \ |\rho(x,s)| \ \leq \ K_2\, a(x)\, [\, |s|^{r+1} + |s|\,], \quad s \in \mathbb{R}, \quad |s| \leq 1, \quad x \in \Omega$$

$$K_3\, a(x)\, |s|^{p+1} \ \leq \ |\rho(x,s)| \ \leq \ K_4\, a(x)\, [\, |s|^{p+1} + |s|\,], \quad s \in \mathbb{R}, \quad |s| > 1, \quad x \in \Omega$$

IV) $$\frac{\partial \rho}{\partial s}(x,s) \geq 0, \quad s \in \mathbb{R}, \quad x \in \Omega.$$

Regarding the stabilization of the wave equation in bounded domains with localized dissipation and Dirichlet boundary condition we mention Zuazua [13] which studied the semilinear wave equation with a linear damping. Nakao [9] studied the wave equation with a localized nonlinear damping. In [10] Nakao studied attractors for a locally damped wave equation. The stabilization of the incompressible wave equation with localized nonlinear damping is considered in Oliveira-Charão [8]. The stabilization of more general systems with Dirichlet boundary condition appear in [2] which studied the system of elasticity with localized nonlinear damping in bounded domains.

In P. Martinez [7] is studied the stabilization of the energy for the wave equation with localized linear internal damping and homogeneous Neumann boundary condition (the system (1) with $\rho(\cdot, s)$ linear) and obtained exponential decay. In this work we prove the uniform stabilization of the total energy with explicit decay rates to the the problem (1) with the function $\rho(\cdot, s)$ nonlinear. When the localized dissipative term is linear for any dimension or linear near the origin in dimension $n = 1, 2$ we obtain exponential decay of the energy. Therefore, the result by Martinez [7] is included in this work. The method we use are energy identities associated to multipliers adapted to the geometry of the domain that come from control theory and estimates that involve differences of energy in order to use the Nakao's Lemma as in [9]. Thus, the Nakao's method is also effective to problems with Neumann condition. However, instead of the unique continuation principle used in Nakao [9], we impose the same geometrical conditions on the domain as in Martinez [7]. Due to this fact the constants

of stabilization can be estimated explicitly. This fact does not occurs when the stabilization is obtained using the unique continuation principle (see Nakao [9]). In general, problems with Neumann boundary conditions requires a more delicate treatment. Since we can not use the Poincaré inequality for problems without Dirichlet boundary condition on the solutions we have also included, as in [7], a potential term in the system to overcome this difficulty. However, we note that it is only necessary to localize this term in a part of the region where the dissipation is effective.

In [3] Cavalcanti studied the exact boundary controllability for the linear wave equation with time-dependent coefficients and the control action of Neumann type. Also, the boundary stabilization for the free wave equation in a bounded domain with a linear dissipative mixed Dirichlet-Neumamm boundary condition is studied by Phung [11] and polynomial decay rate is obtained. More recently, Laziecka-Toundykov [6] studied the wave equation with nonlinear localized damping, nonlinear source term and mixed Dirichlet-Neumann boundary conditions. In that paper they do not include the case with Neumann boundary condition on the total boundary of the domain. Moreover, they also assume an extra condition that the norms of u_t and ∇u are bounded in L^{p_1} and L^{p_2}, respectively. For hyperbolic linear systems, which include the wave equation, with discontinuous coefficients Gómes-Kapitovov [5] proved the uniform exponential stabilization and exact boundary controllability with Dirichlet and mixed Dirichlet-Neumann dissipative conditions, on two different parts of the boundary with empty intersection.

With respect to the wave equation in an exterior domain with Neumann boundary condition, we cite Aloui [1] which proved in odd dimension under a microlocal geometrical condition, the exponential decay of the local energy for the linear wave equation with Neumann boundary damping. For more general problems in exterior domains or in $\mathbb{R}^n$ we mention Charão-Ikehata [4] and references therein.

2 Geometrical Hypotheses

Let Ω be a bounded domain in $\mathbb{R}^N$. We assume that Ω satisfies the following geometric conditions which appear in Martinez [7].

1. The set Ω is of class C^2.

2. There exists positive constant δ, subdomains $\Omega_j \subset \Omega$, $1 \leq j \leq J$, $J \in \mathbb{N}$, $J > 0$, with Lipchitz boundaries $\partial\Omega_j$, associated functions $\phi_j \in C^2(\Omega_j)$ and constants μ_j such that

$$\begin{aligned} \Delta\phi_j - \mu_j &\geq \delta \text{ in } \Omega_j \\ 2\lambda_{1,j} - \Delta\phi_j + \mu_j &\geq \delta \text{ in } \Omega_j \end{aligned} \tag{2}$$

where $\lambda_{1,j}(x)$ is the smallest eigenvalue of the Jacobian matrix $D^2\phi_j(x)$.

3. For $i \neq j$ the sets Ω_i and Ω_j are disjoint.

Given $1 \leq j \leq J$, we consider the sets,

$$\Gamma_j(\phi_j) = \{x \in \partial\Omega_j : \frac{\partial \phi_j}{\partial \eta_j} \neq 0\},$$

where η_j is the outward normal at $x \in \partial\Omega_j$.

Moreover, given a set $Q \subset \mathbb{R}^N$, we denote $V_\epsilon(Q) = \{x \in \mathbb{R}^n \, : \, d(x, Q) < \epsilon\}$ and assume that there exists $\epsilon > 0$ such that

$$\Omega \cap V_\epsilon[\cup_{j=1}^J \Gamma_j(\phi_j) \cup (\Omega \backslash \cup_{j=1}^J \Omega_j)] \subset \omega \tag{3}$$

where ω is defined in the introduction and it is the set where the dissipation is effective.

Examples of such domains appear in Martinez [7].

3 Main results

The energy of a solution $u(x,t)$ of equation (1) at time t is defined as

$$E(t) = \frac{1}{2}\int_\Omega [|u_t|^2 + |\nabla u|^2 + aqu^2]dx.$$

Lemma 3.1. *The energy functional verifies*

$$E(T) - E(S) = \int_S^T \int_\Omega u_t \rho(x, u_t) dx dt. \tag{4}$$

for $0 \leq S \leq T$.

The proof of this lemma is obtained multiplying the equation (1) by u_t and integrate on $\Omega \times [S, T]$.

So, the energy $E(t)$ is a non–increasing function of t, due to hypothesis (I) on the function $\rho(x, s)$. Because of this, it is possible to show that $E(t)$ decays to zero at an uniform rate.

Also, any solution with initial data $u(\cdot, 0) \in \mathcal{H}_1$ and $u_t(\cdot, 0) \in L^2(\Omega)$ will satisfy $u \in L^\infty(\mathbb{R}_+, \mathcal{H}_1)$ and $u_t \in L^\infty(\mathbb{R}_+, L^2)$, where

$$\mathcal{H}_1 = \{u \in H^1(\Omega) \ / \ \frac{\partial u}{\partial \eta}|_{\partial\Omega} = 0 \ \}.$$

Our main result requires more regularity and some uniform bounds on the solution $u(x,t)$. To this end, we consider the space

$$V = \{v \in H^2(\Omega) \ / \ \frac{\partial v}{\partial \eta}|_{\partial\Omega} = 0\}$$

and, throughout all this work, we assume that the initial data $u(x,0) = u_0(x) \in V$ and $u_t(x,0) = u_1(x) \in \mathcal{H}_1$. Then, using the hypotheses on the function $\rho(x,s)$ and, for example, the Galerkin method or semigroups theory it is possible to obtain the existence of a unique solution $u(x,t)$, satisfying $u \in L^\infty(\mathbb{R}^+, V)$ and $u_t \in L^\infty(\mathbb{R}^+, \mathcal{H}_1)$.

The main result of this paper is given by the following theorem.

Theorem 3.1. *The energy associated with the solution of (1) satisfies*

$$E(t) \leq C(1+t)^{-\lambda_i} \qquad t \geq 0$$

where C is a positive constant independent of t and the decay rate λ_i $(i = 1,2,3$ or $4)$ is given according to the following cases.

Case 1 $r > 0,\ 0 < p \leq \dfrac{2}{N-2},\ N \geq 3$: $\lambda_1 = \min\left\{\dfrac{2}{r}, \dfrac{4(p+1)}{p(N-2)}\right\}$.

If $p = 0$ then $\lambda_1 = \dfrac{2}{r}$. If $r = 0$ then $\lambda_1 = \dfrac{4(p+1)}{4(N-2)}$.

When $N = 1,2$ the decay rate is $\lambda_1 = \dfrac{2}{r}$ for $r > 0$ and $0 \leq p < +\infty$.

The decay rate is exponential if $r = p = 0$ and $N \geq 1$ or $r = 0$, $0 < p < +\infty$ and $N = 1,2$.

Case 2 $-1 < r < 0,\ 0 < p \leq \dfrac{2}{N-2},\ N \geq 3$: $\lambda_2 = \min\left\{\dfrac{-2(r+1)}{r}, \dfrac{4(p+1)}{p(N-2)}\right)$

and $\lambda_2 = -\dfrac{2(r+1)}{r}$ for $N \geq 3$ and $p = 0$ or $N = 1,2$ and $0 \leq p < +\infty$.

Case 3 $r > 0,\ -1 < p < 0,\ N \geq 3$: $\lambda_3 = \min\left\{\dfrac{2}{r}, \dfrac{4}{p(2-N)}\right\}$.

If $N = 1,2$ and $r > 0$ then $\lambda_3 = \dfrac{2}{r}$.

If $r = 0$ and $N = 1,2$ the decay rate is exponential.

If $r = 0$ and $N \geq 3$ the decay rate is $\lambda_3 = \dfrac{4}{p(2-N)}$.

Case 4 $-1 < r < 0,\ -1 < p < 0,\ N \geq 3$: $\lambda_4 = \min\left\{\dfrac{-2(r+1)}{r}, \dfrac{4}{p(2-N)}\right\}$

and $\lambda_4 = \dfrac{-2(r+1)}{r}$ *for $N = 1,2$.*

Before giving the proof, we need to construct several estimates.

4 Preliminary estimates

In this section we present some identities and estimates for the energy $E(t)$ which are useful to obtain the decay rates in the previous theorem.

We need the Mini-Max principle.

Lemma 4.1 (Mini-Max Principle). *The solution $u = u(x,t)$ of the problem (1) satisfies*

$$\int_\Omega |u|^2 dx \le \int_\Omega [\ |\nabla u|^2 + aqu^2] dx \le CE(t),\ t \ge 0$$

for some positive constant C which depends only on Ω and the function aq.

Proof. Since $a(x)q(x) \not\equiv 0$ is a continuous and nonnegative function on $\overline{\Omega}$, the elliptic problem

$$\begin{cases} -\triangle w + a(x)q(x)w = \lambda w \ \text{ in } \ \Omega\ , \\ \\ \dfrac{\partial w}{\partial \eta} = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

has a strictly positive first eigenvalue λ_0. By the mini-max principle we have that

$$\lambda_0 \le \frac{\displaystyle\int_\Omega [\ |\nabla w|^2 + aqw^2] dx}{\displaystyle\int_\Omega |w|^2 dx}$$

for all $w \in H^1(\Omega)$ such that $w \not\equiv 0$ and $\dfrac{\partial w}{\partial \eta} = 0$ on $\partial\Omega$. This implies the statement of the lemma. □

The following energy identity is similar to that one in Martinez [7] for $\rho(x,s) = a(x)s$.

Lemma 4.2. *Let $\mathcal{O} \subseteq \Omega$ be an open Lipschitz domain and $h : \mathcal{O} \to \mathbb{R}^N$ a C^1 function. Given $0 \le S < T < \infty$, the solution u of the problem (1) satisfies*

$$\begin{aligned} \int_S^T \int_{\partial\mathcal{O}} (2\frac{\partial u}{\partial \eta} h \cdot \nabla u \ &+ \ h \cdot \eta(u_t^2 - |\nabla u|^2))\ d\Gamma\ dt \\ &= \ (\int_\mathcal{O} 2u_t h \cdot \nabla u\ dx)\Big|_S^T + \int_S^T \int_\mathcal{O} 2(\rho(x,u_t) + a\ q\ u) h \cdot \nabla u\ dx\ dt \\ &+ \ \int_S^T \int_\mathcal{O} (\ div\ (h)(u_t^2 - |\nabla u|^2) + 2\sum_{i,k} D_i h_k D_i u D_k u)\ dx\ dt\ , \end{aligned}$$

where h_k are the components of the function h, $D_i = \frac{\partial}{\partial x_i}$ and $\eta = \eta(x)$ is the outward unit normal at $x \in \partial\mathcal{O}$.

Proof. The proof of this lemma follows by using the standard multiplier $M(u) \equiv h \cdot \nabla u$ and integrating on $\mathcal{O} \times [S,T]$. □

As in Martinez [7], we now introduce the sets

$$Q_i \equiv V_{\epsilon_i}[\cup_{j=1}^{n}\Gamma_j(\phi_j) \cup (\Omega \setminus \cup_{j=1}^{n}\Omega_j)], \tag{5}$$

for arbitrary fixed numbers: $0 \leq \epsilon_0 < \epsilon_1 < \epsilon_2 < \epsilon$ and $i = 0, 1, 2$. The number $\epsilon > 0$ is defined in (3) by the geometric conditions which localize the dissipative term $\rho(x, u_t)$ (see (3)).

The following energy estimate is fundamental to obtain the proof of Theorem 3.1.

Lemma 4.3. *Let $0 \leq S < T < \infty$. Let u be the solution of problem (1). Then, the associated energy $E(t)$ satisfies*

$$\begin{aligned} 2\delta \int_S^T E(t)\, dt \quad &\leq \quad -\left[\int_\Omega u_t M(u) dx\right]_S^T - \int_S^T \int_\Omega (\rho + a\ q\ u) M(u)\ dxdt \\ &+ \quad C\int_S^T \int_{\Omega\cap Q_1} (u_t + |\nabla u|^2 + u^2)\ dxdt + \delta \int_S^T \int_\Omega aqu^2\ dxdt \end{aligned} \tag{6}$$

Proof. The proof is similar as in Martinez [7] and it uses the geometrical hypotheses in Section 2, the Lemma 4.2 with $\mathcal{O} = \Omega_j$ and $h = \psi_j \nabla\phi_j$ to obtain the identity

$$\begin{aligned} &\int_S^T \int_{\partial\Omega_j} [2\frac{\partial u}{\partial \eta_j} \psi_j \nabla\phi_j \cdot \nabla u + \psi_j \nabla\phi_j \cdot \eta_j (u_t^2 - |\nabla u|^2)] d\Gamma dt \\ = \quad &\int_{\Omega_j} 2u_t \psi_j \nabla\phi_j \cdot \nabla u dx \Big|_S^T + 2\int_S^T \int_{\Omega_j} (\rho(x, u_t) + a\ q\ u)\psi_j \nabla\phi_j \cdot \nabla u\ dxdt \\ + \quad &\int_S^T \int_{\Omega_j} [\text{div}\ (\psi_j \nabla\phi_j)(u_t^2 - |\nabla u|^2)\ + 2\sum_{i,k} D_i(\psi_j \nabla\phi_j)_k D_i u D_k u\] d\Gamma dt \end{aligned}$$

where $(\psi_j \nabla\phi_j)_k$ is the k–th component of $\psi_j \nabla\phi_j$.

Moreover, to obtain the estimate (4.3) it is also necessary to use the Lemma 4.1 and the identity

$$\begin{aligned} -\int_S^T \int_\Omega m\ u(\rho + a\ q\ u)\ dxdt &= (\ \int_\Omega m\ u\ u_t\ dx\)\Big|_S^T \\ + \int_S^T \int_\Omega m(|\nabla u|^2 - u_t^2)\ dxdt + \int_S^T \int_\Omega u\nabla m \cdot \nabla u\ dxdt, & \end{aligned}$$

which is obtained by multiplying the equation in (1) by $m(x)u$ and integrate over $(S, T) \times \Omega$, where $m : \overline{\Omega} \to \mathbb{R}$ be a C^2–function such that $m(x) = \mu_j$ in Ω_j (constant on each Ω_j) for $1 \leq j \leq J$. □

Next we estimate each one of the terms in the right hand side of the inequality (6) which appears in Lemma 4.3.

Lemma 4.4. *Let $0 \leq S < T < \infty$. The solution u of (1) satisfies*

$$\left| \left[\int_\Omega u_t M(u) dx \right]_S^T \right| \leq C[E(T) + E(S)]$$

for some positive constant C.

Proof. It follows from standard calculations and the definition of $M(u) = 2h \cdot \nabla u + mu$. □

Lemma 4.5. *There exists a positive constant C such that*

$$\left| \int_S^T \int_\Omega aquM(u) dx \right| \leq \frac{\delta}{2} \int_S^T E(t) dt + \frac{C}{\delta} \int_S^T \int_\Omega aqu^2 \, dxdt$$

with $C > 0$ a constant and $\delta > 0$ appears in the estimate of Lemma 4.3 and the geometrical hypotheses (2).

Proof. Using the definition of $M(u)$, we have that for $L > 0$,

$$\left| \int_S^T \int_\Omega aquM(u) dxdt \right| \leq \frac{\delta}{2L} \int_S^T \int_\Omega M(u)^2 \, dxdt + \frac{L}{2\delta} \int_S^T \int_\Omega (aqu)^2 \, dxdt$$

$$\leq \frac{C\delta}{2L} \int_S^T \int_\Omega [|\nabla u|^2 + |u|^2] dxdt + \frac{C}{\delta} \int_S^T \int_\Omega aqu^2 \, dxdt$$

$$\leq \frac{C\delta}{L} \int_S^T E(t) dt + \frac{C}{\delta} \int_S^T \int_\Omega aqu^2 \, dxdt$$

where we have used the Lemma 4.1 and the fact that $aq = a(x)q(x) \in L^\infty(\Omega)$.

By choosing $L = 2C$ the lemma follows. □

Next, we want to estimate the integral

$$\int_S^T \int_\Omega \rho M(u) dxdt$$

according to the cases of Theorem 3.1.

We note that

$$\left| \int_S^T \int_\Omega \rho M(u) \, dxdt \right| \leq C \int_S^T \int_\Omega |\rho| \, [|\nabla u| + |u|] dxdt = C(I_1 + I_2) \qquad (7)$$

where

$$I_1 = \int_S^T \int_{\Omega_1^*} |\rho| \, [|\nabla u| + |u|] \, dxdt \quad \text{and} \quad I_2 = \int_S^T \int_{\Omega_2^*} |\rho| \, [|\nabla u| + |u|] dxdt.$$

Here we have denoted

$$\Omega_1^* = \{x \in \Omega / |u_t| \leq 1\} \quad \text{and} \quad \Omega_2^* = \Omega \backslash \Omega_1^* \, .$$

Then, for $R > 0$, using Lemma 4.1 we have

$$|I_1| \leq \frac{R}{2\delta} \int_S^T \int_{\Omega_1^*} \rho^2 dxdt + \frac{\delta C^2}{R} \int_S^T \int_\Omega [|\nabla u|^2 + |u|^2] dxdt$$

$$\leq \frac{R}{2\delta} \int_S^T \int_{\Omega_1^*} \rho^2 dxdt + \delta \frac{C_1}{R} \int_S^T E(t)dt,$$

with C_1 a positive constant.

Now, we choose $R = 2C_1$ to obtain that

$$|I_1| \leq C \int_S^T \int_{\Omega_1^*} \rho^2 dxdt + \delta/2 \int_S^T E(t)dt \tag{8}$$

for $0 < S < T < +\infty$.

Lemma 4.6. *Let $0 \leq S < T < \infty$.*

i) If $0 \leq r < +\infty$ then

$$|I_1| \leq C \left[E(S) - E(T)\right]^{\frac{2}{r+2}} + \delta/2 \int_S^T E(t)dt$$

ii) If $-1 < r < 0$ then

$$|I_1| \leq C \left[E(S) - E(T)\right]^{\frac{2r+2}{r+2}} + \delta/2 \int_S^T E(t)dt$$

with C a positive constant which depends on $|T - S|$, $\|a\|_\infty$ and $|\Omega|$.

Proof. i) Since $r \geq 0$, $|u_t| \leq 1$ in Ω_1^*, the hypothesis (III) on $\rho(\cdot, s)$and the fact that $a = a(x) \in L^\infty(\Omega)$ imply that

$$\int_S^T \int_{\Omega_1^*} \rho^2 dxdt \leq \int_S^T \int_{\Omega_1^*} Ca(x)[|u_t|^{2r+2} + |u_t|^2] \, dxdt$$

$$\leq C \int_S^T \int_{\Omega_1^*} a(x)|u_t|^2 dxdt \leq C \left[\int_S^T \int_{\Omega_1^*} a(x) \, |u_t|^{r+2}\right]^{\frac{2}{r+2}}$$

$$\leq C \left[\int_S^T \int_\Omega \rho(x, u_t) u_t \, dxdt\right]^{\frac{2}{r+2}} \leq C \left[E(S) - E(T)\right]^{\frac{2}{r+2}}$$

where we have used the energy identity (4) in Lemma 3.1 and Hölder's inequality.

ii) For $-1 < r < 0$, using the hypothesis (III) on $\rho(\cdot, s)$, we also have

$$\int_S^T \int_{\Omega_1^*} \rho^2 dxdt \leq C \int_S^T \int_{\Omega_1^*} a(x)[|u_t|^{2r+2} + |u_t|^2]dxdt$$

$$\leq C \int_S^T \int_{\Omega_1^*} a(x)|u_t|^{2r+2} dxdt \leq C \left[\int_S^T \int_{\Omega_1^*} a(x)|u_t|^{r+2} dxdt\right]^{\frac{2r+2}{r+2}} |T-S|^{\frac{-r}{r+2}}$$

$$\leq C \left[\int_S^T \int_{\Omega} \rho(x,u_t)u_t dxdt\right]^{\frac{2r+2}{r+2}} \leq C[E(S) - E(T)]^{\frac{2r+2}{r+2}}$$

Combining the above estimates with (8) the lemma follows.

□

Next we also estimate

$$I_2 = \int_S^T \int_{\Omega_2^*} |\rho|[|\nabla u| + |u|]dxdt$$

in terms of energy differences.

Lemma 4.7. *Let* $0 \leq S < T < \infty$.

i) If $0 \leq p \leq \dfrac{2}{N-2}$, $N \geq 3$ *then :* $I_2 \leq C[E(S)-E(T)]^{\frac{p+1}{p+2}} E(S)^{\frac{4+p(2-N)}{4(p+2)}}$.

ii) If $-1 < p < 0$ *and* $N \geq 3$ *then :* $I_2 \leq C\,[E(S)-E(T)]^{\frac{2}{4+p(2-N)}} \sqrt{E(S)}$.

iii) If $0 \leq p < +\infty$ *and* $N = 1$ *or 2 then :* $I_2 \leq C[E(S)-E(T)]^{\frac{p+1}{p+2}} E(S)^{\frac{1}{p+2}}$.

iv) If $-1 < p < 0$ *and* $N = 1$ *or 2 then :* $I_2 \leq C[E(S) - E(T)]^{1/2} \sqrt{E(S)}$.

The constant C is positive and depends on $\|a\|_\infty, |T-S|$ and the initial data.

Proof. (i) Since $p \geq 0$ and $|u_t| \geq 1$ in Ω_2^*, using the hypothesis (III) on $\rho(\cdot, s)$ we have

$$I_2 \leq C \int_S^T \int_{\Omega_2^*} a(x)|u_t|^{p+1}[|\nabla u| + |u|]dxdt$$

$$\leq C \left[\int_S^T \int_{\Omega} \rho(x,u_t)u_t dxdt\right]^{\frac{p+1}{p+2}} \left[\int_S^T \int_{\Omega} (|\nabla u|^{p+2} + |u|^{p+2})dxdt\right]^{\frac{1}{p+2}}$$

$$\leq C[E(S) - E(T)]^{\frac{p+1}{p+2}} \left[\int_S^T \int_{\Omega} |\nabla u|^{p+2} + |u|^{p+2} dxdt\right]^{\frac{1}{p+2}}$$

At this point, we use the Gagliardo-Niremberg Inequality (see for example Nakao [9]) to obtain

$$\|\nabla u\|_{L^{p+2}(\Omega)} \leq C\|\nabla u\|^{\theta}_{H^1(\Omega)} \|\nabla u\|^{1-\theta}_{L^2(\Omega)} \leq C\|u\|^{\theta}_{H^2(\Omega)} \|\nabla u\|^{1-\theta}_{L^2(\Omega)} \tag{9}$$

$$\leq C\|\nabla u\|^{1-\theta}_{L^2(\Omega)} \leq CE^{\frac{1-\theta}{2}}(t)$$

due to the fact that $u \in L^{\infty}(\mathbb{R}^+, H^2(\Omega))$ (Existence theorem), where $\theta = \dfrac{Np}{2(p+2)}$ is such that $0 < \theta < 1$ because, for $N \geq 3$, we have that

$$p \leq \frac{2}{N-2} < \frac{4}{N-2}.$$

Now, by the Sobolev embedding's theorem, for $N \geq 3$, we have

$$\|u\|_{L^{p+2}(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \tag{10}$$

because $0 \leq p \leq \dfrac{2}{N-2}$.

Substituting the estimates (9) and (10) in the last estimate for I_2, we conclude that

$$I_2 \leq C[E(S) - E(T)]^{\frac{p+1}{p+2}}[E(S)^{\frac{1-\theta}{2}} + E(S)^{1/2}]$$

$$\leq C[E(S) - E(T)]^{\frac{p+1}{p+2}} E(S)^{\frac{1-\theta}{2}} = C[E(S) - E(T)]^{\frac{p+1}{p+2}} E(S)^{\frac{4+p(2-N)}{4(p+2)}}.$$

because the function $E(t)$ is bounded and nonincreasing in t. The constant $C > 0$ depends on the initial data and on $|T - S|$.

ii) In this case $(-1 < p < 0)$ using the hypothesis (III) on $\rho(x, s)$ we have

$$I_2 \leq C \int_S^T \int_{\Omega_2^*} a(x)\,|u_t|[|\nabla u| + |u|]dxdt$$

$$\leq C \left(\int_S^T \int_{\Omega_2^*} a(x)|u_t|^2 dxdt \right)^{1/2} \left(\int_S^T \int_{\Omega} (|\nabla u|^2 + |u|^2)dxdt \right)^{1/2}$$

$$\leq C \left(\int_S^T \int_{\Omega_2^*} a(x)|u_t|^2\, dxdt \right)^{1/2} |T - S|^{1/2} \sqrt{E(S)} \tag{11}$$

due to Lemma 4.1 and the fact that $E(t)$ is a nonincreasing function.

Using Hölder's inequality we obtain

$$\left(\int_S^T \int_{\Omega_2^*} a(x)|u_t|^2 dxdt\right)^{1/2}$$

$$\leq C\left(\int_S^T \int_{\Omega_2^*} a(x)|u_t|^{p+2} dxdt\right)^{\frac{2}{4+p(2-N)}} \left(\int_S^T \int_{\Omega_2^*} |u_t|^{\frac{2N}{N-2}} dxdt\right)^{\frac{p(2-N)}{8+2p(2-N)}} \tag{12}$$

$$\leq C\left(\int_S^T \int_{\Omega} \rho(x,u_t)u_t dxds\right)^{\frac{2}{4+p(2-N)}}$$

because u_t belongs to $L^\infty(\mathbb{R}^+, H^1(\Omega))$ which is continuously embedded in $L^\infty(\mathbb{R}^+, L^{\frac{2N}{N-2}}(\Omega))$ for $N \geq 3$. The positive constant C in (12) depends on the initial data and $|T-S|$.

Now, using the energy identity (4) in (12) and combining with (11) we obtain the estimate (ii).

iii) From the proof of item (i), we have that

$$I_2 \leq C[E(S)-E(T)]^{\frac{p+1}{p+2}} \left[\int_S^T \int_{\Omega} (|\nabla u|^{p+2} + |u|^{p+2}) dxdt\right]^{\frac{1}{p+2}}.$$

Using the Sobolev Imbedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$, $q \geq 1$, it follows that

$$I_2 \leq C[E(S)-E(T)]^{\frac{p+1}{p+2}} \left[\int_S^T \int_{\Omega} |\nabla u|^{p+2} dxdt\right]^{\frac{1}{p+2}}.$$

Now, applying Gagliardo-Niremberg Lemma with $\theta = \dfrac{p}{p+2}$ we obtain

$$\|\nabla u\|_{L^{p+2}(\Omega)} \leq \|\nabla u\|^{\theta}_{H^1(\Omega)} \|\nabla u\|^{1-\theta}_{L^2(\Omega} \leq C\|u\|^{\theta}_{H^2(\Omega)} E(t)^{\frac{1-\theta}{2}}.$$

This proves item (iii) since that $\dfrac{1-\theta}{2} = \dfrac{1}{p+2}$ and $u \in L^\infty(\mathbb{R}^+, H^2(\Omega))$.

(iv) From the proof of item (ii) we have

$$I_2 \leq C\left(\int_S^T \int_{\Omega_2^*} a(x)|u_t|^2 dxdt\right)^{1/2} |T-S|^{1/2}\sqrt{E(S)}$$

$$\leq C\left(\int_S^T \int_{\Omega_2^*} a(x)\,|u_t|^{2+p} dxdt\right)^{1/2} \sqrt{E(S)}$$

because in this case p is negative and u_t belongs to $L^\infty(\mathbb{R}^+, H^1(\Omega))$ which is contained in $L^\infty(\mathbb{R}^+ \times \Omega)$ for $N = 1, 2$. Then Item (iv) follows from the hypotheses (III) on $\rho(\cdot, s)$ and the energy identity (4) of Lemma (3.1).

The Lemma 4.7 is proved. □

Next, combining the estimates for I_1 and I_2, in Lemmas 4.6 and 4.7 with estimate (7) we obtain the following

Lemma 4.8. *Let $0 \leq S < T < +\infty$. The solution u satisfies*

$$\left| \int_S^T \int_\Omega \rho M(u) dx dt \right| \leq C D_i^N + \frac{\delta}{2} \int_S^T E(t) dt \tag{13}$$

where, according the cases $i = 1, 2, 3, 4$ we have

Case 1

$$D_1^N = (\triangle E)^{\frac{2}{r+2}} + (\triangle E)^{\frac{p+1}{p+2}} E(S)^{\frac{4+p(2-N)}{4(p+2)}}$$

for $0 \leq r < +\infty$, $0 \leq p \leq \frac{2}{N-2}$, $N \geq 3$.

Case 2

$$D_2^N = (\triangle E)^{\frac{2r+2}{r+2}} + (\triangle E)^{\frac{p+1}{p+2}} E(S)^{\frac{4+p(2-N)}{4(p+2)}}$$

for $-1 < r < 0$ and $0 \leq p \leq \frac{2}{N-2}$, $N \geq 3$.

Case 3

$$D_3^N = (\triangle E)^{\frac{2}{r+2}} + (\triangle E)^{\frac{2}{4+p(2-N)}} \sqrt{E(S)}$$

for $0 \leq r < +\infty$ and $-1 < p < 0$, $N \geq 3$.

Case 4

$$D_4^N = (\triangle E)^{\frac{2r+2}{r+2}} + (\triangle E)^{\frac{2}{4+p(2-N)}} \sqrt{E(S)}$$

for $-1 < r < 0$, $-1 < p < 0$ and $N \geq 3$.

Case 1a

$$D_1^N = (\triangle E)^{\frac{2}{r+2}} + (\triangle E)^{\frac{p+1}{p+2}} E(S)^{\frac{1}{p+2}}$$

for $N = 1, 2$.

Case 2a

$$D_2^N = (\triangle E)^{\frac{2r+2}{r+2}} + (\triangle E)^{\frac{p+1}{p+2}} E(S)^{\frac{1}{p+2}}$$

for $N = 1, 2$.

Case 3a

$$D_3^N = (\triangle E)^{\frac{2}{r+2}} + (\triangle E)^{1/2} E(S)^{1/2}$$

for$N = 1, 2$.

Case 4a

$$D_4^N = (\triangle E)^{\frac{2r+2}{r+2}} + (\triangle E)^{1/2} E(S)^{1/2}$$

for$N = 1, 2$.

The constant C in estimate (13) depends on $\|a\|_\infty$, $|T-S|, |\Omega|$ and the initial data.

Also, we have used the notation

$$\triangle E = E(S) - E(T).$$

Now, substituting the estimates of Lemmas 4.4, 4.5 and 4.8 in the estimate of Lemma 4.3, we obtain the following main estimate

$$\delta \int_S^T E(t)dt \leq C \left\{ [E(T) + E(S)] + D_i^N + \int_S^T \int_\Omega aqu^2 dxdt \right.$$
$$\left. + \int_S^T \int_{\Omega \cap Q_1} (u_t^2 + u^2 + |\nabla u|^2) dxdt \right\} \tag{14}$$

with C a positive constant depending on $\|a\|_\infty$, $|T-S|$, $|\Omega|$ and the initial data.

Next we estimate $\displaystyle\int_S^T \int_{\Omega \cap Q_1} |\nabla u|^2 dxdt$. To do this we take a function $\xi : \mathbb{R}^n \to \mathbb{R}$ such that

$$0 \leq \xi \leq 1 \,, \qquad \xi = 1 \ \text{ in } \ Q_1 \,, \qquad \xi = 0 \ \text{ in } \ \mathbb{R}^n / Q_2$$

where the sets Q_1 and Q_2 are defined in (5).

By multiplying the equation in (1) by ξu and integrating, we obtain

$$\int_S^T \int_\Omega (-\xi u)(\rho + aqu) dxdt = \int_S^T \int_\Omega \xi u (u_{tt} - \triangle u) dxdt =$$

$$\left[\int_\Omega \xi u u_t dx \right]_S^T + \int_S^T \int_\Omega \xi [|\nabla u|^2 - u_t^2] dxdt = \frac{1}{2} \int_S^T \int_\Omega u^2 \triangle \xi dxdt$$

where u is the solution of (1).

Then, by the definition of $\xi = \xi(x)$ we have

$$\int_S^T \int_{\Omega \cap Q_1} |\nabla u|^2 dxdt \leq$$

$$\leq \int_S^T \int_\Omega \left[\xi u_t^2 + \frac{1}{2} u^2 \triangle \xi \right] dxdt - \left[\int_\Omega \xi u u_t dx \right]_S^T - \int_S^T \int_\Omega \xi u [\rho + aqu] dxdt \tag{15}$$

$$\leq C \int_S^T \int_{\Omega \cap Q_2} [u_t^2 + u^2] dxdt + C[E(T) + E(S)] + \int_S^T \int_\Omega |u| [|\rho| + aq|u|] dxdt$$

That is,

$$\int_S^T \int_{\Omega \cap Q_1} |\nabla u|^2 dxdt \leq C \int_S^T \int_{\Omega \cap Q_2} [u_t^2 + u^2] dxdt + C[E(T) + E(S)] +$$

$$+ \int_S^T \int_\Omega |u||\rho| dxdt + \int_S^T \int_\Omega aqu^2 dxdt. \tag{16}$$

Here we note that similar estimates as in Lemma 4.8 hold for the integral

$$\int_S^T \int_\Omega |u||\rho| dxdt. \tag{17}$$

Therefore, substituting (16) in (14) and using estimates as in Lemma 4.8 for (17), we obtain that

$$\frac{\delta}{2}\int_S^T E(t)dt \leq C\left\{E(T)+E(S)+\int_S^T\int_\Omega aqu^2dxdt + D_i^N \right. \\ \left. +\int_S^T\int_{\Omega\cap Q_2}[u_t^2+u^2]dxdt\right\} \tag{18}$$

with C a positive constant depending on $\|a\|_\infty$, $|T-S|$, the initial data and $|\Omega|$. The functions $D_i^N = D_i^N(t)$ are given in Lemma 4.8.

Now,we note that

$$\int_S^T\int_{\Omega\cap Q_2}[u_t^2+u^2]dxdt \leq \frac{1}{a_0}\int_S^T\int_\Omega a(x)[u_t^2+u^2]dxdt$$

because $\Omega\cap Q_2 \subset w$ and $a=a(x)\geq a_0$ in w by hypothesis.

Due to this and the fact that $q\in C(\overline{\Omega})$ it results

$$\delta\int_S^T E(t)dt \leq C\left\{E(T)+E(S)+D_i^N+\int_S^T\int_\Omega a(x)[u_t^2+u^2]dxdt\right\}. \tag{19}$$

Next, we need estimate the integral,

$$\int_S^T\int_\Omega au^2dxdt.$$

To do this, we consider the solution z of the elliptic problem

$$\begin{cases} -\triangle z + aqz = au & \text{in} \quad \Omega \\ \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{20}$$

We multiply the equation in (20) by z to obtain

$$\int_\Omega(|\nabla z|^2+aqz^2)dx = \int_\Omega auzdx. \tag{21}$$

By hypothesis, we have that $g(x)=a(x)q(x)$ is non negative and $g\not\equiv 0$. Then, using the Mini-Max principle, there exists $C>0$ such that

$$\|z\|^2 \leq C\int_\Omega(|\nabla z|^2+aqz^2)dx. \tag{22}$$

Combining (21) and (22) and using the Cauchy-Schwartz inequality it follows that

$$\int_\Omega z^2 dx = \|z\|^2 \leq C \int_\Omega a(x) u^2. \tag{23}$$

Similarly, taking time derivative in (20) we obtain

$$\int_\Omega |z_t|^2 \leq C \int_\Omega a(x) u_t^2. \tag{24}$$

On the other hand, multiplying the equation in (1) by z, integrating and using Green's formula, it results

$$\begin{aligned}
0 &= \int_S^T \int_\Omega z(u_{tt} - \triangle u + \rho(x,u_t) + aqu) dxdt = \left[\int_\Omega z u_t dx\right]_S^T \\
&- \int_S^T \int_\Omega z_t u_t dxdt - \int_S^T \int_\Omega u \triangle z dxdt + \int_S^T \int_\Omega z[\rho(x,u_t) + aqu] dxdt \\
&= \left[\int_\Omega z u_t dx\right]_S^T - \int_S^T \int_\Omega z_t u_t dxdt - \int_S^T \int_\Omega u[aqz - au] dxdt \\
&+ \int_S^T \int_\Omega z[\rho(x,u_t) + aqu] dxdt
\end{aligned}$$

with the last equality due to (20).

Hence,

$$\int_S^T \int_\Omega a(x) u^2 dxdt = -\left[\int_\Omega z u_t dx\right]_S^T + \int_S^T \int_\Omega z_t u_t dxdt - \int_S^T \int_\Omega z\rho dxdt.$$

Using (23) and Cauchy-Schwartz, we obtain

$$\int_S^T \int_\Omega a(x) u^2 dxdt \leq C[E(T) + E(S)] + \int_S^T \int_\Omega [z_t u_t - z\rho] dxdt. \tag{25}$$

Also, using (24) we have

$$\begin{aligned}
\left|\int_S^T \int_\Omega z_t u_t dxdt\right| &\leq \int_S^T \|z_t\| \, \|u_t\| dt \qquad (26) \\
\leq C \int_S^T \left(\int_\Omega a(x) u_t^2 dx\right)^{1/2} \|u_t\| dt &\leq \frac{C}{2\eta} \int_S^T \int_\Omega a(x) u_t^2 dxdt + \eta \int_S^T E(t) dt
\end{aligned}$$

with $\eta > 0$ to be chosen later.

Finally, we make

$$\left|\int_S^T \int_\Omega z\rho dxdt\right| \leq \int_S^T \int_{\Omega_1^*} |z|\ |\rho| dxdt + \int_S^T \int_{\Omega_2^*} |z|\ |\rho| dxdt \tag{27}$$

where Ω_1^* and Ω_2^* are defined in the proof of Lemma 4.5.

Then, using (23) we estimate

$$\begin{aligned}\int_S^T \int_{\Omega_1^*} |z|\rho dxdt &\leq \left(\int_S^T \int_\Omega au^2 dxdt\right)^{1/2} \left(\int_S^T \int_{\Omega_1^*} \rho^2\right)^{1/2} \\ &\leq \lambda \int_S^T \int_\Omega au^2 dxdt + \frac{1}{4\lambda} \int_S^T \int_{\Omega_1^*} \rho^2 dxdt \end{aligned} \tag{28}$$

with $\lambda > 0$ to be chosen.

Combining (26), (27) and (28) with (25), we obtain

$$\int_S^T \int_\Omega au^2 dxdt \leq C[E(T) + E(S)] + \frac{C}{2\eta} \int_S^T \int_\Omega au_t^2 dxdt$$

$$+\eta \int_S^T E(t)dt + \lambda \int_S^T \int_\Omega au^2 dxdt + \frac{1}{4\lambda} \int_S^T \int_{\Omega_1^*} \rho^2 dxdt + 2\int_S^T \int_{\Omega_2^*} |z|\ |\rho| dxdt. \tag{29}$$

To estimate the last integral in (29), we use the fact that $au \in L^\infty(\mathbb{R}^+, L^2(\Omega))$ and Elliptic Regularity in (20) to have that $z \in H^2(\Omega)$ and $\|z\|_{H^2(\Omega)} \leq C\|u\|_{H^2(\Omega)}$ ($C > 0$ constant).

Also, from (21) and (23) it is eazy to see that

$$\|z\|_{H^1(\Omega)} \leq C\|u\| \tag{30}$$

with $C > 0$ a constant depending on $\|a\|_\infty$.

Using (30) and Sobolev imbedding,

$$H^1(\Omega) \subset L^r(\Omega)$$

for $2 \leq r \leq \dfrac{2N}{N-2}$, $N \geq 3$, it follows that

$$\|z\|_{L^{p+2}(\Omega)} \leq C\|u\| \leq CE(t)^{1/2}, \quad t \geq 0, \tag{31}$$

where we have used the Lemma 4.1.

Finally, by using that $|u_t| \geq 1$ in Ω_2^*, we estimate

$$J \equiv 2\int_S^T \int_{\Omega_2^*} |z|\ |\rho| dxdt \leq C\int_S^T \int_{\Omega_2^*} |z|a(x)|u_t|^{p+1} dxdt$$
$$\leq C\left(\int_S^T \int_{\Omega_2^*} a(x)|u_t|^{p+2}\right)^{\frac{p+1}{p+2}} \left(\int_S^T \int_\Omega |z|^{p+2} dxdt\right)^{\frac{1}{p+2}}$$
$$\leq C\left(\int_S^T \int_\Omega \rho(x,u_t)u_t dxdt\right)^{\frac{p+1}{p+2}} E(t)^{1/2} \tag{32}$$
$$\leq C(\triangle E)^{\frac{p+1}{p+2}} E(S)^{1/2}|T-S|^{\frac{1}{p+2}}$$
$$\leq C(\triangle E)^{\frac{p+1}{p+2}} \sqrt{E(S)} \leq C(\triangle E)^{\frac{p+1}{p+2}} E(S)^{\frac{4+p(2-N)}{4(p+2)}}$$

where we have used that $N \geq 3$, the estimate (31) with the fact that $E(t)$ is nonincreasing. This holds for $0 \leq p \leq \dfrac{2}{N-2}$.

Now, due to the fact that $H^1(\Omega) \subset L^\infty(\Omega) \subset L^r(\Omega)$, for $r \geq 1$ and $N = 1, 2$, the estimate

$$J \leq C(\triangle E)^{\frac{p+1}{p+2}} \sqrt{E(S)}$$

holds for $N = 1, 2$.

Then

$$J \leq C(\triangle E)^{\frac{p+1}{p+2}} E(S)^{\frac{1}{2+p}} \tag{33}$$

holds for $0 \leq p < \infty$ and $N = 1, 2$.

¿From (32) and (33), we conclude that the estimates for J are the same as in Lemma 4.11 for I_2 in the cases (i) and (iii) for $p \geq 0$.

It is easy to control J with the same estimates for I_2, in the cases (ii) and (iv) for $-1 < p < 0$.

That is,

$$J \leq C(\triangle E)^{\frac{2}{4+p(2-N)}} \sqrt{E(S)} \tag{34}$$

for $-1 < p < 0$ and $N \geq 3$, and

$$J \leq C(\triangle E)^{1/2} \sqrt{E(S)} \tag{35}$$

for $-1 < p < 0$ and $N = 1, 2$.

At this point we take $\lambda = \frac{1}{2}$ in (29) to obtain that

$$\int_S^T \int_\Omega a(x)u^2 dxdt \leq C[E(T) + E(S)] + \frac{C}{\eta}\int_S^T \int_\Omega a(x)u_t^2 dxdt \tag{36}$$
$$+2\eta \int_S^T E(t)dt + \int_S^T \int_{\Omega_1^*} \rho^2 dxdt + J.$$

Substituting estimate (36) in (19), it follows that

$$\delta \int_S^T E(t)dt \leq C \left[E(T) + E(S) + D_i^N + \int_S^T \int_\Omega a(x) u_t^2 dxdt \right.$$

$$\left. +2\eta \int_S^T E(t)dt + \int_S^T \int_{\Omega_1^*} \rho^2 dxdt + J \right]$$

with η arbitrary.

Taking $\eta > 0$ such that $\delta - 2\eta C > 0$ we obtain

$$\int_S^T E(t)dt \leq C \left[E(T) + E(S) + D_i^N + \int_S^T \int_\Omega a(x) u_t^2 dxdt + \int_S^T \int_{\Omega_1^*} \rho^2 dxdt \right]$$

since J can be estimated in terms of D_i^N due to (32), (33), (34) and (35), where D_i^N are given in Lemma 4.8, for $i = 1, 2, 3$ or 4.

Lemma 4.6 and its proof give that the integral

$$\int_S^T \int_{\Omega_1^*} \rho^2 dxdt$$

can be estimated in terms of D_i^N.

Thus, we have proved the following proposition.

Proposition 4.1. *Given $0 < S < T < +\infty$, the solution $u = u(x,t)$ of (1) satisfies*

$$\int_S^T E(t)dt \leq C_0 \left[E(T) + E(S) + D_i^N + \int_S^T \int_\Omega a(x) u_t^2 dxdt \right] \tag{37}$$

with $C_0 > 0$ a positive constant which depends on $\|a\|_\infty, |T-S|, |\Omega|$ and the initial data, with $i = 1, 2, 3$ or 4 according to the cases in Lemma 4.8.

5 The energy bounded by energy differences

Due to the fact that $E(t)$ is a nonincreasing function, we have that

$$\int_S^T E(t)dt \geq E(T)(T-S).$$

Taking $S = t$ and $T = t + T_0$ for $t \geq 0$ and a fix $T_0 > 0$ we obtain from (37)

$$T_0 E(t+T_0) \leq C_0 \left[E(t+T_0) + E(t) + D_i^N + \int_t^{t+T_0} \int_\Omega a(x) u_t^2 dxdt \right].$$

Now we fix T_0 such that $T_0 \geq 2C_0 + 1$. Then we obtain that

$$E(t) \leq C \left[\triangle E(t) + D_i^N + \int_t^{t+T_0} \int_\Omega a(x) u_t^2 dx dt \right], \quad t \geq 0 \tag{38}$$

where

$$\triangle E(t) = E(t) - E(t + T_0)$$

and C is a positive constant which depends only on T_0, $\|a\|_\infty$, $|\Omega|$ and the initial data.

Now, D_i^N are given in terms of $\triangle E(t)$ instead of $\triangle E = E(S) - E(T)$ as in Lemma 4.8.

The idea here is to estimate the integral in (38) in terms of $\triangle E(t)$.

In fact, we can write

$$\int_t^{t+T_0} \int_\Omega a(x) u_t^2 dx dt = \int_t^{t+T_0} \int_{\Omega_1^*} a(x) u_t^2 + \int_t^{t+T_0} \int_{\Omega_2^*} a(x) u_t^2. \tag{39}$$

Then, for $p \geq 0$ and $N \geq 1$ we have

$$\begin{aligned} J_2 \equiv \int_t^{t+T_0} \int_{\Omega_2^*} a(x) u_t^2 dx dt &\leq \int_t^{t+T_0} \int_{\Omega_2^*} a(x) u_t^{2+p} dx dt \\ &\leq C \int_t^{t+T_0} \int_\Omega \rho(x, u_t) u_t dx dt \leq C \triangle E(t), \end{aligned} \tag{40}$$

due to the hypotheses on $\rho(x, s)$, the energy identity (4) and the fact that $|u_t| \geq 1$ in Ω_2^*.

We also have

$$J_1 = \int_t^{t+T_0} \int_{\Omega_1^*} a(x) u_t^2 dx dt \leq C \left(\int_t^{t+T_0} \int_{\Omega_1^*} a(x) u_t^{r+2} dx dt \right)^{\frac{2}{r+2}}$$

for $r \geq 0$, where $C > 0$ depends only on T_0, $\|a\|_\infty$, $|\Omega|$ and r.

Then, using the hypotheses on $\rho(x, s)$ and the energy identity(4) we obtain that

$$J_1 \leq C(\triangle E(t))^{\frac{2}{r+2}}, \tag{41}$$

for $r \geq 0$.

Therefore, from (39), (40) and (41), we get

$$I \equiv \int_t^{t+T_0} \int_\Omega a(x) u_t^2 dx dt \leq C[\triangle E(t) + (\triangle E(t))^{\frac{2}{r+2}}]$$

for $r \geq 0$ and $p \geq 0$.

So, the integral I, in Case 1, can be estimated in terms of $\triangle E(t)$.

In the same way, similarly as in Lemmas 4.6 and 4.7, we can see that this fact is true for the other cases. Then, combining these conclusions with estimate (38) we can see that

$$E(t) \leq C[\triangle E(t) + D_i^N(t)] \ , \quad t \geq 0 \tag{42}$$

where C is a positive constant independent of t which can be estimated explicitly and D_i^N is given in Lemma 4.8 according to the four cases $i = 1, 2, 3, 4$.

Finally, we use the following Young inequality

$$AB^{\frac{1}{\alpha}} \leq CA^{\alpha'} + \frac{1}{2}B$$

with $A, B \geq 0$, $\alpha > 1$ and α' such that $\dfrac{1}{\alpha'} + \dfrac{1}{\alpha} = 1$, to estimate in (42) the expression of D_i^N, $i = 1, 2, 3, 4$, which depends on the power nonlinearity p. Making this we obtain the next result.

Proposition 5.1. *The energy $E(t)$ satisfies,*

$$E(t) \leq Cd_i^N(t) \ , \quad t \geq 0$$

with C a positive constant independent of t , where according to the four cases:

Case 1: $r \geq 0$, $0 \leq p \leq \dfrac{2}{N-2}, N \geq 3$:

$$d_1^N(t) = \triangle E(t) + (\triangle E(t))^{\frac{2}{r+2}} + (\triangle E(t))^{\frac{4(p+1)}{4+p(N+2)}}$$

and

$$d_1^N(t) = \triangle E(t) + (\triangle E(t))^{\frac{2}{r+2}}$$

for $r \geq 0, p \geq 0$ and $N = 1, 2$.

Case 2: $-1 < r < 0$, $0 \leq p \leq \dfrac{2}{N-2}$, $N \geq 3$:

$$d_2^N(t) = \triangle E(t) + (\triangle E(t))^{\frac{2r+2}{r+2}} + (\triangle E(t))^{\frac{4(p+1)}{4+p(N+2)}}$$

and

$$d_2^N(t) = \triangle E(t) + (\triangle E(t))^{\frac{2r+2}{r+2}}$$

for $r \geq 0, p \geq 0$ and $N = 1, 2$.

Case 3: $r \geq 0$, $-1 < p < 0$, $N \geq 3$

$$d_i^N(t) = \triangle E(t) + (\triangle E(t))^{\frac{2}{r+2}} + (\triangle E(t))^{\frac{4}{4+p(2-N}}$$

and

$$d_i^N(t) = \triangle E(t) + (\triangle E(t))^{\frac{2}{r+2}}$$

for $r \geq 0$, $-1 < p < 0$ *and* $N = 1, 2$.

Case 4: $-1 < r < 0$, $-1 < p < 0$, $N \geq 3$

$$d_i^N(t) \leq \triangle E(t) + (\triangle E(t))^{\frac{2r+2}{r+2}} + (\triangle E(t))^{\frac{4}{4+p(2-N)}}$$

and

$$d_i^N(t) = \triangle E(t) + (\triangle E(t))^{\frac{2r+2}{r+2}}$$

for $-1 < r < 0$, $-1 < p < 0$ *and* $N = 1, 2$.

6 Proof of Theorem 3.1:

We need the following Nakao's Lemma which appears in [9].

Lemma 6.1. *Let* $E(t) : \mathbb{R}^+ \to \mathbb{R}$ *be a non increasing and non negative function satisfying*

$$\sup_{t \leq s \leq t+T} E(S)^{1+\delta} \leq C_1[E(t) - E(t+T)]$$

for all $t \geq 0$, *with* $\delta > 0, T > 0, C_1 > 0$ *fixed numbers.*

Then

$$E(t) \leq C_2 E(0)(1+t)^{-\frac{1}{\delta}},\ t \geq 0$$

with $C_2 > 0$ *a constant.*

If $\delta = 0$, *there exist* $\alpha > 0$ *and* $C_3 > 0$ *constants such that*

$$E(t) \leq C_3 e^{-\alpha t},\ t \geq 0\ .$$

Now, to obtain the proof we use in the estimates given by Proposition 5.1 the fact that $E(t)$ is nonincreasing in t.

Thus,

Case 1: $r > 0$, $0 < p \leq \dfrac{2N}{N-2}$, $N \geq 3$

In this case it holds that

$$E(t) \leq C(\triangle E(t))^{\gamma_1}, \qquad \gamma_1 = \min\left\{\frac{2}{r+2}\ , \frac{4(p+1)}{4+p(N+2)}\right\}$$

Then, writing

$$\frac{1}{\gamma_1} = 1 + \delta_1,\ \delta_1 = \max\left\{\frac{r}{2}\ , \frac{p(N-2)}{4(p+1)}\right\}$$

we have

$$\sup_{t \leq s \leq t+T_0} E(s)^{1+\delta_1} \leq C \triangle E(t) = C[E(t) - E(t+T_0].$$

Then, applying Nakao's lemma, we obtain

$$E(t) \leq C(1+t)^{-1/\delta_1} = C(1+t)^{-\lambda_1}$$

with

$$\lambda_1 = \min\left\{\frac{2}{r}\,, \frac{4(p+1)}{p(N-2)}\right\}.$$

If $r = p = 0$ the decay rate is exponential, because in this case (see Nakao's Lemma):

$$E(t) \leq C\triangle E(t), \quad t \geq 0.$$

If $r > 0$ and $p = 0$, $\lambda_1 = \dfrac{2}{r}$. If $r = 0$ and $p > 0$, $\lambda_1 = \dfrac{4(p+1)}{p(N-2)}$.

Finally, when $N = 1, 2$ from Proposition 5.1 we have

$$E(t) \leq C(\triangle E(t))^{\frac{2}{r+2}}$$

for $r \geq 0$ and $p \geq 0$.

Then, in this case λ_1 is given by

$$\lambda_1 = \frac{2}{r} \quad (for \;\; r > 0)$$

and the decay rate is exponential if $r = 0$ and $p \geq 0$.

The cases 2, 3 and 4 are proven in the same way.

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GENERALIZED HAUSDORFF MATRICES AS BOUNDED OPERATORS OVER $\mathcal{A}_k$

EKREM SAVAŞ* AND HAMDULLAH ŞEVLİ**

ABSTRACT. In this paper we prove a theorem which shows that a generalized Hausdorff matrix is a bounded operator on $\mathcal{A}_k$, defined below by (2); i.e., $\left(H^{\beta},\mu\right)\in B\left(\mathcal{A}_k\right)$.

Let σ_n^{α} denote the nth terms of the transform of a Cesáro matrix (C,α) of a sequence (s_n). In 1957 Flett [3] made the following definition. A series $\sum a_n$, with partial sums s_n, is said to be absolutely (C,α) summable of order $k\geq 1$, written $\sum a_n$ is summable $|C,\alpha|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_{n-1}^{\alpha}-\sigma_n^{\alpha}\right|^k<\infty. \tag{1}$$

He also proved the following inclusion theorem. If series $\sum a_n$ is summable $|C,\alpha|_k$, it is summable $|C,\beta|_r$ for each $r\geq k\geq 1$, $\alpha>-1$, $\beta>\alpha+1/k-1/r$. It then follows that, if one chooses $r=k$, then a series $\sum a_n$ which is $|C,\alpha|_k$ summable is also $|C,\beta|_k$ summable for $k\geq 1$, $\beta>\alpha>-1$.

Let $\sum a_n$ be an infinite series with partial sums (s_n). Define

$$\mathcal{A}_k:=\left\{(s_n)_{n=0}^{\infty}\ :\ \sum_{n=1}^{\infty} n^{k-1}\left|a_n\right|^k<\infty\ ;\ a_n=s_n-s_{n-1}\right\} \tag{2}$$

for $k\geq 1$.

A matrix T is said to be a bounded linear operator on $\mathcal{A}_k$, written $T\in B\left(\mathcal{A}_k\right)$, if $T:\mathcal{A}_k\to\mathcal{A}_k$.

If one sets $\alpha=0$ in the inclusion statement involving (C,α) and (C,β), then one obtains the fact that $(C,\beta)\in B\left(\mathcal{A}_k\right)$ for each $\beta>0$.

In 1970, using definition (1) and (2), Das [1] defined the concept of absolutely kth power conservative. Now we introduce some terminology about this concept and we establish concerning $\mathcal{A}_k$ spaces.

Let A be a sequence to sequence transformation mapping the sequence (s_n) into (t_n). If, whenever (s_n) converges absolutely, (t_n) converges absolutely, A is called absolutely conservative. If the absolute convergence of (s_n) implies absolute convergence of (t_n) to the same limit, A is called absolutely regular. For some given $k\geq 1$ if

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$$\sum_{n=1}^{\infty} n^{k-1} |s_n - s_{n-1}|^k < \infty \tag{3}$$

implies

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty \tag{4}$$

then A is called absolutely kth power conservative.

Using the definition of $\mathcal{A}_k$ space, we may write this definition in another form as follows.

If A is a mapping from $\mathcal{A}_k$ to $\mathcal{A}_k$; i.e., $A \in B(\mathcal{A}_k)$ then A is called absolutely kth power conservative.

For $k = 1$ condition (3) guarantees the convergence of (s_n). Note that when $k > 1$, (3) does not necessarily imply the convergence of (s_n). For example, take

$$s_n = \sum_{v=1}^{n} \frac{1}{v \log(v+1)}.$$

Then (3) holds but (s_n) does not converge. Thus, since the limit of (s_n) need not exist, we can't introduce the concept of absolute kth power regularity when $k > 1$.

From Knopp and Lorentz [7], Morley [8] we konow that a conservative Hausdorff transformation is absolutely conservative. The following theorem generalizes this result.

Theorem 1. [1] *Let $k \geq 1$. Then a conservative Hausdorff transformation is absolutely kth power conservative.*

Since Euler, Hölder and (C, α) methods are special Hausdorff methods, we obtain the following corollaries.

Corollary 1. *Let $0 < \alpha \leq 1$ and $q = (1-\alpha)/\alpha$. Then each (E, q) is a map from $\mathcal{A}_k$ to $\mathcal{A}_k$; i.e., $(E, q) \in B(\mathcal{A}_k)$.*

Proof. We know that (E, q), $\alpha \in (0, 1]$ and $q = (1-\alpha)/\alpha$, is a Hausdorff method generated by the sequence $\mu_n = \alpha^n$. Also, we know that this method is regular and so conservative. Hence from Theorem 1 $E_\alpha : \mathcal{A}_k \to \mathcal{A}_k$. □

From Corollary 1, each Euler transformation is absolutely kth power conservative.

Corollary 2. *Let $\alpha > 0$. Then H_α is a transformation from A_k into A_k; i.e., $H_\alpha \in B(\mathcal{A}_k)$.*

Proof. It is known that Hölder method H_α is a Hausdorff method generated by the moment sequence $\mu_n = (n+1)^{-\alpha}$ for $\alpha > 0$. Also it is known that H_α is regular and so conservative for $\alpha \geq 0$. Hence for $\alpha > 0$ $H^\alpha : \mathcal{A}_k \to \mathcal{A}_k$ by Theorem 1. □

Let $\alpha > 0$. Then Hölder transformation is absolutely kth power conservative by Corollary 2.

Corollary 3. *Let $\alpha > 0$. Then $(C, \alpha) \in B(\mathcal{A}_k)$.*

Proof. We know that (C,α) is a Hausdorff method generated by the moment sequence $\mu_n = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}^{-1}$ for $\alpha > 0$. For $\alpha \geq 0$ (C,α) is regular and conservative. So for $\alpha > 0$, $(C,\alpha) : \mathcal{A}_k \to \mathcal{A}_k$ by Theorem 1. □

In [9] it is shown that $(C,\alpha) \in B(\mathcal{A}_k)$ for each $\alpha > -1$. Therefore being conservative is not a necessary condition for a matrix to map $\mathcal{A}_k$ to $\mathcal{A}_k$. Because we know that (C,α) is not conservative for $\alpha < 0$.

By Theorem 1, if a Hausdorff transformation (H,μ) is conservative, then $(H,\mu) \in B(\mathcal{A}_k)$. Now, we will obtain this result for the generalized Hausdorff matrices defined independently by Endl [2] and Jakimovski [4].

Let β be a real number, let (μ_n) be a real sequence, and let Δ be the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^n(\mu_k) = \Delta(\Delta^{n-1}\mu_k)$. Then the infinite matrix $\left(H^{(\beta)}, \mu_n^{(\beta)}\right) = \left(H^{\beta}, \mu\right) = \left(h_{nk}^{(\beta)}\right)$ is defined by

$$h_{nk}^{(\beta)} := \begin{cases} \begin{pmatrix} n+\beta \\ n-k \end{pmatrix} \Delta^{n-k}\mu_k^{(\beta)} & , \quad 0 \leq k \leq n\,, \\ 0 & , \quad k > n\,, \end{cases}$$

and the associated matrix method is called a generalized Hausdorff matrix and generalized Hausdorff method, respectively. The moment sequence $\mu_n^{(\beta)}$ is given by

$$\mu_n^{(\beta)} = \int_0^1 t^{n+\beta} d\chi(t), \tag{5}$$

where $\chi(t) \in BV[0,1]$. We shall consider here only nonnegative β. The case $\beta = 0$ corresponds to ordinary Hausdorff summability.

Theorem 2. *If the generalized Hausdorff transformation $\left(H^{\beta}, \mu\right)$ is conservative, then it is absolutely kth power conservative; i.e., $\left(H^{\beta}, \mu\right) \in B(\mathcal{A}_k)$.*

We need these Lemmas to prove the theorem.

Lemma 1. *Let $k \geq 1$, $n \geq v$ and $\beta \geq 0$. Then*

$$E_{n+\beta}^{k-1} E_{n-v}^{v+\beta-1} \leq E_{v+\beta}^{k-1} E_{n-v}^{v+\beta+k-2}. \tag{6}$$

Proof. For v fixed, we define a sequence (f_n) by

$$f_n = \frac{E_{n+\beta}^{k-1} E_{n-v}^{v+\beta-1}}{E_{v+\beta}^{k-1} E_{n-v}^{v+\beta+k-2}}.$$

Then

$$f_n = \frac{(n+\beta+k-1)(v+\beta)}{(n+\beta)(v+\beta+k-1)},$$

and

$$\frac{f_{n+1}}{f_n} = \frac{(n+\beta+k)(n+\beta)}{(n+\beta+k-1)(n+\beta+1)} = \frac{(n+\beta)^2 + k(n+\beta)}{(n+\beta)^2 + k(n+\beta) + (k-1)}.$$

$f_{n+1}/f_n < 1$; i.e., (f_n) is decreasing. Since

$$f_v = \frac{(v+\beta+k-1)(v+\beta)}{(v+\beta)(v+\beta+k-1)} = 1,$$

and since $n \geq v$, then $f_n \leq 1$. Hence for $n \geq v$, (6) is satisfied. □

Lemma 2. [6] *For* $0 \leq t \leq 1$ *and* $\beta \geq 0$

$$\sum_{n=0}^{m} \binom{m+\alpha}{n} (1-t)^n t^{m+\alpha-n} \leq 1.$$

Proof of Theorem 2. Let (t_n) denote the $\left(H^\beta, \mu\right)$ transform of (s_n); i.e.,

$$t_n = \sum_{v=0}^{n} h_{nv}^{\beta} s_v.$$

We will show that

$$\sum_{n=1}^{\infty} n^{k-1} |a_n|^k < \infty \Rightarrow \sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{7}$$

We write

$$t_n = \sum_{v=0}^{n} b_v.$$

Then $b_n = t_n - t_{n-1}$. For $k \geq 1$

$$E_n^{k-1} = \binom{n+k-1}{n} = \binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!\,(k-1)!} = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}.$$

Then

$$E_n^{k-1} \approx \frac{n^{k-1}}{\Gamma(k)} \approx n^{k-1},$$

$$n^{k-1} \approx E_n^{k-1} \approx E_{n+\beta}^{k-1}.$$

Hence (7) is equivalent to

$$\sum_{n=1}^{\infty} E_{n+\beta}^{k-1} |a_n|^k < \infty \Rightarrow \sum_{n=1}^{\infty} E_{n+\beta}^{k-1} |b_n|^k < \infty. \tag{8}$$

For $0 \leq t \leq 1$ define

$$\phi_n(t) = \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta} (1-t)^{n-v} a_v. \tag{9}$$

Using Hölder's inequality,

$$\begin{aligned} |\phi_n(t)|^k &= \left| \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta} (1-t)^{n-v} a_v \right|^k \\ &\leq \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta} (1-t)^{n-v} |a_v|^k \times \left\{ \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta} (1-t)^{n-v} \right\}^{k-1}. \end{aligned}$$

From Lemma 2

$$\begin{aligned} \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta} (1-t)^{n-v} &= \sum_{v=1}^{n} \binom{n+\beta-1}{n-v} t^{v+\beta} (1-t)^{n-v} \\ &= \sum_{i=0}^{n-1} \binom{n+\beta-1}{n-i-1} t^{i+\beta+1} (1-t)^{n-i-1} \\ &= t \sum_{i=0}^{n-1} \binom{n-1+\beta}{n-1-i} t^{i+\beta} (1-t)^{n-1-i} \\ &= O(t). \end{aligned}$$

Hence

$$|\phi_n(t)|^k = O(1)\ t^{k-1} \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta} (1-t)^{n-v} |a_v|^k.$$

Using Lemma 1 we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} E_{n+\beta}^{k-1} |\phi_n(t)|^k &= O(1) \sum_{n=1}^{\infty} E_{n+\beta}^{k-1} t^{k-1} \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta} (1-t)^{n-v} |a_v|^k \\ &= O(1)\ t^{k-1} \sum_{v=1}^{\infty} t^{v+\beta} |a_v|^k \sum_{n=v}^{\infty} E_{n+\beta}^{k-1} E_{n-v}^{v+\beta-1} (1-t)^{n-v} \\ &= O(1)\ t^{k-1} \sum_{v=1}^{\infty} t^{v+\beta} |a_v|^k E_{v+\beta}^{k-1} \sum_{n=v}^{\infty} E_{n-v}^{v+\beta+k-2} (1-t)^{n-v} \\ &= O(1)\ t^{k-1} \sum_{v=1}^{\infty} t^{v+\beta} |a_v|^k E_{v+\beta}^{k-1} t^{-v-\beta-k+1} \\ &= O(1) \sum_{v=1}^{\infty} E_{v+\beta}^{k-1} |a_v|^k. \end{aligned} \tag{10}$$

It is known by Das [1] and Jakimovski [5] that, if (t_n) and (τ_n) are the (H, μ_n) transformation of (s_n) and (na_n), respectively, then

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$$n\left(t_n - t_{n-1}\right) = \tau_n.$$

A similar result can be proved for $\left(H^{\beta}, \mu\right)$, which we now do.

$$\begin{aligned}
\tau_n &= \sum_{k=0}^{n} h_{nk}^{(\beta)} \left(k+\beta\right) a_k \\
&= \sum_{k=0}^{n} h_{nk}^{(\beta)} \left(k+\beta\right) \left(s_k - s_{k-1}\right) \\
&= \sum_{k=0}^{n} h_{nk}^{(\beta)} \left(k+\beta\right) s_k - \sum_{k=1}^{n} h_{nk}^{(\beta)} \left(k+\beta\right) s_{k-1} \\
&= \sum_{k=0}^{n} h_{nk}^{(\beta)} \left(k+\beta\right) s_k - \sum_{k=0}^{n-1} h_{n,k+1}^{(\beta)} \left(k+\beta+1\right) s_k \\
&= \sum_{k=0}^{n} \left(\begin{array}{c} n+\beta \\ n-k \end{array} \right) \Delta^{n-k} \mu_k^{(\beta)} \left(k+\beta\right) s_k \\
&\quad - \sum_{k=0}^{n-1} \left(\begin{array}{c} n+\beta \\ n-k-1 \end{array} \right) \Delta^{n-k-1} \mu_{k+1}^{(\beta)} \left(k+\beta+1\right) s_k.
\end{aligned}$$

Since

$$\begin{aligned}
\left(k+\beta\right) \left(\begin{array}{c} n+\beta \\ n-k \end{array} \right) &= \frac{\left(k+\beta\right) \Gamma\left(n+\beta-1\right)}{(n-k)! \Gamma\left(k+\beta-1\right)} \\
&= \frac{\left(n+\beta\right) \Gamma\left(n+\beta\right)}{(n-k)! \Gamma\left(k+\beta\right)} \\
&= \left(n+\beta\right) \left(\begin{array}{c} n+\beta-1 \\ n-k \end{array} \right),
\end{aligned}$$

we write

$$\begin{aligned}
\tau_n &= \left(n+\beta\right) \left[\sum_{k=0}^{n} \left(\begin{array}{c} n+\beta-1 \\ n-k \end{array} \right) \Delta^{n-k} \mu_k^{(\beta)} s_k - \sum_{k=0}^{n-1} \left(\begin{array}{c} n+\beta-1 \\ n-k-1 \end{array} \right) \Delta^{n-k-1} \mu_{k+1}^{(\beta)} s_k \right] \\
&= \left(n+\beta\right) \sum_{k=0}^{n} \left[\left(\begin{array}{c} n+\beta \\ n-k \end{array} \right) - \left(\begin{array}{c} n+\beta-1 \\ n-k-1 \end{array} \right) \right] \Delta^{n-k} \mu_k^{(\beta)} s_k \\
&\quad - \sum_{k=0}^{n-1} \left(\begin{array}{c} n+\beta-1 \\ n-k-1 \end{array} \right) \Delta^{n-k-1} \mu_{k+1}^{(\beta)} s_k \\
&= \left(n+\beta\right) \left[t_n - \sum_{k=0}^{n} \left(\begin{array}{c} n+\beta-1 \\ n-k-1 \end{array} \right) \left(\Delta^{n-k-1} \mu_k^{(\beta)} s_k - \Delta^{n-k-1} \mu_{k+1}^{(\beta)} s_k \right) \right. \\
&\quad \left. - \sum_{k=0}^{n-1} \left(\begin{array}{c} n+\beta-1 \\ n-k-1 \end{array} \right) \Delta^{n-k-1} \mu_{k+1}^{(\beta)} s_k \right]
\end{aligned}$$

$$= (n+\beta)\left[t_n - \sum_{k=0}^{n} \left(\begin{array}{c} n+\beta-1 \\ n-k-1 \end{array} \right) \Delta^{n-k-1}\mu_k^{(\beta)} s_k\right]$$
$$= (n+\beta)\left(t_n - t_{n-1}\right).$$

Thus

$$\begin{aligned} b_n =& \frac{1}{n+\beta}\tau_n \\ =& \frac{1}{n+\beta}\sum_{v=0}^{n} \left(\begin{array}{c} n+\beta \\ n-v \end{array} \right) \left(\Delta^{n-v}\mu_v^{(\beta)}\right)(v+\beta)\, a_v \\ =& \sum_{v=0}^{n} \left(\begin{array}{c} n+\beta-1 \\ n-v \end{array} \right) \Delta^{n-v}\mu_v^{(\beta)} a_v \\ =& \sum_{v=0}^{n} E_{n-v}^{v+\beta-1}\Delta^{n-v}\mu_v^{(\beta)} a_v. \end{aligned}$$

Since $\left(H^{\beta}, \mu\right)$ is conservative, $\mu_n^{(\beta)}$ as given by (5) is a moment sequence and

$$\Delta^{n-v}\mu_v^{(\beta)} = \int_0^1 t^{v+\beta}\left(1-t\right)^{n-v} d\chi\left(t\right)$$

from Endl [2]. Using (9) we see that

$$\begin{aligned} b_n =& \sum_{v=1}^{n} E_{n-v}^{v+\beta-1} \int_0^1 t^{v+\beta}\left(1-t\right)^{n-v} d\chi\left(t\right) a_v \\ =& \int_0^1 \left(\sum_{v=1}^{n} E_{n-v}^{v+\beta-1} t^{v+\beta}\left(1-t\right)^{n-v} a_v\right) d\chi\left(t\right) \\ =& \int_0^1 \phi_n\left(t\right) d\chi\left(t\right). \end{aligned}$$

Using Minkowski's inequality and (10) we have

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$$\left\{\sum_{n=1}^{\infty} E_{n+\beta}^{k-1}\left|b_n\right|^k\right\}^{1/k} = \left\{\sum_{n=1}^{\infty} E_{n+\beta}^{k-1}\left|\int_0^1 \phi_n(t)\, d\chi(t)\right|^k\right\}^{1/k}$$
$$\leq \int_0^1 |d\chi(t)| \left\{\sum_{n=1}^{\infty} E_{n+\beta}^{k-1}\left|\phi_n(t)\right|^k\right\}^{1/k}$$
$$= O(1)\int_0^1 |d\chi(t)| \left\{\sum_{v=1}^{\infty} E_{v+\beta}^{k-1}\left|a_v\right|^k\right\}^{1/k}.$$

Hence the proof is complete. □

If we take $\beta = 0$ in Theorem 2, we get Theorem 1 as a corollary.

We use $C_\alpha^{(\beta)}$, $H_\alpha^{(\beta)}$, and $E_\alpha^{(\beta)}$ to denote the corresponding Endl generalizations of the (C,α), H_α, and E_α methods, respectively. For example $C_\alpha^{(\beta)}$ has moment sequence

$$\mu_n^{(\beta)} = \int_0^1 t^{n+\beta}\alpha(1-t)^{\alpha-1}\, dt.$$

Corollary 4. *Let $0 < \alpha \leq 1$, $q = (1-\alpha)/\alpha$, and $\beta \geq 0$. Then each $\left(E^{(\beta)}, q\right)$ is a map $\mathcal{A}_k$ to $\mathcal{A}_k$; i.e., $E_\alpha^{(\beta)} \in B(\mathcal{A}_k)$.*

Proof. We know from Endl [2] that $E_\alpha^{(\beta)}$ is a generalized Hausdorff method generated by the sequence $\mu_n = \alpha^{n+\beta}$ and that this method is conservative for $0 < \alpha \leq 1$, $q = (1-\alpha)/\alpha$, and $\beta \geq 0$. Hence from Theorem 2 $\quad E_\alpha^{(\beta)} : \mathcal{A}_k \to \mathcal{A}_k$. □

From Corollary 4 for $\beta \geq 0$ each generalized Euler transformation is absolutely kth power conservative. If we take $\beta = 0$ in Corollary 4, we get Corollary 1.

Corollary 5. *Let $\alpha > 0$ and $\beta \geq 0$. Then $H_\alpha^{(\beta)}$ is a transformation from $\mathcal{A}_k$ into $\mathcal{A}_k$; i.e., $H_\alpha^{(\beta)} \in B(\mathcal{A}_k)$.*

Proof. It is known by Endl [2] that $H_\alpha^{(\beta)}$ is a generalized Hausdorff method generated by the moment sequence $\mu_n = (n+\beta+1)^{-\alpha}$ and that this transformation is conservative for $\alpha > 0$, $\beta \geq 0$. Hence for $\alpha > 0$, $\beta \geq 0$ $\quad H_\alpha^{(\beta)} : \mathcal{A}_k \to \mathcal{A}_k$ by Theorem 2. □

Let $\alpha > 0$. Then generalized Hölder transformation is absolutely kth power conservative by Corollary 5. If we take $\beta = 0$ in Corollary 5, we get Corollary 2.

Corollary 6. *Let $\alpha > 0$ and $\beta \geq 0$. Then $C_\alpha^{(\beta)} \in B(\mathcal{A}_k)$.*

Proof. We know from Endl [2] that $C_\alpha^{(\beta)}$ is a generalized Hausdorff method generated by the moment sequence $\mu_n = \left(\begin{array}{c} n+\alpha+\beta \\ n \end{array}\right)^{-1}$ and that this method is conservative for $\alpha > 0$, $\beta \geq 0$. So for $\alpha > 0$, $\beta \geq 0$ $\quad C_\alpha^{(\beta)} : \mathcal{A}_k \to \mathcal{A}_k$ by Theorem 2. □

From Corollary 6 $C_\alpha^{(\beta)}$ is absolutely kth power conservative. If we take $\beta = 0$ in Corollary 6, we obtain Corollary 3.

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Some stability and boundedness results to nonlinear differential equations of Liénard type with finite delay

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Abstract: This paper gives sufficient conditions to ensure the stability and boundedness of solutions to a class of second order nonlinear delay differential equations of Liénard type. With the help of a Lyapunov functional, we obtain three new results and give some explanatory examples on the subject.

1. Introduction

As well-known, the area of differential equations is an old but durable subject that remains alive and useful to a wide class of engineers, scientists and mathematicians. In particular, many actual systems have the property aftereffect, i.e. the future states depend not only on the present, but also on the past history. Aftereffect is believed to occur in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory and so on. During investigations in applied sciences, economy and etc. some practical problems related to physics, mechanics, engineering technique fields, etc. are associated with some second order linear or nonlinear differential equations. Some of these equations are Liénard type equations. So far, qualitative properties of Liénard type equations with delay; in particular, stability, boundedness, asymptotic behavior, etc. of solutions have been widely studied in the literature. For instance, one can refer to the papers or books of Barnett [1], Burton ([2], [3]), Burton and Zhang [4], Caldeira-Saraiva [5], Cantarelli [6], Èl'sgol'ts [7], Èl'sgol'ts and. Norkin [8], Furumochi [9], Gao and Zhao [10], Hale [11], Hara and Yoneyama ([12], [13]), Hatvani [14], Heidel ([15],[16]), Huang and Yu [17], Jitsuro and Yusuke [18], Kato ([19], [20]), Kolmanovskii and Myshkis [21], Krasovskiì ([22], [23]), Li [24], Li and Wen [25], Lin [26], Liu and Huang [27], Liu and Xu [28], Liu [29], Ludeke [30], Luk [31], Malyseva [33], Muresan [34], Nakajima [35], Nápoles Valdés [36], Qian [37], Sugie [38], Sugie and Amano [39], Sugie et al. [40], C. Tunç and E. Tunç [41], Utz [42], Yang [43], Yoshizawa [44], Zhang ([45], [46]), Zhang and Yan [47], Zhou and Jiang [48], Zhou and Liu, [49], Zhou and Xiang [50], Wei and Huang [51], Wiandt ([52], [53]) and the references listed in these sources for some works and applications performed on the above mentioned subjects. To the best of our knowledge, for the sake of brevity, some results obtained in the literature on stability and boundedness of solutions of Liénard type equations can be summarized as follows:

In 1953, Krasovskiì [22] considered the system

$$\frac{dx}{dt} = f_1(x) + ay,$$

$$\frac{dy}{dt} = bx + f_2(y).$$

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and derived simple sufficient conditions which ensure the asymptotic stability of the null solution $x = y = 0$ under arbitrary initial conditions.
Later, in 1970, Burton [2] considered nonlinear differential equation

$$x'' + f(x)h(x')x' + g(x) = e(t),$$

and the stability of solutions for the case $e(t) = 0$ and boundeness of solutions for the case $e(t) \neq 0$ were investigated by the author.
In 1970 and 1972, Heidel ([15], [16]) considered Liénard equation

$$x'' + p(x)|x'|^{\alpha} x' + g(x) = 0$$

and the forced generalized Liénard equation

$$x'' + f(x)h(x')x' + g(x)k(x') = e(t),$$

respectively. By using Lyapunov functions [32], the author showed a necessary and sufficient condition for global asymptotic stability of the origin $(0,0)$ of the first equation and boundedness of solutions $x(t)$ and $x'(t)$, and global asymptotic stability of the trivial solution of the second equation.
Later, in 1976, Luk [31] obtained some results on the boundedness of solutions of Liénard equation with delay

$$x''(t) + \mu f(x(t))x'(t) + g(x(t-r)) = 0,$$

subject to different classes of initial values.
After that, in 1985 and 1988, Hara and Yoneyama ([12], [13]) considered the system

$$x' = y - F(x),$$

$$y' = -g(x).$$

They gave a detailed and interesting discussion of stability, boundedness, oscillation and periodicity of solutions, and conditions based on the Poincaré-Bendixson theorem were also given to ensure global asymptotic stability of the zero solution of the system.
Besides, in 1998, Li [24] considered the second-order differential equation

$$x'' + f(x)x' + g(x) = 0.$$

The author established necessary and sufficient conditions for global asymptotic stability of the zero solution of this equation and corrected some erroneous conditions of Krasovskiì [22].
In 1992 and 1993, Zhang ([45], [46]) also considered the second order Liénard equation with delay

$$x'' + f(x)x' + g(x(t-h)) = 0.$$

In the first paper, the author established necessary and sufficient conditions for the boundedness of all solutions of this delay differential equation, and also gave necessary and sufficient conditions for all solutions and their derivatives to converge to zero. In the second paper, the author searched some necessary and sufficient conditions for uniform boundedness

of the solutions and their derivatives, and the global asymptotic stability of the zero solution of this equation on $[0,\infty)$.

In 1993, Hatvani [14] established sufficient conditions for stability of the zero solution of the differential equation

$$x'' + a(t)f(x,x')x' + b(t)g(x) = 0.$$

In 1994, Qian [37] considered the nonlinear system

$$x' = [h(y) - F(x)]/a(x),$$
$$y' = -a(x)g(x),$$

and proved global asymptotic stability of the trivial solution of this system subject to the certain assumptions.

In 1995, Liu [29] gave some necessary and sufficient conditions for the global stability of the Liénard equation system

$$x' = y - F(x),$$
$$y' = -g(x).$$

In the same year, Gao and Zhao [10] investigated the system

$$x' = \varphi(y) - F(x),$$
$$y' = -g(x).$$

Their investigation focused on the question of whether or not that equilibrium point is asymptotically stable. The authors developed three criteria, each pertaining to the functions φ, F, g and each necessary and sufficient for the required asymptotic stability to hold.

In 1995, Wiandt [52] considered the equation

$$x'' + f(x)x' + g(x) = 0.$$

It was conjectured that the condition $x\int_0^x f(s)ds \geq 0$ is necessary for the global asymptotic stability of the zero solution of this equation. By the use of the Poincaré-Bendixson theorem and the LaSalle invariance principle, a counter example was given to this conjecture.

In 1996, Cantarelli [6] dealt with the generalized Liénard equation

$$x'' + a(t)f(x)x' + p(t)g(x) = 0.$$

The author studied the Lyapunov stability [32] of the origin by means of Lyapunov functions. Three theorems are given that use three different techniques.

In 1998, Sugie et al. [40] gave some necessary and sufficient conditions under which the zero solution of Liénard-type system

$$x' = h(y) - F(x),$$
$$y' = -g(x)$$

is global asymptotic stable.
In 1999, Zhou and Xiang [50] considered the retarded Liénard-type equation

$$x'' + f_1(x)x' + f_2(x)(x')^2 + \varphi(x) + g(x(t-h)) = 0 .$$

Using the Lyapunov functional method, the author obtained some sufficient conditions to ensure the stability and boundedness of solutions of this equation.
In 2000, Zhang and Yan [47] gave sufficient conditions for boundedness and asymptotic stability of the delay Liénard equation

$$x'' + f(x)x' + ax'(t-\tau) + g(x(t-\tau)) = 0 .$$

In 2003, Liu and Huang [27] considered the standard Liénard equation

$$x'' + f_1(x)x' + f_2(x)(x')^2 + g(x(t-h)) = e(t) .$$

They obtained some new sufficient conditions, as well as some new necessary and sufficient conditions, for all solutions and their derivatives to be bounded.
In 2003, Zhou and Jiang [48] considered the Liénard equation

$$x''(t) + f_1(x(t))x'(t) + f_2(x(t))x'(t-\tau) + f_3(x(t))(x'(t))^2 + \varphi(x(t)) + g(x(t-\tau)) = 0 .$$

Using appropriate Lyapunov functionals, the authors obtained stability and boundedness results for solutions of this equation.
In 2004, Jitsuro and Yusuke [18] have studied the global asymptotic stability of solutions of the non-autonomous system of Lienard type:

$$x' = y - F(x), \ (f(x) = dF(x)/dx),$$

$$y' = -p(t)g_1(x) - q(t)g_2(x) .$$

The main result of Jitsuro and Yusuke [18] was proved by means of phase plane analysis with a Lyapunov function.
More recently, in 2007, C. Tunç and E. Tunç [41] obtained a result on the uniform boundedness and convergence of solutions of second order nonlinear differential equations of the form

$$x'' + a(t)f(x,x')x' + b(t)g(x) = p(t,x,x') .$$

In view of the above discussion, it can be seen that so far qualitative properties of Liénard type equations with delay or without delay have been discussed by many authors in the literature and some different tools have also been used to prove the results in these works.

In this paper, we consider the real second order Liénard type equation with delay:

$$x''(t) + \varphi(t, x(t-r), x'(t-r)) + b(t)g(x(t-r))$$

$$= p(t, x(t), x(t-r), x'(t)), \qquad \textbf{(1)}$$

in which r is a positive constant, that is, r fixed delay; φ, b, g and p are continuous functions in their respective domains; $\varphi(t,x,0) = g(0) = 0$ and the derivatives $b'(t)$ and

$\frac{dg}{dx} \equiv g'(x)$ exist and are also continuous. The primes in equation (1) denote differentiation with respect to t. Throughout the paper $x(t)$ and $y(t)$ are respectively abbreviated as x and y. We investigate here the uniform asymptotic stability of zero solution of equation (1) when $p(t,x(t),x(t-r),x'(t))=0$ in (1), and establish two results on the boundedness of the solutions of equation (1) when $p(t,x(t),x(t-r),x'(t))\neq 0$ in (1). It is worth mentioning that, with respect to the observations in the literature, it is not found any research on stability and boundedness of solutions of equation (1). To prove our main results we introduce a Lyapunov functional. It should be noted that the motivation for the present work has been inspired basically by the papers mentioned above and that exist in the relevant literature. Equation (1) and the assumptions will be established are different than that in the papers or books mentioned above.

Throughout the paper, instead of equation (1), we consider the equivalent system

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\varphi(t,x(t-r),y(t-r)) - b(t)g(x(t)) \\ &\quad + b(t)\int_{t-r}^{t} g'(x(s))y(s)ds + p(t,x(t),x(t-r),y(t)), \end{aligned} \tag{2}$$

which was obtained from (1).

2. Preliminaries

In order to reach our main results, first, we will give some basic definitions and some important stability and boundedness criteria for the general non-autonomous delay differential system. Consider the general non-autonomous delay differential system

$$\dot{x} = f(t,x_t),\ x_t = x(t+\theta),\ -r \le \theta \le 0,\ t \ge 0, \tag{3}$$

where $f:[0,\infty)\times C_H \to \Re^n$ is a continuous mapping, $f(t,0)=0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C,\|.\|)$ is the Banach space of continuous function $\phi:[-r,0]\to\Re^n$ with supremum norm, $r>0$, C_H is the open H-ball in C; $C_H := \{\phi\in(C[-r,0],\Re^n):\|\phi\|<H\}$. Standard existence theory, see Burton [3], shows that if $\phi\in C_H$ and $t\ge 0$, then there is at least one continuous solution $x(t,t_0,\phi)$ such that on $[t_0,t_0+\alpha)$ satisfying equation (3) for $t>t_0$, $x_t(t,\phi)=\phi$ and α is a positive constant. If there is a closed subset $B\subset C_H$ such that the solution remains in B, then $\alpha=\infty$. Further, the symbol $|.|$ will denote the norm in $\Re^n$ with $|x| = \max_{1\le i\le n}|x_i|$.

Definition 1. (See [3].) A continuous function $W:[0,\infty)\to[0,\infty)$ with $W(0)=0$, $W(s)>0$ if $s>0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i an integer.)

Definition 2. (See [3].) A function $V:[0,\infty)\times D\to[0,\infty)$ is called positive definite if $V(t,0)=0$ and if there is a wedge W_1 with $V(t,x)\geq W_1(|x|)$, and is called decrescent if there is a wedge W_2 with $V(t,x)\leq W_2(|x|)$.

Definition 3. (See [44].) A function $x(t_0,\phi)$ is said to be a solution of (3) with the initial condition $\phi\in C_H$ at $t=t_0$, $t_0\geq 0$, if there is a constant $A>0$ such that $x(t_0,\phi)$ is a function from $[t_0-h,t_0+A]$ into $\Re^n$ with the properties:

(i) $x_t(t_0,\phi)\in C_H$ for $t_0\leq t<t_0+A$,

(ii) $x_{t_0}(t_0,\phi)=\phi$,

(iii) $x(t_0,\phi)$ satisfies (3) for $t_0\leq t<t_0+A$.

Definition 4. (See [3].) Let $f(t,0)=0$. The zero solution of equation (3) is:

(a) stable if for each $\varepsilon>0$ there is a $\delta>0$ such that $[t\geq 0,\ \|\phi\|<\delta,\ t\geq t_0]$ implies that $|x(t,t_0,\phi)|<\varepsilon$.

(b) asymptotically stable if it is stable and if for each $t\geq 0$ there is an $\eta>0$ such that $\|\phi\|<\eta$ implies that $x(t,t_0,\phi)\to 0$ as $t\to\infty$.

Definition 5. (See [3].) A continuous functional $V:[0,\infty)\times C_H\to[0,\infty)$, which is locally Lipschitizian in ϕ, is called a Lyapunov functional for equation (3) if there is a wedge W with

(a) $W(|\phi(0)|)\leq V(t,\phi)$, $V(t,0)=0$, and

(b) $\dot{V}_{(3)}(t,x_t)=\limsup_{h\to 0}\frac{1}{h}[V(t+h,x_{t+h}(t_0,\phi))-V(t,x_t(t_0,\phi))]\leq 0$.

Theorem 1. (See ([3].) If there is a Lyapunov functional for the equation (3) and wedges satisfying:

(i) $W_1(|\phi(0)|)\leq V(t,\phi)\leq W_2(\|\phi\|)$, where $W_1(r)$ and $W_2(r)$ are wedges,

(ii) $\dot{V}_{(3)}(t,x_t)\leq 0$,

then the zero solution of equation (3) is uniformly stable.

Theorem 2. (See ([3].) If there is a Lyapunov functional $V(t,\phi)$ for equation (3) and wedges W_1, W_2 and W_3 such that

(i) $W_1(|\phi(0)|)\leq V(t,\phi)\leq W_2(\|\phi\|)$, (where $W_1(r)$ and $W_2(r)$ are wedges,)

(ii) $\dot{V}_{(3)}(t,x_t)\leq -W_3(|x(t)|)$,

then the zero solution of equation (3) is uniformly asymptotically stable.

Lemma. (See [3].) Let $V(t,x_t):[0,\infty)\times C_H\to\Re$ be a continuous functional satisfying a local Lipschitz condition. If

(i) $W(|x(t)|) \le V(t,x_t) \le W_1(|x(t)|) + W_2(\int_{t-r}^{t} W_3(|x(s)|)ds)$, and

(ii) $\dot{V}_{(3)} \le -W_3(|x(t)|) + M$ for some $M > 0$, where $W(r)$ and W_i, $(i = 1, 2, 3)$, are wedges, then the solutions of the equation (3) are uniform-bounded and uniform-ultimately bounded for bound B.

Theorem 3. (See [44].) If $f(t,\phi)$ in (3) is continuous in t, ϕ, for every $\phi \in C_{H_1}$, $H_1 < H$, and t_0, $0 \le t_0 < c$, where c is a positive constant, then there exist a solution of (3) with initial value ϕ at $t = t_0$, and this solution has a continuous derivative for $t > t_0$.

2. Main results

For the case $p(t, x(t), x(t-r), y(t)) = 0$ in (1), the first main result of this paper is the following:

Theorem 4. In addition to the basic assumptions imposed on the functions b, f and g, we suppose the following assumptions hold (α_1, α_2-some arbitrary positive constants, ε_0 and ε_1 are some sufficiently small positive constants):

(i) $B \ge b(t) \ge b_0 \ge 1$ for all $t \in \Re^+$, $\Re^+ = [0,\infty)$; and $b'(t) < 0$ (,where B and b_0 are some constants).

(ii) $0 \le \dfrac{\varphi(t, x(t-r), y(t-r))}{y(t)} - \alpha_1 \le \varepsilon_0$ for all $t \in \Re^+$, $x(t-r)$, $y(t-r)$ and $y(t)$ $(y(t) \ne 0)$, and $\varphi(t, x(t-r), 0) = 0$.

(iii) $g(0) = 0$ and $0 \le g'(x) - \alpha_2 \le \varepsilon_1$ for all x.

Then the zero solution of equation (1) are uniformly asymptotically stable, provided that

$$r < \frac{\alpha_1}{B(\alpha_2 + \varepsilon_1)}.$$

Proof. To verify the main results of this paper, we introduce a Lyapunov functional $V_0 = V_0(t, x_t, y_t)$, which is defined by:

$$V_0(t, x_t, y_t) = b(t)\int_0^x g(s)ds + \frac{1}{2}y^2 + \delta\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds. \qquad \textbf{(4)}$$

In view of conditions of (i) and (iii) of Theorem 4 and the functional $V_0(t, x_t, y_t)$ given by (4), we see that

$$V_0 \ge b_0\int_0^x g(s)ds + \frac{1}{2}y^2 + \delta\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds,$$

$$\ge \left(\frac{b_0\alpha_2}{2}\right)x^2 + \frac{y^2}{2} \ge D_1(x^2 + y^2),$$

where $D_1 = \min\left\{\frac{b_0\alpha_2}{2}, \frac{1}{2}\right\}$. Similarly, making use of conditions (i) and (iii) of Theorem 4, it follows that

$$V_0 \le B\int_0^x g(\xi)d\xi + \frac{y^2}{2} + \delta\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$$

$$\le \left(\frac{B\alpha_2 + B\varepsilon_1}{2}\right)x^2 + \frac{y^2}{2} + \delta\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$$

$$\le D_2(x^2 + y^2) + \delta\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds,$$

where $D_2 = \max\left\{\frac{B(\alpha_2 + \varepsilon_1)}{2}, \frac{1}{2}\right\}$.

Evaluating the time derivative of the functional $V_0(t, x_t, y_t)$ along the system (2), that is, $\frac{d}{dt}V_0(t, x_t, y_t)$, we get

$$\frac{d}{dt}V_0(t, x_t, y_t) = -\left[\frac{\varphi(t, x(t-r), y(t-r))}{y(t)}\right]y^2(t) + b(t)y(t)\int_{t-r}^{t} g'(x(s))y(s)ds$$

$$+ b'(t)\int_0^x g(x(s))ds + \delta r y^2 - \delta\int_{t-r}^{t} y^2(s)ds. \qquad \textbf{(5)}$$

Now, in view of the assumptions $\frac{\varphi(t, x(t-r), y(t-r))}{y(t)} \ge \alpha_1$, $b(t) \le B$ and $\alpha_2 \le g'(x) \le \varepsilon_1 + \alpha_2$ of Theorem 4 and the inequality $2|ab| \le a^2 + b^2$, we obtain from (5) that

$$\frac{d}{dt}V_0(t, x_t, y_t) \le -\alpha_1 y^2(t) + \frac{B(\alpha_2 + \varepsilon_1)r}{2}y^2(t) + \frac{B(\alpha_2 + \varepsilon_1)}{2}\int_{t-r}^{t} y^2(s)ds$$

$$+ \left(\frac{b'(t)\alpha_2}{2}\right)x^2(t) + \delta r\, y^2(t) - \delta\int_{t-r}^{t} y^2(s)ds$$

$$= -\left[\alpha_1 - \left(\frac{B(\alpha_2 + \varepsilon_1)}{2} + \delta\right)r\right]y^2(t) + \left(\frac{b'(t)\alpha_2}{2}\right)x^2(t)$$

$$-\left(\delta - \frac{B(\alpha_2 + \varepsilon_1)}{2}\right)\int_{t-r}^{t} y^2(s)ds. \qquad \textbf{(6)}$$

Let us choose $\delta = \frac{B(\alpha_2 + \varepsilon_1)}{2}$. Then, we have from (6) that

$$\frac{d}{dt}V_0(t, x_t, y_t) \le -\left[\alpha_1 - \left(\frac{B(\alpha_2 + \varepsilon_1)}{2} + \delta\right)r\right]y^2(t) + \left(\frac{b'(t)\alpha_2}{2}\right)x^2(t) \le 0 \qquad \textbf{(7)}$$

provided $r < \dfrac{\alpha_1}{B(\alpha_2 + \varepsilon_1)}$.

In view of the above discussion and (7), one can conclude that the zero solution of equation (1) is uniformly asymptotically stable.

Example 1. Consider the second order nonlinear delay differential equation

$$x''(t) + \frac{t^2+2}{t^2+1}\varphi(x(t-r), x'(t-r)) + \frac{e^t+1}{e^t}x(t) = 0, \tag{8}$$

where r is a positive constant, that is, r is fixed delay, $t \in [0,\infty)$, φ is a continuous function, $\varphi(x(t-r),0) = 0$ and

$$\frac{\varphi(x(t-r), y(t-r))}{y(t)} \geq a > 0 \text{ for all } x(t-r) \quad y(t-r) \text{ and } y(t), \ (y(t) \neq 0).$$

Equation (8) can be transformed into an equivalent system of the form

$$x'(t) = y(t),$$

$$y'(t) = -\frac{t^2+2}{t^2+1}\varphi(x(t-r), y(t-r)) - \frac{e^t+1}{e^t}x(t). \tag{9}$$

We define the Lyapunov functional

$$V_1(t, x_t, y_t) = (1+e^{-t})\ \frac{x^2}{2} + \frac{y^2}{2} + \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(\eta)\, d\eta\, ds \tag{10}$$

to verify the stability of the solution $x = 0$ of equation (8), where λ is a positive constant which will be determined later. It is clear that the Lyapunov functional $V_1(t, x_t, y_t)$ is positive definite; $V_1(t,0,0) = 0$, and we have from (10) that

$$0 < \frac{x^2}{2} + \frac{y^2}{2} \leq V_1(t, x_t, y_t). \tag{11}$$

Similarly, we obtain from (10) that

$$V_1(t, x_t, y_t) \leq x^2 + \frac{y^2}{2} + \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(\eta)\, d\eta\, ds.$$

Now, the time derivative of the functional $V_1 = V_1(t, x_t, y_t)$ in (10) with respect to the system (9) can be calculated as:

$$\frac{d}{dt}V_1(t, x_t, y_t) = -\frac{e^{-t}}{2}x^2(t) - \frac{t^2+2}{2(t^2+1)}\left[\frac{\varphi(x(t-r), y(t-r))}{y(t)}\right] y^2(t)$$

$$+ \lambda r y^2(t) - \lambda \int_{t-r}^{t} y^2(s)\, ds.$$

Making use of the assumption $\frac{\varphi(x(t-r), y(t-r))}{y(t)} \geq a > 0$, $(y(t) \neq 0)$, it follows that

$$\frac{d}{dt}V_1(t, x_t, y_t) \leq -\frac{e^{-t}}{2}x^2(t) - \frac{a}{2}y^2(t) + \lambda r y^2(t) - \lambda \int_{t-r}^{t} y^2(s)ds$$

$$= -\frac{e^{-t}}{2}x^2(t) - \left(\frac{a}{2} - \lambda r\right) y^2(t) - \lambda \int_{t-r}^{t} y^2(s)ds .$$

If we choose $\lambda = \frac{L}{2}$, then the last inequality implies that

$$\frac{d}{dt}V_1(t, x_t, y_t) \leq -\frac{e^{-t}}{2}x^2(t) - \alpha\, y^2(t) \leq 0 \quad \text{provided} \quad r < \frac{a}{L}.$$

Thus, under the above discussion, one can say that the zero solution of equation (8) is uniformly asymptotically stable.

For the case $p(t, x(t), y(t), y(t-r)) \neq 0$ in (1), the second main result of this paper is the following:

Theorem 5. In addition to the basic assumptions imposed on the functions b, f, g and p, we suppose that there are arbitrary positive constants α_1, α_2 and sufficiently small positive constants ε_0 and ε_1 such that the following assumptions hold

(i) $B \geq b(t) \geq b_0 \geq 1$ for all $t \in \Re^+$ and $b'(t) < 0$ (, where B and b_0 are some constants).

(ii) $0 \leq \frac{\varphi(t, x(t-r), y(t-r))}{y(t)} - \alpha_1 \leq \varepsilon_0$ for all $t \in \Re^+$, $x(t-r)$, $y(t-r)$ and $y(t)$ and $(y(t) \neq 0)$, and $\varphi(t, x(t-r), 0) = 0$.

(iii) $g(0) = 0$ and $0 \leq g'(x) - \alpha_2 \leq \varepsilon_1$ for all x.

(iv) $\left|p(t, x(t), x(t-r), y(t)\right| \leq K$ for all t, $x(t)$, $x(t-r)$ and $y(t)$, where K is finite positive constant. Then, every solution (x_t, y_t) of the system (2) is uniform-bounded and uniforml-ultimately bounded provided that $r < \frac{\alpha_1}{B(\alpha_2 + \varepsilon_1)}$. .

Proof. We use the Lyapunov functional given by (4) to complete the proof of Theorem 5. By differentiating the functional $V_0 = V_0(t, x_t, y_t)$ throughout the system (2), we get

$$\frac{d}{dt}V_0(t, x_t, y_t) = -\left[\frac{\varphi(t, x(t-r), y(t-r))}{y(t)}\right] y^2(t) + b(t)y(t)\int_{t-r}^{t} g'(x(s))y(s)ds$$

$$+ b'(t)\int_{0}^{x} g(x(s))ds + y(t)\,p(t, x(t), x(t-r), y(t))$$

$$+\delta r y^2(t) - \delta \int_{t-r}^{t} y^2(s)ds . \qquad \textbf{(12)}$$

Using the assumptions of Theorem 5 and the inequality $2|ab| \le a^2 + b^2$, we obtain from (12) that

$$\frac{d}{dt}V_0(t,x_t,y_t) \le -\alpha_1 y^2(t) + \frac{B(\alpha_2+\varepsilon_1)r}{2} y^2(t) + \frac{B(\alpha_2+\varepsilon_1)}{2} \int_{t-r}^{t} y^2(s)ds$$

$$+ \left(\frac{b'(t)\alpha_2}{2}\right)x^2(t) + \delta r y^2(t) - \delta \int_{t-r}^{t} y^2(s)ds$$

$$+ |y(t)\,p(t,x(t),x(t-r),y(t))|$$

$$= -\left[\alpha_1 - \left(\frac{B(\alpha_2+\varepsilon_1)}{2} + \delta\right) r\right] y^2(t) + \left(\frac{b'(t)\alpha_2}{2}\right)x^2(t)$$

$$+ |y(t)|\,|p(t,x(t),x(t-r),y(t))| + \left(\frac{B(\alpha_2+\varepsilon_1)}{2} - \delta\right) \int_{t-r}^{t} y^2(s)ds$$

$$\le -\left[\alpha_1 - \left(\frac{B(\alpha_2+\varepsilon_1)}{2} + \delta\right) r\right] y^2(t) + \left(\frac{b'(t)\alpha_2}{2}\right)x^2(t)$$

$$+ K|y(t)| + \left(\frac{B(\alpha_2+\varepsilon_1)}{2} - \delta\right) \int_{t-r}^{t} y^2(s)ds . \qquad \textbf{(13)}$$

If we choose $\delta = \frac{B(\alpha_2+\varepsilon_1)}{2}$, then we have from (13) for some constants $\rho > 0$ and $\gamma > 0$ that

$$\frac{d}{dt}V_0(t,x_t,y_t) \le -\left(\frac{\gamma\alpha_2}{2}\right)x^2 - \rho y^2 + K|y| \quad \text{provided} \quad r < \frac{\alpha_1}{B(\alpha_2+\varepsilon_1)}, \qquad \textbf{(14)}$$

where $-\gamma = \sup_{0\le t<\infty} b'(t)$.

Let $\alpha = \min\left\{\frac{\gamma\alpha_2}{2}, \rho\right\}$. When we choose $K = k\alpha$, we get from (14) that

$$\frac{d}{dt}V_0(t,x_t,y_t) \le -\alpha(x^2+y^2) + k\alpha|y|$$

$$\le -\frac{\alpha}{2}(x^2+y^2) - \frac{\alpha}{2}(|y|-k)^2 + \frac{\alpha}{2}k^2$$

$$\le -\frac{\alpha}{2}(x^2+y^2) + \frac{\alpha}{2}k^2 \quad \text{for some constants } k > 0 \text{ and } \alpha > 0 .$$

If we take $W_3(r)=\frac{\alpha r^2}{2}$ and $M=\frac{\alpha}{2}k^2$, then it can be easily verified that the Lyapunov functional $V_0(t,x_t,y_t)$ satisfies condition (ii) of the lemma. In view of the assumptions of Theorem 5, by a similar argument proceeded above, one can easily show that the Lyapunov functional $V_0(t,x_t,y_t)$ satisfies the first part of the lemma, (i).
The proof of this theorem is now complete.

Example 2. Consider the second order nonlinear delay differential equation

$$x''(t)+\frac{t^2+2}{t^2+1}\left(1+e^{-x^2(t-r)-x'^2(t-r)}\right)x'(t)+\frac{e^t+1}{e^t}x(t)=\frac{1}{1+t^2+x^2(t)+x^2(t-r)+x'^2(t)}, \quad \textbf{(15)}$$

where r is a positive constant, $t\in[0,\infty)$. Equation (15) is equivalent to the system:

$$x'(t)=y(t),$$

$$y'(t)=-\frac{t^2+2}{t^2+1}\left(1+e^{-x^2(t-r)-y^2(t-r)}\right)y(t)-\frac{e^t+1}{e^t}x(t)+\frac{1}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)}. \quad \textbf{(16)}$$

It follows from the system (15) that

$$\frac{\varphi(t,x(t-r),y(t-r))}{y(t)}=\frac{t^2+2}{t^2+1}\left(1+e^{-x^2(t-r)-y^2(t-r)}\right),\ b(t)=\frac{e^t+1}{e^t}\ ,\ g(x(t))=x(t) \text{ and}$$

$$p(t,x(t),x(t-r),y(t))=\frac{1}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)}.$$

Clearly, the functions φ, b, g and p satisfy the assumptions of Theorem 5. Namely, $1\le\frac{t^2+2}{t^2+1}\left(1+e^{-x^2(t-r)-y^2(t-r)}\right)\le 4$, the case for the function $g(x)=x$ is clear, that is, $\alpha_2\le g'(x)=1\le\alpha_2+\varepsilon_1$, $b'(t)=-e^{-t}<0$ for all $t\in[0,\infty)$ and $\left|p(t,x(t),x(t-r),y(t)\right|$
$=\frac{1}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)}\le 1=K$.

We define the Lyapunov functional

$$V_2(t,x_t,y_t)=(1+e^{-t})\frac{x^2}{2}+\frac{y^2}{2}+\lambda\int_{-r}^{0}\int_{t+s}^{t}y^2(\eta)d\eta ds \quad \textbf{(17)}$$

to verify the boundedness of solutions of equation (15), where λ is a positive constant which will be determined later. The Lyapunov functional $V_2(t,x_t,y_t)$ given by (17) is positive definite; $V_0(t,0,0)=0$ and

$$0<\frac{x^2}{2}+\frac{y^2}{2}\le V_2(t,x_t,y_t).$$

In view of (17), it can be seen that

$$V_2(t,x_t,y_t) \le x^2+\frac{y^2}{2}+\lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\eta)d\eta ds.$$

Now, the time derivative of the functional $V_2 = V_2(t,x_t,y_t)$ with respect to the system (16) can be calculated as follows:

$$\frac{d}{dt}V_2(t,x_t,y_t) = -\frac{e^{-t}}{2}x^2 - \frac{t^2+2}{t^2+1}\left(1+e^{-x^2(t-r)-y^2(t-r)}\right)y^2(t)$$

$$+\frac{y(t)}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)} + \lambda r y^2(t) - \lambda\int_{t-r(t)}^{t} y^2(s)ds.$$

Hence

$$\frac{d}{dt}V_2(t,x_t,y_t) \le -\frac{e^{-t}}{2}x^2 - y^2(t) + \lambda r y^2(t) - \lambda\int_{t-r}^{t} y^2(s)ds + \frac{1+y^2(t)}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)}$$

$$\le -\frac{e^{-t}}{2}x^2 - (1-\lambda r)y^2 - \lambda\int_{t-r}^{t} y^2(s)ds + M$$

since $\dfrac{1+y^2(t)}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)} \le M$, $M>0$.

Le us choose $\lambda = 2$. Hence

$$\frac{d}{dt}V_2(t,x_t,y_t) \le -\frac{e^{-t}}{2}x^2 - \alpha y^2 + M \text{ provided } r < 2^{-1}.$$

Thus, under the above discussion, all solutions of equation (15) are uniform-bounded and uniform-ultimately bounded.

Our last result is the following theorem:

Theorem 6. Suppose the following conditions are satisfied:

(i) Conditions (i)-(iii) of Theorem 5 hold.

(ii) $\left|p(t,x(t),x(t-r),y(t)\right| \le q(t)$,

where $\max q(t) < \infty$ and $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of integrable Lebesgue functions.

Then, there exists a finite positive constant K such that the solution $x(t)$ of equation (1) defined by the initial functions

$$x(t) = \phi(t),\ x'(t) = \phi'(t)$$

satisfies the inequalities

$$\left|x(t)\right| \le K,\ \left|x'(t)\right| \le K$$

for all $t \ge t_0$, where $\phi \in C^1([t_0-r,t_0],\Re)$, provided that $r < \dfrac{\alpha_1}{B(\alpha_2+\varepsilon_1)}$.

Proof. For the proof of this theorem use the Lyapunov functional $V_0 = V_0(t, x_t, y_t)$ defined by (4). Clearly, we have

$$D_1(x^2 + y^2) \le V_0(t, x_t, y_t).$$

Evaluating the time derivative of the functional $V_0(t, x_t, y_t)$ along the solution $(x(t), y(t), z(t))$ of the system (2) and making use of the assumptions Theorem 6, we get

$$\frac{d}{dt}V_0(t, x_t, y_t) \le \left|y(t)\,p(t, x(t), x(t-r), y(t))\right|$$

$$\le |y|q(t) \le q(t) + y^2 q(t).$$

Since

$$y^2 \le D_1^{-1} V_0(t, x_t, y_t),$$

we have

$$\frac{d}{dt}V(x_t, y_t, z_t) \le \left(1 + D_1^{-1}V(t, x_t, y_t)\right)q(t).$$

Now, integrating the last inequality from 0 to t, using the assumption $q \in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, we obtain

$$V(t, x_t, y_t) \le V(0, x_0, y_0) + A + D_1^{-1}\int_0^t \left(V_0(s, x_s, y_s)\right)q(s)ds$$

$$\le \left(V(0, x_0, y_0) + A\right)\exp\left(D_1^{-1}\int_0^t q(s)ds\right)$$

$$= \left(V(0, x_0, y_0) + A\right)\exp\left(D_1^{-1}A\right) = K_1 < \infty, \qquad \textbf{(18)}$$

where $K_1 > 0$ is a constant, $K_1 = \left(V(0, x_0, y_0) + A\right)\exp\left(D_1^{-1}A\right)$ and $A = \int_0^\infty q(s)ds$.

Now, the inequalities $(x^2 + y^2) \le D_1^{-1}V_0(t, x_t, y_t)$ and (18) together imply that

$$x^2 + y^2 \le D_1^{-1}V_0(t, x_t, y_t)\ K,$$

where $K = K_1 D_1^{-1}$. Hence, we conclude that

$$|x(t)| \le K,\ |y(t)| \le K$$

for all $t \ge t_0$. That is,

$$|x(t)| \le K,\ |x'(t)| \le K$$

for all $t \ge t_0$.

The proof of the theorem is now complete.

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A P_1-P_3-NZT FEM for solving general elastic multi-structure problems*

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Abstract

A new finite element method is introduced for solving general elastic multi-structure problems, where displacements on bodies, longitudinal displacements on plates, longitudinal displacements and rotational angles on rods are discretized by conforming linear elements, transverse displacements on rods and plates are discretized respectively by Hermite elements of third order and Zienkiewicz-type elements due to Wang, Shi, and Xu, and the discrete generalized displacement fields in individual elastic members are coupled together by some feasible interface conditions. The optimal error estimate in the energy norm is established for the method, which is also validated by some numerical examples.

Keywords: Elastic multi-structures; New Zienkiewicz-type elements; Error estimates

1 Introduction

Elastic multi-structures are usually assembled by elastic substructures of the same or different dimensions (bodies, plates, rods, etc.) with proper junctions, which are widely used in automobile and aeroplane structures, and motion- and force- transmitting machines and mechanisms. In the past few decades, many researchers have paid much attention to mathematical modeling and numerical solutions for elastic multi-structures composed of only two elastic members. We refer to [2, 5, 7, 10, 13, 15] and references therein for details. However, there are few considerations about general elastic multi-structure problems. Feng and Shi [8, 9] established mathematical models for general elastic multi-structure problems by the variational principle together with reasonable presentation for the interface conditions among substructures. The corresponding mathematical theory and a P_1-P_3-Morley FEM were developed in [11, 12] by Huang, Shi, and Xu. The purpose of this paper is to introduce and analyze a new finite element method for solving general elastic multi-structure

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problems, where displacements on bodies, longitudinal displacements on plates, longitudinal displacements and rotational angles on rods are discretized by conforming linear elements, transverse displacements on rods and plates are discretized respectively by Hermite elements of third order and Zienkiewicz-type elements due to Wang, Shi, and Xu [17], and the discrete generalized displacement fields in individual elastic members are coupled together by some feasible interface conditions. (The method is called simply as a P_1-P_3-NZT FEM in what follows.)

The unique solvability of the method is guaranteed by the Lax-Milgram lemma after deriving generalized Korn's inequalities in some nonconforming element spaces over elastic multi-structures. The optimal error estimate in the energy norm is obtained, which is also validated by some numerical examples.

We end this section by introducing some notation for later uses. Throughout this paper, the symbol "$\lesssim \cdots$" stands for "$\leq C\cdots$" with a generic positive constant C independent of corresponding parameters and functions under considerations, which may take different values in different appearances. We adopt standard notation for Sobolev spaces [1, 14], and use the same index and summation conventions as given in [11,12]. That means, Latin indices i, j, l take their values in the set $\{1, 2, 3\}$, while the capital Latin indices I, J, L (resp. K) take their values in the set $\{1, 2\}$ (resp. $\{2, 3\}$). The summation is implied when a Latin index (or a capital Latin index) appears exactly two times. For a sum $\sum_{t\in\Lambda} a_t$, if a_t is not defined for some $t_0 \in \Lambda$, a_{t_0} is taken to be zero automatically.

2 The mathematical model and the P_1-P_3-NZT FEM

Let there be given N_3 body members $\Omega^3 := \{\alpha_1, \cdots, \alpha_{N_3}\}$, N_2 plate members $\Omega^2 := \{\beta_1, \cdots, \beta_{N_2}\}$, and N_1 rod (beam) members $\Omega^1 := \{\gamma_1, \cdots, \gamma_{N_1}\}$. They are rigidly connected to form an elastic multi-structure [9, 12]:

$$\Omega = \{\alpha_1, \cdots, \alpha_{N_3}; \beta_1, \cdots, \beta_{N_2}; \gamma_1, \cdots, \gamma_{N_1}\}.$$

As in [9, 11], assume that Ω fulfills the following four conditions:

- Each body member α is a bounded polyhedron and each plate member β is a bounded polygon.
- Ω is geometrically connected in the sense that for any two points in Ω, one can connect them by a continuous path consisting of a finite number of line segments each of which belongs to some elastic member in Ω.
- For any two adjacent elastic members $\mathcal{A}$ and $\mathcal{B}$, the dimension of the intersection $\bar{\mathcal{A}} \cap \bar{\mathcal{B}}$ can only differ from the dimensions of these two members by one dimension at most; for example, a body member can only have body or plate members as its adjacent elastic members.
- Ω is geometrically conforming in the sense that if $\mathcal{A}$ and $\mathcal{B}$ are two adjacent elastic members in Ω with the same dimension, then $\partial\mathcal{A} \cap \partial\mathcal{B}$ should be the common boundary of $\mathcal{A}$ and $\mathcal{B}$.

It is mentioned that if an elastic multi-structure does not satisfy the last two conditions, one may transform it into a new one satisfying such conditions by adding or changing some individual elastic members. We refer to [9] for details along this line.

Denote all proper boundary area elements of bodies by

$$\Gamma^2 := \{\beta_{N_2+1}, \cdots, \beta_{N_2'}\} = \Gamma_1^2 \cup \Gamma_2^2,\ \Gamma_1^2 := \{\beta_{N_2+1}, \cdots, \beta_{N_2+M_2}\},\ \Gamma_2^2 := \{\beta_{N_2+M_2+1}, \cdots, \beta_{N_2'}\},$$

where Γ_1^2 consists of all external proper boundary area elements while Γ_2^2 consists of all interfaces of bodies. Analogously, denote all proper boundary line segments of plates by

$$\Gamma^1 := \{\gamma_{N_1+1}, \cdots, \gamma_{N_1'}\} = \Gamma_1^1 \cup \Gamma_2^1,\ \Gamma_1^1 := \{\gamma_{N_1+1}, \cdots, \gamma_{N_1+M_1}\},\ \Gamma_2^1 := \{\gamma_{N_1+M_1+1}, \cdots, \gamma_{N_1'}\},$$

where Γ_1^1 consists of all external boundary lines while Γ_2^1 consists of all interfaces of plates. Denote all boundary points of rods by $\Gamma^0 := \{\delta_1, \cdots, \delta_{N_0}\}$, and all corner points of proper boundaries of plates by $\Gamma_3^0 := \{\delta_{N_0+1}, \cdots, \delta_{N_0'}\}$ (except those in Γ^0). Let $\Gamma^0 = \Gamma_1^0 \cup \Gamma_2^0$ with

$$\Gamma_1^0 := \{\delta_1, \cdots, \delta_{M_0}\},\ \Gamma_2^0 := \{\delta_{M_0+1}, \cdots, \delta_{N_0}\}.$$

Here Γ_1^0 consists of all external boundary points while Γ_2^0 consists of all common boundary points. An element of Ω^3, $\Omega^2 \cup \Gamma^2$, $\Omega^1 \cup \Gamma^1$, and $\Gamma^0 \cup \Gamma_3^0$ is called respectively a body, area, line, and point element.

Introduce a right-handed orthogonal system (x_1, x_2, x_3) in the space R^3, whose orthonormal basis vectors are denoted by $\{\boldsymbol{e}_i\}_{i=1}^3$. With each elastic member in ω, we associate a local right-handed coordinate system $(x_1^\omega, x_2^\omega, x_3^\omega)$ as follows. ($\{\boldsymbol{e}_i^\omega\}_{i=1}^3$ represent the related orthonormal basis vectors.) For a body member $\alpha \in \Omega^3$, its local coordinate system is chosen as the global system (x_1, x_2, x_3), and let $\boldsymbol{n}^\alpha$ be the unit outward normal to the boundary $\partial\alpha$ of α. For a plate member $\beta \in \Omega^2$, x_1^β and x_2^β are its longitudinal directions, and x_3^β the transverse direction. Moreover, along the boundary $\partial\beta$ of β, a unit tangent vector $\boldsymbol{t}^\beta$ is selected such that $\{\boldsymbol{n}^\beta, \boldsymbol{t}^\beta, \boldsymbol{e}_3^\beta\}$ forms a right-handed coordinate system, where $\boldsymbol{n}^\beta$ denotes the unit outward normal to $\partial\beta$ in the longitudinal plane, and $\boldsymbol{e}_3^\beta$ the unit transverse vector of β. It is noted that similar notation is also used for a subdomain of α or β in what follows. For a rod line element $\gamma \in \Omega^1$, x_1^γ is the longitudinal direction, x_2^γ and x_3^γ are the transverse directions, and the origin of the local coordinates is located at an endpoint of γ. For a line element $\gamma \in \Gamma^1$, let $\boldsymbol{e}_1^\gamma$ be a unit vector representing the longitudinal direction of γ.

For any two elements $\beta \in \Omega^2 \cap \Gamma^2$ and $\alpha \in \Omega^3$, $\alpha \in \partial^{-1}\beta$ means that β is a boundary element of α. For any two elements $\beta \in \Omega^2$ and $\gamma \in \Omega^1 \cup \Gamma^1$, define

$$\varepsilon(\beta, \gamma) := \begin{cases} 0 & \text{if } \gamma \notin \partial\beta, \\ 1 & \text{if } \gamma \in \partial\beta, \text{ and } \boldsymbol{e}_1^\gamma \text{ and } \boldsymbol{t}^\beta \text{ have the same direction on } \gamma, \\ -1 & \text{if } \gamma \in \partial\beta, \text{ and } \boldsymbol{e}_1^\gamma \text{ and } \boldsymbol{t}^\beta \text{ have the opposite direction on } \gamma. \end{cases}$$

The symbols $\beta \in \partial^{-1}\gamma$, $\gamma \in \partial^{-1}\delta$, $\varepsilon(\alpha, \beta)$ and $\varepsilon(\gamma, \delta)$ are understood in the similar manners.

We impose the clamped conditions on a line element $\gamma_{N_1+1} \in \partial\beta_1$:

$$\boldsymbol{u}^{\beta_1} = \boldsymbol{0},\ \partial_{\boldsymbol{n}^{\beta_1}} u_3^{\beta_1} = 0 \quad \text{on } \gamma_{N_1+1},$$

and impose the force and moment free conditions on all kinds of proper boundaries of Ω except γ_{N_1+1}. Here $\boldsymbol{u}^{\beta_1}$ denotes the displacement field on the plate member β_1 (see the following descriptions for details). It is noted that all derivations in this paper can be extended naturally to problems with other boundary conditions after some straightforward modifications.

Since Ω is rigidly connected, the admissible space $\boldsymbol{V}$ of generalized displacement fields consists of all functions

$$\boldsymbol{v} := \left\{\{\boldsymbol{v}^{\alpha}\}_{\alpha\in\Omega^3},\, \{\boldsymbol{v}^{\beta}\}_{\beta\in\Omega^2},\, \{\boldsymbol{v}^{\gamma}\}_{\gamma\in\Omega^1},\, \{v_4^{\gamma}\}_{\gamma\in\Omega^1}\right\}$$

in $\prod_{\alpha\in\Omega^3}\boldsymbol{W}(\alpha)\times\prod_{\beta\in\Omega^2}\boldsymbol{W}(\beta)\times\prod_{\gamma\in\Omega^1}\boldsymbol{W}(\gamma)\times\prod_{\gamma\in\Omega^1}H^1(\gamma)$ which fulfill the following interface conditions [9,12]:

$$\boldsymbol{v}^{\alpha}=\boldsymbol{v}^{\alpha'} \quad \text{on } \beta\ \forall\,\alpha,\alpha'\in\partial^{-1}\beta\ \forall\,\beta\in\Gamma_2^2; \tag{2.1}$$

$$\boldsymbol{v}^{\beta}=\boldsymbol{v}^{\beta'},\quad \varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^{\beta}}v_3^{\beta}=\varepsilon(\beta',\gamma)\partial_{\boldsymbol{n}^{\beta'}}v_3^{\beta'} \quad \text{on } \gamma\ \forall\,\beta,\beta'\in\partial^{-1}\gamma\ \forall\,\gamma\in\Gamma_2^1; \tag{2.2}$$

$$v_i^{\gamma}\boldsymbol{e}_i^{\gamma}=v_i^{\gamma'}\boldsymbol{e}_i^{\gamma'},\quad v_{i+3}^{\gamma}\boldsymbol{e}_i^{\gamma}=v_{i+3}^{\gamma'}\boldsymbol{e}_i^{\gamma'} \quad \text{on } \delta\ \forall\,\gamma,\gamma'\in\partial^{-1}\delta\cap\Omega^1\ \forall\,\delta\in\Gamma_2^0, \tag{2.3}$$

where $v_5^{\gamma}:=-dv_3^{\gamma}/dx_1^{\gamma}$, $v_6^{\gamma}:=dv_2^{\gamma}/dx_1^{\gamma}$;

$$\boldsymbol{v}^{\alpha}=\boldsymbol{v}^{\beta} \quad \text{on } \beta\ \forall\,\alpha\in\partial^{-1}\beta\ \forall\,\beta\in\Omega^2; \tag{2.4}$$

$$\boldsymbol{v}^{\beta}=\boldsymbol{v}^{\gamma},\quad -\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^{\beta}}v_3^{\beta}=v_4^{\gamma} \quad \text{on } \gamma\ \forall\,\beta\in\partial^{-1}\gamma,\ \forall\,\gamma\in\Omega^1. \tag{2.5}$$

Here

$$\boldsymbol{v}^{\alpha}:=v_i^{\alpha}\boldsymbol{e}_i^{\alpha},\quad \boldsymbol{v}^{\beta}:=v_i^{\beta}\boldsymbol{e}_i^{\beta},\quad \boldsymbol{v}^{\gamma}:=v_i^{\gamma}\boldsymbol{e}_i^{\gamma};$$

$$\boldsymbol{W}(\alpha):=(H^1(\alpha))^3,\ \boldsymbol{W}(\beta):=(H_*^1(\beta))^2\times H_*^2(\beta),\ \boldsymbol{W}(\gamma):=H^1(\gamma)\times(H^2(\gamma))^2,$$

$$H_*^1(\beta_1):=H_0^1(\beta_1;\gamma_{N_1+1}),\ H_*^2(\beta_1):=H_0^2(\beta_1;\gamma_{N_1+1}),$$
$$H_*^1(\beta):=H^1(\beta),\ H_*^2(\beta):=H^2(\beta)\ \text{for each}\ \beta\in\Omega^2\backslash\beta_1.$$

Therefore, under the action of the applied generalized load field

$$\boldsymbol{f} := \left\{\{\boldsymbol{f}^{\alpha}\}_{\alpha\in\Omega^3},\, \{\boldsymbol{f}^{\beta}\}_{\beta\in\Omega^2},\, \{\boldsymbol{f}^{\gamma}\}_{\gamma\in\Omega^1},\, \{f_4^{\gamma}\}_{\gamma\in\Omega^1}\right\},$$

the generalized displacement field of the equilibrium configuration

$$\boldsymbol{u} := \left\{\{\boldsymbol{u}^{\alpha}\}_{\alpha\in\Omega^3},\, \{\boldsymbol{u}^{\beta}\}_{\beta\in\Omega^2},\, \{\boldsymbol{u}^{\gamma}\}_{\gamma\in\Omega^1},\, \{u_4^{\gamma}\}_{\gamma\in\Omega^1}\right\}$$

of Ω is governed by the following problem [9,12]: Find $\boldsymbol{u}\in\boldsymbol{V}$ such that

$$D(\boldsymbol{u},\boldsymbol{v})=F(\boldsymbol{v})\ \ \forall\,\boldsymbol{v}\in\boldsymbol{V}, \tag{2.6}$$

where

$$F(\boldsymbol{v}):=\sum_{\alpha\in\Omega^3}F^{\alpha}(\boldsymbol{v})+\sum_{\beta\in\Omega^2}F^{\beta}(\boldsymbol{v})+\sum_{\gamma\in\Omega^1}F^{\gamma}(\boldsymbol{v}),$$

$$F^{\alpha}(\boldsymbol{v}):=\int_{\alpha}\boldsymbol{f}^{\alpha}\cdot\boldsymbol{v}^{\alpha}d\alpha,\quad F^{\beta}(\boldsymbol{v}):=\int_{\beta}\boldsymbol{f}^{\beta}\cdot\boldsymbol{v}^{\beta}d\beta,\ F^{\gamma}(\boldsymbol{v}):=\int_{\gamma}\boldsymbol{f}^{\gamma}\cdot\boldsymbol{v}^{\gamma}d\gamma+\int_{\gamma}f_4^{\gamma}v_4^{\gamma}d\gamma;$$

moreover, for $\boldsymbol{w}=\left\{\{\boldsymbol{w}^{\alpha}\}_{\alpha\in\Omega^3},\, \{\boldsymbol{w}^{\beta}\}_{\beta\in\Omega^2},\, \{\boldsymbol{w}^{\gamma}\}_{\gamma\in\Omega^1},\, \{w_4^{\gamma}\}_{\gamma\in\Omega^1}\right\}\in\boldsymbol{V}$,

$$D(\boldsymbol{v},\boldsymbol{w}):=\sum_{\alpha\in\Omega^3}D^{\alpha}(\boldsymbol{v},\boldsymbol{w})+\sum_{\beta\in\Omega^2}D^{\beta}(\boldsymbol{v},\boldsymbol{w})+\sum_{\gamma\in\Omega^1}D^{\gamma}(\boldsymbol{v},\boldsymbol{w}),$$

where

$$D^\alpha(\boldsymbol{v},\boldsymbol{w}) := \int_\alpha \sigma_{ij}^\alpha(\boldsymbol{v})\varepsilon_{ij}^\alpha(\boldsymbol{w})d\alpha,$$

$$\varepsilon_{ij}^\alpha(\boldsymbol{v}) := (\partial_i v_j^\alpha + \partial_j v_i^\alpha)/2, \quad \partial_i v_j^\alpha := v_{j,i}^\alpha = \partial v_j^\alpha/\partial x_i^\alpha,$$
$$\sigma_{ij}^\alpha(\boldsymbol{v}) := \tfrac{E_\alpha}{1+\nu_\alpha}\varepsilon_{ij}^\alpha(\boldsymbol{v}) + \tfrac{E_\alpha\nu_\alpha}{(1+\nu_\alpha)(1-2\nu_\alpha)}(\varepsilon_{ll}^\alpha(\boldsymbol{v}))\delta_{ij},\ 1\le i,j\le 3;$$

$$D^\beta(\boldsymbol{v},\boldsymbol{w}) := \int_\beta [\mathcal{Q}_{IJ}^\beta(\boldsymbol{v})\varepsilon_{IJ}^\beta(\boldsymbol{w})d\beta + \mathcal{M}_{IJ}^\beta(\boldsymbol{v})\mathcal{K}_{IJ}^\beta(\boldsymbol{w})]d\beta,$$

$$\varepsilon_{IJ}^\beta(\boldsymbol{v}) := (\partial_I v_J^\beta + \partial_J v_I^\beta)/2,\ \partial_I v_J^\beta := v_{J,I}^\beta = \tfrac{\partial v_J^\beta}{\partial x_I^\beta},$$
$$\mathcal{Q}_{IJ}^\beta(\boldsymbol{v}) := \tfrac{E_\beta h_\beta}{1-\nu_\beta^2}((1-\nu_\beta)\varepsilon_{IJ}^\beta(\boldsymbol{v}) + \nu_\beta(\varepsilon_{LL}^\beta(\boldsymbol{v}))\delta_{IJ}),\ 1\le I,J\le 2,$$

$$\mathcal{K}_{IJ}^\beta(\boldsymbol{v}) := -\partial_{IJ}v_3^\beta = -\tfrac{\partial^2 v_3^\beta}{\partial x_I^\beta \partial x_J^\beta},$$
$$\mathcal{M}_{IJ}^\beta(\boldsymbol{v}) := \tfrac{E_\beta h_\beta^3}{12(1-\nu_\beta^2)}((1-\nu_\beta)\mathcal{K}_{IJ}^\beta(\boldsymbol{v}) + \nu_\beta(\mathcal{K}_{LL}^\beta(\boldsymbol{v}))\delta_{IJ});$$

$$D^\gamma(\boldsymbol{v},\boldsymbol{w}) := \int_\gamma [\mathcal{Q}_1^\gamma(\boldsymbol{v})\varepsilon_{11}^\gamma(\boldsymbol{w}) + \mathcal{M}_i^\gamma(\boldsymbol{v})\mathcal{K}_i^\gamma(\boldsymbol{w})]d\gamma,$$

$$\varepsilon_{11}^\gamma(\boldsymbol{v}) := dv_1^\gamma/dx_1^\gamma, \quad \mathcal{Q}_1^\gamma(\boldsymbol{v}) := E_\gamma A_\gamma \varepsilon_{11}^\gamma(\boldsymbol{v}),$$

$$\mathcal{K}_2^\gamma(\boldsymbol{v}) := -d^2v_3^\gamma/(dx_1^\gamma)^2, \quad \mathcal{K}_3^\gamma(\boldsymbol{v}) := d^2v_2^\gamma/(dx_1^\gamma)^2,$$
$$\mathcal{M}_2^\gamma(\boldsymbol{v}) := E_\gamma I_{22}^\gamma \mathcal{K}_2^\gamma(\boldsymbol{v}) + E_\gamma I_{23}^\gamma \mathcal{K}_3^\gamma(\boldsymbol{v}),$$
$$\mathcal{M}_3^\gamma(\boldsymbol{v}) := E_\gamma I_{32}^\gamma \mathcal{K}_2^\gamma(\boldsymbol{v}) + E_\gamma I_{33}^\gamma \mathcal{K}_3^\gamma(\boldsymbol{v}),$$
$$I_{23}^\gamma = I_{32}^\gamma,$$

$$\mathcal{K}_1^\gamma(\boldsymbol{v}) := dv_4^\gamma/dx_1^\gamma, \quad \mathcal{M}_1^\gamma(\boldsymbol{v}) := \frac{E_\gamma}{2(1+\nu_\gamma)} J_\gamma \mathcal{K}_1^\gamma(\boldsymbol{v}).$$

Here $E_\omega > 0$ and $\nu_\omega \in (0, 1/2)$ denote Young's modulus and Poisson's ratio of the elastic member $\omega = \alpha$, β, γ, respectively; h_β is the thickness of plate β; A_γ is the area of the cross section, $I_{\cdot\cdot}^\gamma$ the moment of inertia of the cross section, and J_γ the geometric torsional rigidity of the cross section; δ_{ij} and δ_{IJ} stand for the usual Kronecker delta.

Following the similar arguments in [11], it is easy to show that problem (2.6) has a unique solution $\boldsymbol{u}$. In what follows, we will always use

$$\boldsymbol{u} := \left\{\{\boldsymbol{u}^\alpha\}_{\alpha\in\Omega^3}, \{\boldsymbol{u}^\beta\}_{\beta\in\Omega^2}, \{\boldsymbol{u}^\gamma\}_{\gamma\in\Omega^1}, \{u_4^\gamma\}_{\gamma\in\Omega^1}\right\}$$

to denote this solution, and assume that for all $\alpha \in \Omega^3$, $\beta \in \Omega^2$ and $\gamma \in \Omega^1$,

$$\boldsymbol{u}^\alpha \in (H^2(\alpha))^3, \boldsymbol{u}^\beta \in (H^2(\beta))^2 \times H^3(\beta), \boldsymbol{u}^\gamma \in H^2(\gamma) \times (H^3(\gamma))^2, u_4^\gamma \in H^2(\gamma),$$
$$\boldsymbol{f}^\alpha \in (L^2(\alpha))^3, \boldsymbol{f}^\beta \in (L^2(\beta))^3, \boldsymbol{f}^\gamma \in (L^2(\gamma))^3, f_4^\gamma \in L^2(\gamma). \tag{2.7}$$

We next introduce a P_1-P_3-NZT FEM for solving problem (2.6). For each $\alpha \in \Omega^3$, let $\mathcal{T}_h^\alpha$ be a shape-regular triangulation of α into open tetrahedrons K^α [4, 6]. Similarly, let $\mathcal{T}_h^\beta := \{K^\beta\}$ and $\mathcal{T}_h^\gamma := \{K^\gamma\}$ be the shape-regular triangulations of plate member $\beta \in \Omega^2$ and rod member $\gamma \in \Omega^1$, respectively. Hence we obtain a total triangulation of Ω,

$$\mathcal{T}_h^{\Omega} := \left\{ \{\mathcal{T}_h^{\alpha}\}_{\alpha\in\Omega^3},\ \{\mathcal{T}_h^{\beta}\}_{\beta\in\Omega^2},\ \{\mathcal{T}_h^{\gamma}\}_{\gamma\in\Omega^1} \right\}.$$

To avoid unnecessary complexity, assume that the mesh sizes of all triangulations for individual elastic members are of the same mesh size h. Moreover, the triangulation $\mathcal{T}_h^{\Omega}$ is matching across interfaces among different geometric elements. For instance, if $\beta \in \Omega^2$ and $\alpha \in \partial^{-1}\beta$, the restriction of the triangulation $\mathcal{T}_h^{\alpha}$ to β is nothing but the triangulation $\mathcal{T}_h^{\beta}$; if $\beta \in \Gamma_2^2$, all the triangulations $\mathcal{T}_h^{\alpha}$ for $\alpha \in \partial^{-1}\beta$ induce the same triangulation on β.

Let $V_h^1(\omega)$ be the conforming P_1 element space associated with the triangulation $\mathcal{T}_h^{\omega}$, where $\omega = \alpha,\ \beta$ or γ is an elastic member of Ω. Let $V_h^{NZT}(\beta)$ and $V_h^H(\gamma)$ be the new Zienkiewicz-type element space and the Hermite element space of three order, respectively [4, 6, 17]. Hence, for each $K^{\beta} \in \mathcal{T}_h^{\beta}$, the local shape function space related to $V_h^{NZT}(\beta)$ is

$$P_{K^{\beta}}^{NZT} := P_2(K^{\beta}) + \mathrm{span}\{q_{ij},\ 1 \le i < j \le 3\},$$

where

$$q_{ij} := \lambda_i^2\lambda_j - \lambda_i\lambda_j^2 + \left(2\left(\lambda_i - \lambda_j\right) + 3\frac{\left(\nabla\lambda_i - \nabla\lambda_j\right)^{\mathsf{T}}\nabla\lambda_k}{\left\|\nabla\lambda_k\right\|^2}\left(2\lambda_k - 1\right) \right)\lambda_1\lambda_2\lambda_3,$$

with $\{\lambda_i\}_{i=1}^3$ the barycentric coordinates of the triangle K^{β}, and the nodal variables are given by

$$\Sigma_{K^{\beta}} := \{v(p_i^{\beta}), \partial_1 v(p_i^{\beta}), \partial_2 v(p_i^{\beta}), 1 \le i \le 3\}. \tag{2.8}$$

For each $K^{\gamma} \in \mathcal{T}_h^{\gamma}$, the local shape function space with respect to $V_h^H(\gamma)$ is $P_3(K^{\gamma})$ equipped with the nodal variables

$$\Sigma_{K^{\gamma}} := \{v(p_I^{\gamma}),\ v'(p_I^{\gamma}), \quad I = 1, 2\}.$$

Here and in what follows, the derivatives are based on the local coordinate system involved, e.g., $\partial_1 v(p_i^{\beta}) := \partial_{x_1^{\beta}} v(p_i^{\beta})$ in the definition (2.8); $P_k(G)$ represents the space of all polynomials with the total degree no more than k over an open set G. The symbol p with or without indices is used to denote a vertex of some individual element of a triangulation. For an area element $\beta \in \Omega^2 \cup \Gamma^2$ (resp. a line element $\gamma \in \Omega^1 \cup \Gamma^1$), $p \in \beta$ (resp. $p \in \gamma$) means that $p \in \bar{\beta}$ (resp. $p \in \bar{\gamma}$) is a vertex of some individual element of the corresponding triangulation.

We introduce the following finite element spaces to describe discrete displacement fields on individual elastic members.

$$\boldsymbol{W}_h(\alpha) := (V_h^1(\alpha))^3 \ \forall\, \alpha \in \Omega^3;$$

$$\boldsymbol{W}_h(\beta) := (V_h^1(\beta))^2 \times V_h^{NZT}(\beta) \ \forall\, \beta \in \Omega^2 \backslash \beta_1,$$
$$\boldsymbol{W}_h(\beta_1) := (V_h^1(\beta_1; \gamma_{N_1+1}))^2 \times V_h^{NZT}(\beta_1; \gamma_{N_1+1}),$$

where

$$V_h^1(\beta_1; \gamma_{N_1+1}) := \{v_h \in V_h^1(\beta_1);\ v_h(p) = 0 \quad \forall\, p \in \gamma_{N_1+1}\}\ ,$$

and

$$V_h^{NZT}(\beta_1; \gamma_{N_1+1}) := \{v_h \in V_h^{NZT}(\beta_1);\ v_h(p) = \partial_I v_h(p) = 0 \quad \forall\, p \in \gamma_{N_1+1}, \quad I = 1, 2\};$$

$$\boldsymbol{W}_h(\gamma) := V_h^1(\gamma) \times (V_h^H(\gamma))^2 \ \forall\, \gamma \in \Omega^1.$$

The discrete rigid conditions related to (2.1)-(2.5) are given below.

$$v_i^{\alpha}(p)\boldsymbol{e}_i^{\alpha} = v_i^{\alpha'}(p)\boldsymbol{e}_i^{\alpha'} \quad \forall\, p \in \beta,\ \forall\, \alpha, \alpha' \in \partial^{-1}\beta\ \forall\, \beta \in \Gamma_2^2; \tag{2.9}$$

$$v_i^{\alpha}(p)\boldsymbol{e}_i^{\alpha} = v_i^{\beta}(p)\boldsymbol{e}_i^{\beta} \quad \forall\, p \in \beta,\ \forall\, \alpha \in \partial^{-1}\beta,\ \forall\, \beta \in \Omega^2; \tag{2.10}$$

for any line element $\gamma \in \Gamma_2^1$ and any two plate members $\beta, \beta' \in \partial^{-1}\gamma$,

$$v_i^{\beta}(p)\boldsymbol{e}_i^{\beta} = v_i^{\beta'}(p)\boldsymbol{e}_i^{\beta'}, \quad \varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^{\beta}} v_3^{\beta}(p) = \varepsilon(\beta',\gamma)\partial_{\boldsymbol{n}^{\beta'}} v_3^{\beta'}(p) \quad \forall\, p \in \gamma; \tag{2.11}$$

for any rod member γ and any plate member $\beta \in \partial^{-1}\gamma$,

$$v_i^{\beta}(p)\boldsymbol{e}_i^{\beta} = v_i^{\gamma}(p)\boldsymbol{e}_i^{\gamma}, \quad -\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^{\beta}} v_3^{\beta}(p) = v_4^{\gamma}(p) \quad \forall\, p \in \gamma; \tag{2.12}$$

$$v_i^{\gamma}(\delta)\boldsymbol{e}_i^{\gamma} = v_i^{\gamma'}(\delta)\boldsymbol{e}_i^{\gamma'},\ v_{i+3}^{\gamma}(\delta)\boldsymbol{e}_i^{\gamma} = v_{i+3}^{\gamma'}(\delta)\boldsymbol{e}_i^{\gamma'}\ \forall\, \gamma, \gamma' \in \partial^{-1}\delta \cap \Omega^1\ \forall\, \delta \in \Gamma_2^0. \tag{2.13}$$

With these notations, we get a total finite element space on Ω,

$$\boldsymbol{V}_h := \left\{ \boldsymbol{v}_h \in \prod_{\alpha\in\Omega^3} \boldsymbol{W}_h(\alpha) \times \prod_{\beta\in\Omega^2} \boldsymbol{W}_h(\beta) \times \prod_{\gamma\in\Omega^1} \boldsymbol{W}_h(\gamma) \times \prod_{\gamma\in\Omega^1} V_h^1(\gamma),\ \boldsymbol{v}_h \text{ satisfies (2.9)-(2.13)} \right\}.$$

Thus, our P_1-P_3-NTZ FEM for solving problem (2.6) is to find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$D_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = F(\boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.14}$$

where

$$\begin{aligned}
D_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &:= \sum_{\alpha\in\Omega^3} D_h^{\alpha}(\boldsymbol{u}_h, \boldsymbol{v}_h) + \sum_{\beta\in\Omega^2} D_h^{\beta}(\boldsymbol{u}_h, \boldsymbol{v}_h) + \sum_{\gamma\in\Omega^1} D_h^{\gamma}(\boldsymbol{u}_h, \boldsymbol{v}_h), \\
D_h^{\alpha}(\boldsymbol{u}_h, \boldsymbol{v}_h) &:= \sum_{K^{\alpha}\in\mathcal{T}_h^{\alpha}} \int_{K^{\alpha}} \sigma_{ij}^{\alpha}(\boldsymbol{u}_h)\, \varepsilon_{ij}^{\alpha}(\boldsymbol{v}_h)\, dK^{\alpha}, \\
D_h^{\beta}(\boldsymbol{u}_h, \boldsymbol{v}_h) &:= \sum_{K^{\beta}\in\mathcal{T}_h^{\beta}} \int_{K^{\beta}} \left[\mathcal{Q}_{IJ}^{\beta}(\boldsymbol{u}_h)\, \varepsilon_{IJ}^{\beta}(\boldsymbol{v}_h) + \mathcal{M}_{IJ}^{\beta}(\boldsymbol{u}_h)\, \mathcal{K}_{IJ}^{\beta}(\boldsymbol{v}_h) \right] dK^{\beta}, \\
D_h^{\gamma}(\boldsymbol{u}_h, \boldsymbol{v}_h) &:= \sum_{K^{\gamma}\in\mathcal{T}_h^{\gamma}} \int_{K^{\gamma}} \left[\mathcal{Q}_1^{\gamma}(\boldsymbol{u}_h)\, \varepsilon_{11}^{\gamma}(\boldsymbol{v}_h) + \mathcal{M}_i^{\gamma}(\boldsymbol{u}_h)\, \mathcal{K}_i^{\gamma}(\boldsymbol{v}_h) \right] dK^{\gamma}.
\end{aligned}$$

We will prove unique solvability of the problem (2.14) and then discuss its error analysis in later sections.

3 Unique solvability of the finite element method

Introduce a broken Sobolev space by

$$\boldsymbol{W} := \left\{ \boldsymbol{v} \in \prod_{\alpha\in\Omega^3} \boldsymbol{W}(\alpha) \times \prod_{\beta\in\Omega^2} \boldsymbol{W}(\beta) \times \prod_{\gamma\in\Omega^1} \boldsymbol{W}(\gamma) \times \prod_{\gamma\in\Omega^1} H^1(\gamma);\ \boldsymbol{v} \text{ satisfies (2.13)} \right\},$$

which is equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}} := \Big\{ \sum_{\alpha\in\Omega^3}\sum_{i=1}^{3}\|v_i^\alpha\|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2}\Big(\sum_{I=1}^{2}\|v_I^\beta\|_{1,\beta}^2 + \|v_3^\beta\|_{2,\beta}^2\Big) + \sum_{\gamma\in\Omega^1}\Big(\|v_1^\gamma\|_{1,\gamma}^2 + \sum_{K=2}^{3}\|v_K^\gamma\|_{2,\gamma}^2 + \|v_4^\gamma\|_{1,\gamma}^2\Big)\Big\}^{1/2}$$

and the seminorm

$$|\boldsymbol{v}|_{\boldsymbol{W}} := \Big\{ \sum_{\alpha\in\Omega^3}\sum_{i=1}^{3}|v_i^\alpha|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2}\Big(\sum_{I=1}^{2}|v_I^\beta|_{1,\beta}^2 + |v_3^\beta|_{2,\beta}^2\Big) + \sum_{\gamma\in\Omega^1}\Big(|v_1^\gamma|_{1,\gamma}^2 + \sum_{K=2}^{3}|v_K^\gamma|_{2,\gamma}^2 + |v_4^\gamma|_{1,\gamma}^2\Big)\Big\}^{1/2}.$$

The estimate below is due to [11].

Lemma 3.1 *For all* $\boldsymbol{v}\in\boldsymbol{W}$,

$$|\boldsymbol{v}|_{\boldsymbol{W}}^2 \le \|\boldsymbol{v}\|_{\boldsymbol{W}}^2 \lesssim \sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3}\|\varepsilon_{ij}^\alpha(\boldsymbol{v})\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2}\big(\|\varepsilon_{IJ}^\beta(\boldsymbol{v})\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v})\|_{0,\beta}^2\big) + \sum_{\gamma\in\Omega^1}\Big(\|\varepsilon_{11}^\gamma(\boldsymbol{v})\|_{0,\gamma}^2 + \sum_{i=1}^{3}\|\mathcal{K}_i^\gamma(\boldsymbol{v})\|_{0,\gamma}^2\Big) + I_\Omega(\boldsymbol{v}), \tag{3.1}$$

where

$$I_\Omega(\boldsymbol{v}) := I_1(\boldsymbol{v}) + I_2(\boldsymbol{v}) + I_3(\boldsymbol{v}) + I_4(\boldsymbol{v}),$$

$$I_1(\boldsymbol{v}) := \sum_{\beta\in\Gamma_2^2}\sum_{\alpha,\alpha'\in\partial^{-1}\beta}\|\boldsymbol{v}^\alpha - \boldsymbol{v}^{\alpha'}\|_{0,\beta}^2,\ I_2(\boldsymbol{v}) := \sum_{\beta\in\Omega_2^2}\sum_{\alpha\in\partial^{-1}\beta}\|\boldsymbol{v}^\alpha - \boldsymbol{v}^\beta\|_{0,\beta}^2,$$

$$\begin{aligned} I_3(\boldsymbol{v}) :&= \sum_{\gamma\in\Gamma_2^1}\sum_{\beta,\beta'\in\partial^{-1}\gamma}\|\boldsymbol{v}^\beta - \boldsymbol{v}^{\beta'}\|_{0,\gamma}^2 + \sum_{\gamma\in\Gamma_2^1}\sum_{\beta,\beta'\in\partial^{-1}\gamma}\|\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta}v_3^\beta - \varepsilon(\beta',\gamma)\partial_{\boldsymbol{n}^{\beta'}}v_3^{\beta'}\|_{0,\gamma}^2 \\ &=: I_{31}(\boldsymbol{v}) + I_{32}(\boldsymbol{v}), \end{aligned}$$

$$\begin{aligned} I_4(\boldsymbol{v}) :&= \sum_{\gamma\in\Omega_2^1}\sum_{\beta\in\partial^{-1}\gamma}\|\boldsymbol{v}^\beta - \boldsymbol{v}^\gamma\|_{0,\gamma}^2 + \sum_{\gamma\in\Omega_2^1}\sum_{\beta,\beta'\in\partial^{-1}\gamma}\|\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta}v_3^\beta + v_4^\gamma\|_{0,\gamma}^2 \\ &=: I_{41}(\boldsymbol{v}) + I_{42}(\boldsymbol{v}), \end{aligned}$$

and Ω_2^1 *and* Ω_2^2 *consist of all rod member connected with plate members and all plate members connected with body members, respectively.*

Denote by $V_h^B(\beta)$ the Bell element space [4,6] with respect to the triangulation $\mathcal{T}_h^\beta$; that means, for each $K^\beta\in\mathcal{T}_h^\beta$ with three vertices $\{p_i^\beta\}_{i=1}^3$, the local shape function space is

$$P_{K^\beta} := \{v\in P_5(K^\beta);\ \partial_{\boldsymbol{n}^{K^\beta}}v\big|_{F^\beta}\in P_3(F^\beta)\quad\forall\, F^\beta\subset\partial K^\beta\}$$

equipped with the nodal variables

$$\Sigma_{K^\beta} := \{v(p_i^\beta),\, \partial_1 v(p_i^\beta),\, \partial_2 v(p_i^\beta),\, \partial_{11} v(p_i^\beta),\, \partial_{12} v(p_i^\beta),\, \partial_{22} v(p_i^\beta), \quad 1 \le i \le 3\}.$$

Define

$$\boldsymbol{W}_h^B(\beta_1) := (V_h^1(\beta_1; \gamma_{N_1+1}))^2 \times V_h^B(\beta_1; \gamma_{N_1+1}),$$

where

$$V_h^B(\beta_1; \gamma_{N_1+1}) := \{v_h \in V_h^B(\beta_1);\ v_h = \partial_{\boldsymbol{n}^{\beta_1}} v_h = 0 \quad \text{on } \gamma_{N_1+1}\},$$

and for each $\beta \in \Omega^2 \backslash \beta_1$,

$$\boldsymbol{W}_h^B(\beta) := (V_h^1(\beta))^2 \times V_h^B(\beta).$$

Therefore, we obtain a finite-dimensional subspace of $\boldsymbol{W}$ given by

$$\boldsymbol{W}_h := \Big\{ \boldsymbol{v}_h \in \prod_{\alpha\in\Omega^3} \boldsymbol{W}_h(\alpha) \times \prod_{\beta\in\Omega^2} \boldsymbol{W}_h^B(\beta) \times \prod_{\gamma\in\Omega^1} \boldsymbol{W}_h(\gamma) \times \prod_{\gamma\in\Omega^1} V_h^1(\gamma);$$
$$\boldsymbol{v}_h \text{ satisfies (2.9)-(2.13)} \Big\}.$$

Lemma 3.2 *For all $\boldsymbol{v}_h \in \boldsymbol{W}_h$,*

$$|\boldsymbol{v}_h|_{\boldsymbol{W}}^2 \lesssim \sum_{\alpha\in\Omega^3} \sum_{i,j=1}^{3} \|\varepsilon_{ij}^\alpha(\boldsymbol{v}_h)\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2} \sum_{I,J=1}^{2} (\|\varepsilon_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2)$$
$$+ \sum_{\gamma\in\Omega^1} \Big(\|\varepsilon_{11}^\gamma(\boldsymbol{v}_h)\|_{0,\gamma}^2 + \sum_{i=1}^{3} \|\mathcal{K}_i^\gamma(\boldsymbol{v}_h)\|_{0,\gamma}^2 \Big). \tag{3.2}$$

Proof. The proof is quite similar to that of Lemma 3.2 in [11]. Because of Lemma 3.1, it suffices to bound $I_\Omega(\boldsymbol{v})$ in (3.1) as desired. By (2.9),

$$I_1(\boldsymbol{v}_h) = 0. \tag{3.3}$$

Introduce interpolation operators $\boldsymbol{\Pi}_h^\beta$ and $\boldsymbol{\Pi}_h^\gamma$ by

$$\boldsymbol{\Pi}_h^\beta \boldsymbol{v}_h^\beta := (I_{1,h}^\beta v_{h,i}^\beta)\boldsymbol{e}_i^\beta \quad \forall\, \boldsymbol{v}_h^\beta \in \boldsymbol{W}_h^B(\beta), \tag{3.4}$$

and

$$\boldsymbol{\Pi}_h^\gamma \boldsymbol{v}_h^\gamma := (I_{1,h}^\gamma v_{h,i}^\gamma)\boldsymbol{e}_i^\gamma \quad \forall\, \boldsymbol{v}_h^\gamma \in \boldsymbol{W}_h(\gamma).$$

Thus, from (2.10), (3.4), and usual error estimates for interpolation operators $I_{1,h}^\beta$ and $I_{1,h}^\gamma$ [6], it follows that

$$I_2(\boldsymbol{v}_h) \le 2 \sum_{\beta\in\Omega_2^2} \|v_{h,3}^\beta - I_{1,h}^\beta\|_{0,\beta}^2 \lesssim h^4 \sum_{\beta\in\Omega^2} \sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2. \tag{3.5}$$

The combination of (2.11) and (3.4) gives

$$\boldsymbol{\Pi}_h^\beta \boldsymbol{v}_h^\beta = \boldsymbol{\Pi}_h^{\beta'} \boldsymbol{v}_h^{\beta'} \ \forall\, \beta, \beta' \in \partial^{-1}\gamma \ \forall\, \gamma \in \Gamma_2^1,$$

from which and error estimate for interpolation operator $\mathbf{\Pi}_h^\beta$ we are led to

$$I_{31}(\boldsymbol{v}_h) \lesssim \sum_{\gamma\in\Gamma_2^1}\sum_{\beta\in\partial^{-1}\gamma} \|\boldsymbol{v}_h^\beta - \mathbf{\Pi}_h^\beta \boldsymbol{v}_h^\beta\|_{0,\gamma}^2 \lesssim \sum_{\gamma\in\Gamma_2^1}\sum_{\beta\in\partial^{-1}\gamma} \|v_{h,3}^\beta - I_{1,h}^\beta v_{h,3}^\beta\|_{0,\infty,\beta}^2$$

$$\lesssim h^2 \sum_{\gamma\in\Gamma_2^1}\sum_{\beta\in\partial^{-1}\gamma} |v_{h,3}^\beta|_{2,\beta}^2 \lesssim h^2 \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2. \tag{3.6}$$

Similarly, from (2.11), (3.4), the mean value theorem, and the local inverse inequality for finite elements, it follows that

$$I_{32}(\boldsymbol{v}_h) \lesssim \sum_{\gamma\in\Gamma_2^1}\sum_{\beta\in\partial^{-1}\gamma}\sum_{F^\beta\subset\gamma} \|\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta - (\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta)(p^{F^\beta})\|_{0,F^\beta}^2$$

$$\lesssim \sum_{\gamma\in\Gamma_2^1}\sum_{\beta\in\partial^{-1}\gamma}\sum_{F^\beta\subset\gamma} h_{K^\beta}^2 |v_{h,3}^\beta|_{2,\infty,K^\beta}^2 \lesssim \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2. \tag{3.7}$$

Here and in what follows, for a point set $\mathcal{E}$, $F^\beta \subset \mathcal{E}$ indicates that F^β is some edge of a triangle K^β in $\mathcal{T}_h^\beta$, and some subset of $\mathcal{E}$ as well.

Arguing as in the above deduction, we also have

$$I_{41}(\boldsymbol{v}_h) \lesssim h^2 \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2 + h^4 \sum_{\gamma\in\Omega_2^1}\sum_{K=2}^{3} \|\mathcal{K}_K^\gamma(\boldsymbol{v}_h)\|_{0,\gamma}^2,$$

$$I_{42}(\boldsymbol{v}_h) \lesssim \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2 + h^2 \sum_{\gamma\in\Omega_2^1} \|\mathcal{K}_1^\gamma(\boldsymbol{v}_h)\|_{0,\gamma}^2.$$

These with (3.1), (3.3), and (3.5)-(3.7) lead to (3.2) directly. □

Next, introduce a connection operator E_h^β from $V_h^{NZT}(\beta)$ into $V_h^B(\beta)$ as follows. For each $v_3^\beta \in V_h^{NZT}(\beta)$, $E_h^\beta v_3^\beta$ is uniquely determined by

$$\begin{cases} (E_h^\beta v_3^\beta)(p) = v_3^\beta(p)\ \forall\, p\in\beta, \\ (\partial_I E_h^\beta v_3^\beta)(p) = (\partial_I v_3^\beta)(p), & 1\le I\le 2,\ \forall\, p\in\beta, \\ (\partial_{IJ} E_h^\beta v_3^\beta)(p) = 0, & 1\le I,J\le 2, \forall\, p\in\beta, \\ (\partial_{\boldsymbol{n}^{K^\beta}} E_h^\beta v_3^\beta)\big|_{F^\beta} \in P_3(F^\beta) & \forall\, F^\beta\subset K^\beta\in\mathcal{T}_h^\beta. \end{cases}$$

Following the arguments for proving Lemma 5.1 in [3], we readily have

Lemma 3.3 *For the connection operator E_h^β given above,*

$$\sum_{K^\beta\in\mathcal{T}_h^\beta} \big|E_h^\beta v_3^\beta\big|_{2,K^\beta}^2 \lesssim \sum_{K^\beta\in\mathcal{T}_h^\beta} \big|v_3^\beta\big|_{2,K^\beta}^2\ \forall\ v_3^\beta \in V_h^{NZT}(\beta).$$

We equip the finite element space $\boldsymbol{V}_h$ with a norm $\|\cdot\|_h$ given by

$$\|\boldsymbol{v}_h\|_h := \Big\{ \sum_{\alpha\in\Omega^3} |\boldsymbol{v}_h^\alpha|_{1,\alpha}^2 + \sum_{\beta\in\Omega^2} |\boldsymbol{v}_h^\beta|_{h,\beta}^2 + \sum_{\gamma\in\Omega^1} |\boldsymbol{v}_h^\gamma|_{h,\gamma}^2 + \sum_{\gamma\in\Omega^1} |v_{h,4}^\gamma|_{1,\gamma}^2 \Big\}^{1/2}$$

for each $\boldsymbol{v}_h = \big\{\{\boldsymbol{v}_h^\alpha\}_{\alpha\in\Omega^3}, \{\boldsymbol{v}_h^\beta\}_{\beta\in\Omega^2}, \{\boldsymbol{v}_h^\gamma\}_{\gamma\in\Omega^1}, \{v_{h,4}^\gamma\}_{\gamma\in\Omega^1}\big\}$. Here

$$|\boldsymbol{v}_h^\beta|_{h,\beta} := \Big\{ \sum_{K^\beta\in\mathcal{T}_h^\beta} \Big(\sum_{I=1}^{2} |v_{h,I}^\beta|_{1,K^\beta}^2 + |v_{h,3}^\beta|_{2,K^\beta}^2 \Big) \Big\}^{1/2}, \ |\boldsymbol{v}_h^\gamma|_{h,\gamma} := \Big\{ |v_{h,1}^\gamma|_{1,\gamma}^2 + \sum_{K=2}^{3} |v_{h,K}^\gamma|_{2,\gamma}^2 \Big\}^{1/2}.$$

It is noted that the above notation will also be used for functions that make it sensible.

Thus, using Lemmas 3.2 and 3.3 and arguing as in the proof of Theorems 1.1-1.2 in [11], we can achieve the following two results directly.

Theorem 3.1 *For each $\boldsymbol{v}_h \in \boldsymbol{V}_h$,*

$$\begin{aligned}\|\boldsymbol{v}_h\|_h^2 \lesssim & \sum_{\alpha\in\Omega^3}\sum_{i,j=1}^{3} \|\varepsilon_{ij}^\alpha(\boldsymbol{v}_h)\|_{0,\alpha}^2 + \sum_{\beta\in\Omega^2}\sum_{I,J=1}^{2} (\|\varepsilon_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2 + \|\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)\|_{0,\beta}^2) \\ & + \sum_{\gamma\in\Omega^1} \Big(\|\varepsilon_{11}^\gamma(\boldsymbol{v}_h)\|_{0,\gamma}^2 + \sum_{i=1}^{3} \|\mathcal{K}_i^\gamma(\boldsymbol{v}_h)\|_{0,\gamma}^2 \Big)\end{aligned}$$

and

$$\|\boldsymbol{v}_h\|_h^2 \lesssim D_h(\boldsymbol{v}_h, \boldsymbol{v}_h).$$

The first estimate can be viewed as generalized Korn's inequalities over the nonconforming finite element space $\boldsymbol{V}_h$.

Theorem 3.2 *The discrete problem (2.14) has a unique solution in $\boldsymbol{V}_h$.*

4 Error analysis

For each $\alpha \in \Omega^3$, let $I_{1,h}^\alpha$ be the usual interpolation operator related to $V_h^1(\alpha)$. Likewise, for each $\beta \in \Omega^2$, let $I_{1,h}^\beta$ and $I_{NZT,h}^\beta$ be the interpolation operators related to $V_h^1(\beta)$ and $V_h^{NZT}(\beta)$, respectively. For each $\gamma \in \Omega^1$, let $I_{1,h}^\gamma$ and $I_{H,h}^\gamma$ be the interpolation operators related to $V_h^1(\gamma)$ and $V_h^H(\gamma)$, respectively. We then define interpolation operators $\boldsymbol{I}_h^\alpha$, $\boldsymbol{I}_h^\beta$, and $\boldsymbol{I}_h^\gamma$ by

$$\boldsymbol{I}_h^\alpha \boldsymbol{v}^\alpha := (I_{1,h}^\alpha v_i^\alpha)\boldsymbol{e}_i^\alpha \quad \forall\, \boldsymbol{v}^\alpha \in (H^2(\alpha))^3,$$

$$\boldsymbol{I}_h^\beta \boldsymbol{v}^\beta := (I_{1,h}^\beta v_I^\beta)\boldsymbol{e}_I^\beta + (I_{NZT,h}^\beta v_3^\beta)\boldsymbol{e}_3^\beta \quad \forall\, \boldsymbol{v}^\beta \in (H^2(\beta))^2 \times H^3(\beta),$$

and

$$\boldsymbol{I}_h^\gamma \boldsymbol{v}^\gamma := (I_{1,h}^\gamma v_1^\gamma)\boldsymbol{e}_1^\gamma + (I_{H,h}^\gamma v_K^\gamma)\boldsymbol{e}_K^\gamma \quad \forall\, \boldsymbol{v}^\gamma \in H^2(\gamma) \times (H^3(\gamma))^2,$$

which induce a global interpolation operator $\boldsymbol{I}_h$ as follows.

$$(\boldsymbol{I}_h \boldsymbol{v})^\alpha := \boldsymbol{I}_h^\alpha \boldsymbol{v}^\alpha \quad \text{on } \alpha \quad \forall\ \boldsymbol{v}^\alpha \in (H^2(\alpha))^3, \forall\, \alpha \in \Omega^3,$$

$$(\boldsymbol{I}_h \boldsymbol{v})^\beta := \boldsymbol{I}_h^\beta \boldsymbol{v}^\beta \quad \text{on } \beta \quad \forall\ \boldsymbol{v}^\beta \in (H^2(\beta))^2 \times H^3(\beta) \forall\, \beta \in \Omega^2,$$

$$(\boldsymbol{I}_h \boldsymbol{v})^\gamma := \{\boldsymbol{I}_h^\gamma \boldsymbol{v}^\gamma, I_{1,h}^\gamma v_4^\gamma\} \quad \text{on } \gamma \quad \forall\ \{\boldsymbol{v}^\gamma,\ v_4^\gamma\} \in H^2(\gamma) \times (H^3(\gamma))^2 \times H^2(\gamma) \forall\, \gamma \in \Omega^1.$$

We have by the error estimates [4, 6, 17] for interpolation operators $I_{1,h}^\alpha$, $I_{1,h}^\beta$, $I_{NZT,h}^\beta$, $I_{1,h}^\gamma$, and $I_{H,h}^\gamma$ that

Lemma 4.1 *For the interpolation operator $\boldsymbol{I}_h$ mentioned above,*

$$\|\boldsymbol{u}-\boldsymbol{I}_h\boldsymbol{u}\|_h \lesssim h\Big\{\sum_{\alpha\in\Omega^3}\sum_{i=1}^{3}|u_i^\alpha|_{2,\alpha}^2+\sum_{\beta\in\Omega^2}\Big(\sum_{I=1}^{2}|u_I^\beta|_{2,\beta}^2+|u_3^\beta|_{3,\beta}^2\Big)+\sum_{\gamma\in\Omega^1}\Big(|u_1^\gamma|_{2,\gamma}^2+\sum_{K=2}^{3}|u_K^\gamma|_{3,\gamma}^2+|u_4^\gamma|_{2,\gamma}^2\Big)\Big\}^{1/2}.$$

Lemma 4.2 presents some equilibrium equations from the variational formulation (2.6), which is contained in [12].

Lemma 4.2 *Let $\boldsymbol{u}\in\boldsymbol{V}$ be the solution of problem (2.6). Assume that the regularity assumption (2.7) holds true. Then*

$$-\sigma_{ij,j}^\alpha(\boldsymbol{u})=f_i^\alpha \quad \text{in } L^2(\alpha) \quad \forall\ \alpha\in\Omega^3, \tag{4.1}$$

$$\sum_{\alpha\in\partial^{-1}\beta}\sigma_{ij}^\alpha(\boldsymbol{u})n_j^\alpha\boldsymbol{e}_i^\alpha=\boldsymbol{0} \quad \text{in } (H^{1/2}(\beta))^3 \quad \forall\ \beta\in\Gamma^2, \tag{4.2}$$

$$-\mathcal{M}_{1,1}^\gamma(\boldsymbol{u})+\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\mathcal{M}_{\boldsymbol{nn}}^\beta(\boldsymbol{u})=f_4^\gamma \quad \text{in } L^2(\gamma) \quad \forall\ \gamma\in\Omega^1, \tag{4.3}$$

$$\sum_{\beta\in\partial^{-1}\gamma}\varepsilon(\beta,\gamma)\mathcal{M}_{\boldsymbol{nn}}^\beta(\boldsymbol{u})=0 \quad \text{in } H^{1/2}(\gamma) \quad \forall\ \gamma\in\Gamma^1\backslash\gamma_{N_1+1}, \tag{4.4}$$

$$\sum_{\gamma\in\partial^{-1}\delta}\varepsilon(\gamma,\delta)\mathcal{M}_i^\gamma(\boldsymbol{u})(\delta)\boldsymbol{e}_i^\gamma=\boldsymbol{0} \quad \forall\ \delta\in\Gamma^0. \tag{4.5}$$

The next result is nothing but the second Strang lemma related to the finite element method (2.14), which can be proved in the standard manner [4, 6].

Lemma 4.3 *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of problem (2.6) and the discrete problem (2.14), respectively. Then*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h \lesssim E_a(\boldsymbol{u})+E_c(\boldsymbol{u}), \tag{4.6}$$

where

$$E_a(\boldsymbol{u}):=\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_h,\ E_c(\boldsymbol{u}):=\sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{|D_h(\boldsymbol{u},\boldsymbol{v}_h)-F(\boldsymbol{v}_h)|}{\|\boldsymbol{v}_h\|_h}.$$

Now, we are ready to establish error estimates for the finite element method (2.14).

Theorem 4.1 *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of problem (2.6) and the discrete problem (2.14), respectively. Assume that (2.7) holds true. Then*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_h \lesssim h\Big\{\sum_{\alpha\in\Omega^3}\sum_{i=1}^{3}|u_i^\alpha|_{2,\alpha}^2+\sum_{\beta\in\Omega^2}\Big(\sum_{I=1}^{2}|u_I^\beta|_{2,\beta}^2+\|u_3^\beta\|_{3,\beta}^2\Big)+\sum_{\gamma\in\Omega^1}\Big(|u_1^\gamma|_{2,\gamma}^2+\sum_{K=2}^{3}|u_K^\gamma|_{3,\gamma}^2+|u_4^\gamma|_{2,\gamma}^2\Big)+\sum_{\beta\in\Omega^2}h^2\|f_3^\beta\|_{0,\beta}^2+\sum_{\gamma\in\Omega_2^1}\Big(\sum_{K=2}^{3}h^2\|f_K^\gamma\|_{0,\gamma}^2+\|f_4^\gamma\|_{0,\gamma}^2\Big)\Big\}^{1/2}. \tag{4.7}$$

Proof. The proof is based mainly on the arguments employed for getting Theorem 1.3 in [11]. At first, from the definition of $\boldsymbol{I}_h$, it is easy to check that $\boldsymbol{I}_h\boldsymbol{u} \in \boldsymbol{V}_h$, so Lemma 4.1 yields

$$E_a(\boldsymbol{u}) \le \|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}\|_h \lesssim h\Big\{ \sum_{\alpha\in\Omega^3}\sum_{i=1}^{3} |u_i^\alpha|_{2,\alpha}^2 + \sum_{\beta\in\Omega^2}\Big(\sum_{I=1}^{2}|u_I^\beta|_{2,\beta}^2 + |u_3^\beta|_{3,\beta}^2\Big) + \sum_{\gamma\in\Omega^1}\Big(|u_1^\gamma|_{2,\gamma}^2 + \sum_{K=2}^{3}|u_K^\gamma|_{3,\gamma}^2 + |u_4^\gamma|_{2,\gamma}^2\Big)\Big\}^{1/2}. \tag{4.8}$$

Hence, it remains to bound $E_c(\boldsymbol{u})$ from Lemma 4.3. For this, we have to use the following integration by parts formulas. For $K^\alpha \in \mathcal{T}_h^\alpha$,

$$\int_{K^\alpha} \sigma_{ij}^\alpha(\boldsymbol{u})\varepsilon_{ij}^\alpha(\boldsymbol{v}_h)dK^\alpha = -\int_{K^\alpha}\sigma_{ij,j}^\alpha(\boldsymbol{u})v_{h,i}^\alpha dK^\alpha + \int_{\partial K^\alpha}\sigma_{ij}^\alpha(\boldsymbol{u})n_j^{K^\alpha}v_{h,i}^\alpha ds^\alpha;$$

for $K^\beta \in \mathcal{T}_h^\beta$,

$$\begin{aligned}
\int_{K^\beta}\mathcal{Q}_{IJ}^\beta(\boldsymbol{u})\varepsilon_{IJ}^\beta(\boldsymbol{v}_h)dK^\beta &= -\int_{K^\beta}\mathcal{Q}_{IJ,J}^\beta(\boldsymbol{u})v_{h,I}^\beta dK^\beta + \int_{\partial K^\beta}\mathcal{Q}_{IJ}^\beta(\boldsymbol{u})n_J^{K^\beta}v_{h,I}^\beta ds^\beta,\\
\int_{K^\beta}\mathcal{M}_{IJ}^\beta(\boldsymbol{u})\mathcal{K}_{IJ}^\beta(\boldsymbol{v}_h)dK^\beta &= \int_{K^\beta}\mathcal{M}_{IJ,J}^\beta(\boldsymbol{u})\partial_I v_{h,3}^\beta dK^\beta - \int_{\partial K^\beta}\Big\{\mathcal{M}_{\boldsymbol{nn}}^{K^\beta}(\boldsymbol{u})\partial_{\boldsymbol{n}^{K^\beta}}v_{h,3}^\beta\\
&\quad + \mathcal{M}_{\boldsymbol{nt}}^{K^\beta}(\boldsymbol{u})\partial_{\boldsymbol{t}^{K^\beta}}v_{h,3}^\beta\Big\}ds^\beta,
\end{aligned}$$

where $\boldsymbol{n}^{K^\beta} := n_I^{K^\beta}\boldsymbol{e}_I^\beta$, $\boldsymbol{t}^{K^\beta} := t_I^{K^\beta}\boldsymbol{e}_I^\beta$, and

$$\mathcal{M}_{\boldsymbol{nn}}^{K^\beta}(\boldsymbol{u}) := \mathcal{M}_{IJ}^\beta(\boldsymbol{u})n_I^{K^\beta}n_J^{K^\beta},\ \mathcal{M}_{\boldsymbol{nt}}^{K^\beta}(\boldsymbol{u}) := \mathcal{M}_{IJ}^\beta(\boldsymbol{u})n_I^{K^\beta}t_J^{K^\beta};$$

for $K^\gamma \in \mathcal{T}_h^\gamma$,

$$\begin{aligned}
\int_{K^\gamma}\mathcal{Q}_1^\gamma(\boldsymbol{u})\varepsilon_{11}^\gamma(\boldsymbol{v}_h)dK^\gamma &= -\int_{K^\gamma}\mathcal{Q}_1^\gamma(\boldsymbol{u})v_{h,1}^\gamma dK^\gamma + \sum_{\delta\in\partial K^\gamma}\varepsilon(K^\gamma,\delta)[\mathcal{Q}_1^\gamma(\boldsymbol{u})v_{h,1}^\gamma](\delta),\\
\int_{K^\gamma}\mathcal{M}_K^\gamma(\boldsymbol{u})\mathcal{K}_K^\gamma(\boldsymbol{v}_h)dK^\gamma &= \int_{K^\gamma}\mathcal{Q}_K^\gamma(\boldsymbol{u})(v_{h,K}^\gamma)'dK^\gamma + \sum_{\delta\in\partial K^\gamma}\varepsilon(K^\gamma,\delta)[\mathcal{M}_K^\gamma(\boldsymbol{u})v_{h,K+3}^\gamma](\delta),
\end{aligned}$$

and

$$\int_{K^\gamma}\mathcal{M}_1^\gamma(\boldsymbol{u})\mathcal{K}_1^\gamma(\boldsymbol{v}_h)dK^\gamma = -\int_{K^\gamma}\mathcal{M}_{1,1}^\gamma(\boldsymbol{u})v_{h,4}^\gamma dK^\gamma + \sum_{\delta\in\partial K^\gamma}\varepsilon(K^\gamma,\delta)[\mathcal{M}_1^\gamma(\boldsymbol{u})v_{h,4}^\gamma](\delta).$$

From the above identities, we have for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ that

$$D_h(\boldsymbol{u},\boldsymbol{v}_h) - F(\boldsymbol{v}_h) = J_1 + J_2 + J_3 + J_4, \tag{4.9}$$

where

$$J_1 := \sum_{\alpha\in\Omega^3}\int_\alpha(-\sigma_{ij,j}^\alpha(\boldsymbol{u}) - f_i^\alpha)v_{h,i}^\alpha d\alpha + \sum_{\beta\in\Gamma^2}\sum_{\alpha\in\partial^{-1}\beta}\sigma_{ij}^\alpha(\boldsymbol{u})n_j^\alpha v_{h,i}^\alpha d\beta,$$

$$J_2 := -\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\sum_{F^\beta\subset\partial K^\beta}\int_{F^\beta}\mathcal{M}^{K^\beta}_{\boldsymbol{nn}}(\boldsymbol{u})\partial_{\boldsymbol{n}^{K^\beta}}v^\beta_{h,3}ds^\beta + \sum_{\gamma\in\Omega^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\int_{K^\gamma}(-\mathcal{M}^\gamma_{1,1}(\boldsymbol{u})-f_4^\gamma)v^\gamma_{h,4}dK^\gamma,$$

$$\begin{aligned}J_3 :=& \sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\Big(-\int_{K^\beta}\mathcal{Q}^\beta_{IJ,J}(\boldsymbol{u})v^\beta_{h,I}dK^\beta+\int_{K^\beta}\mathcal{M}^\beta_{IJ,J}(\boldsymbol{u})\partial_I v^\beta_{h,3}dK^\beta\\ &-\int_{K^\beta}f_i^\beta v^\beta_{h,i}dK^\beta-\sum_{F^\beta\subset\partial K_h^\beta}\int_{F^\beta}\mathcal{M}^{K^\beta}_{\boldsymbol{nt}}(\boldsymbol{u})\partial_{\boldsymbol{t}^{K^\beta}}v^\beta_{h,3}ds^\beta\Big)\\ &+\sum_{\beta\in\Omega^2}\sum_{\alpha\in\partial^{-1}\beta}\int_\beta\sigma^\alpha_{ij}(\boldsymbol{u})n^\alpha_j v^\alpha_{h,i}d\beta+\sum_{\beta\in\Omega^2}\int_{\partial\beta}\mathcal{Q}^\beta_{IJ}(\boldsymbol{u})n^\beta_J v^\beta_{h,I}d\gamma\\ &+\sum_{\gamma\in\Omega^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\Big(\int_{K^\gamma}\mathcal{Q}^\gamma_i(\boldsymbol{u})(v^\gamma_{h,i})'dK^\gamma-\int_{K^\gamma}f_i^\gamma v^\gamma_{h,i}dK^\gamma\Big),\\ J_4 :=& \sum_{\gamma\in\Omega^1}\sum_{\delta\in\partial\gamma}\varepsilon(\gamma,\delta)[\mathcal{M}^\gamma_i(\boldsymbol{u})v^\gamma_{h,i+3}](\delta).\end{aligned}$$

From (4.1) and (4.2), it is easy to see that $J_1 = 0$. As with the term J_2, we first rewrite it as

$$\begin{aligned}J_2 =& \Big\{-\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\sum_{F^\beta\subset\partial K_h^\beta\setminus\partial\beta}\int_{F^\beta}\mathcal{M}^{K^\beta}_{\boldsymbol{nn}}(\boldsymbol{u})\partial_{\boldsymbol{n}^{K^\beta}}v^\beta_{h,3}ds^\beta\Big\}\\ &+\Big\{-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\Gamma^1}\int_{F^\beta}\mathcal{M}^\beta_{\boldsymbol{nn}}(\boldsymbol{u})\partial_{\boldsymbol{n}^\beta}v^\beta_{h,3}ds^\beta\Big\}\\ &+\Big\{-\sum_{\beta\in\Omega^2}\sum_{F^\beta\subset\Omega_2^1}\int_{F^\beta}\mathcal{M}^{K^\beta}_{\boldsymbol{nn}}(\boldsymbol{u})\partial_{\boldsymbol{n}^{K^\beta}}v^\beta_{h,3}ds^\beta\\ &+\sum_{\gamma\in\Omega_2^1}\sum_{K^\gamma\in\mathcal{T}_h^\gamma}\int_{K^\gamma}(-\mathcal{M}^\gamma_{1,1}(\boldsymbol{u})-f_4^\gamma)v^\gamma_{h,4}dK^\gamma\Big\}\\ &+\Big\{\sum_{\gamma\in\Omega_1^1}\int_{K^\gamma}(-\mathcal{M}^\gamma_{1,1}(\boldsymbol{u})-f_4^\gamma)v^\gamma_{h,4}dK^\gamma\Big\}\\ =:& J_{21}+J_{22}+J_{23}+J_{24}.\end{aligned}\tag{4.10}$$

Observe that the integral $\int_{F^\beta}\nabla v^\beta_{h,3}dF^\beta$ is continuous across the interior edge F^β of the triangular subdivision $\mathcal{T}^\beta$ [17]. Therefore,

$$J_{21} = -\sum_{\beta\in\Omega^2}\sum_{K^\beta\in\mathcal{T}_h^\beta}\sum_{F^\beta\subset\partial K_h^\beta\setminus\partial\beta}\int_{F^\beta}R_0^{F^\beta}\big(\mathcal{M}^{K^\beta}_{\boldsymbol{nn}}(\boldsymbol{u})\big)R_0^{F^\beta}\big(\partial_{\boldsymbol{n}^{K^\beta}}v^\beta_{h,3}\big)ds^\beta.$$

Here and in what follow, $R_0^{F^\beta} := I - P_0^{F^\beta}$, with $P_0^{F^\beta}$ the L^2-orthogonal projection operator from $L^2(F^\beta)$ onto the space of constants on F^β. By the trace theorem and the scaling argument, it holds that [16]

$$\|R_0^{F^\beta} g\|_{0,F^\beta} \lesssim h_{K^\beta}^{1/2} |g|_{1,K^\beta} \ \forall\, g \in H^1(K^\beta), \tag{4.11}$$

where h_{K^β} denotes the diameter of a triangle K^β.

Using (4.11) and the Cauchy-Schwarz inequality we easily have

$$|J_{21}| \lesssim h \Big(\sum_{\beta\in\Omega^2} |u_3^\beta|_{3,\beta}^2 \Big)^{1/2} \|\boldsymbol{v}_h\|_h. \tag{4.12}$$

On the other hand, from the identity (7) in [17] it follows that

$$P_0^{F^\beta}(\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta) = \frac{1}{2}(\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta(p_1^{F^\beta}) + \varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta(p_2^{F^\beta})), \tag{4.13}$$

where $p_1^{F^\beta}$ and $p_2^{F^\beta}$ are two endpoints of the edge F^β. In addition, we have by the definition of $\boldsymbol{V}_h$ that

$$\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta(p) = \varepsilon(\beta',\gamma)\partial_{\boldsymbol{n}^{\beta'}} v_{h,3}^{\beta'}(p) \ \forall\, p \in \gamma \in \Gamma_2^1 \ \forall\, \beta,\beta' \in \partial^{-1}\gamma, \tag{4.14}$$

$$\partial_{\boldsymbol{n}^{\beta_1}} v_{h,3}^{\beta_1}(p) = 0 \ \forall\, p \in \gamma_{N_1+1}. \tag{4.15}$$

Hence, it follows from (4.13)-(4.15) and the equilibrium equation (4.4) that

$$J_{22} = -\sum_{\beta\in\Omega^2} \sum_{F^\beta\subset\Gamma^1} \int_{F^\beta} R_0^{F^\beta}\big(\varepsilon(\beta,\gamma)\mathcal{M}_{\boldsymbol{nn}}(\boldsymbol{u})\big) R_0^{F^\beta}\big(\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta\big) ds^\beta,$$

from which and employing the similar argument for getting (4.12) we find

$$|J_{22}| \lesssim h \Big(\sum_{\beta\in\Omega^2} |u_3^\beta|_{3,\beta}^2 \Big)^{1/2} \|\boldsymbol{v}_h\|_h \tag{4.16}$$

For the estimate of J_{23}, we have by the interface condition (2.12) and relation (4.13) that

$$-P_0^{F^\beta}(\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta) = \frac{1}{2}(v_{h,4}^\gamma(p_1^{F^\beta}) + v_{h,4}^\gamma(p_2^{F^\beta})) \ \forall\, K^\gamma = F^\beta \subset \partial\beta \cap \Omega_2^1.$$

This with the equilibrium equation (4.3) implies

$$\begin{aligned}
J_{23} =& -\sum_{\beta\in\Omega^2} \sum_{F^\beta\subset\Omega_2^1} \int_{F^\beta} \varepsilon(\beta,\gamma)\mathcal{M}_{\boldsymbol{nn}}^{K^\beta}(\boldsymbol{u})(\partial_{\boldsymbol{n}^{K^\beta}} v_{h,3}^\beta ds^\beta - P_0^{F^\beta}(\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta)) ds^\beta \\
&+ \sum_{\gamma\in\Omega_2^1} \sum_{K^\gamma\in\mathcal{T}_h^\gamma} \int_{K^\gamma} (-\mathcal{M}_{1,1}^\gamma(\boldsymbol{u}) - f_4^\gamma)(v_{h,4}^\gamma - 1/2(v_{h,4}^\gamma(p_1^{F^\beta}) + v_{h,4}^\gamma(p_2^{F^\beta}))) dK^\gamma \\
=& -\sum_{\beta\in\Omega^2} \sum_{F^\beta\subset\gamma\in\Omega_2^1} \int_{F^\beta} R_0^{F^\beta}\big(\varepsilon(\beta,\gamma)\mathcal{M}_{\boldsymbol{nn}}^\beta(\boldsymbol{u})\big) R_0^{F^\beta}\big(\varepsilon(\beta,\gamma)\partial_{\boldsymbol{n}^\beta} v_{h,3}^\beta\big) ds^\beta \\
&+ \frac{1}{2}\sum_{\gamma\in\Omega_2^1} \sum_{K^\gamma\in\mathcal{T}_h^\gamma} \int_{K^\gamma} (-\mathcal{M}_{1,1}^\gamma(\boldsymbol{u}) - f_4^\gamma)((v_{h,4}^\gamma - v_{h,4}^\gamma(p_1^{F^\beta})) + (v_{h,4}^\gamma - v_{h,4}^\gamma(p_2^{F^\beta}))) dK^\gamma.
\end{aligned} \tag{4.17}$$

Therefore, using the Cauchy-Schwarz inequality and estimate (4.11) we get

$$|J_{23}| \lesssim h\Big\{ \sum_{\beta\in\Omega^2} |u_3^\beta|_{3,\beta}^2 + \sum_{\gamma\in\Omega_2^1} (|u_4^\gamma|_{2,\gamma}^2 + \|f_4^\gamma\|_{0,\gamma}^2)\Big\}^{1/2} \|\boldsymbol{v}_h\|_h. \tag{4.18}$$

For the term J_{24}, we have by the equilibrium equation (4.3) that

$$-\mathcal{M}_{1,1}^\gamma(\boldsymbol{u}) = f_4^\gamma \text{ in } L^2(\gamma)\ \forall\,\gamma\in\Omega_1^1,$$

so $J_{24}=0$. Combining this with (4.10), (4.12), (4.16), and (4.18) we get

$$|J_2| \lesssim h\Big\{ \sum_{\beta\in\Omega^2} |u_3^\beta|_{3,\beta}^2 + \sum_{\gamma\in\Omega_2^1} (|u_4^\gamma|_{2,\gamma}^2 + \|f_4^\gamma\|_{0,\gamma}^2)\Big\}^{1/2} \|\boldsymbol{v}_h\|_h. \tag{4.19}$$

Furthermore, since each function in $\boldsymbol{V}_h$ is continuous at all vertices of the triangulation $\mathcal{T}_h^\Omega$ and vanishes at all vertices on γ_{N_1+1}, using the equilibrium equation (4.5) and following the same arguments for bounding the terms I_3 and I_4 in [11], we have $J_4=0$ and

$$|J_3| \lesssim h\Big\{ \sum_{\beta\in\Omega^2} (|u_3^\beta|_{3,\beta}^2 + h^2\|f_3^\beta\|_{0,\beta}^2) + \sum_{\gamma\in\Omega_2^1}\sum_{K=2}^{3} (|u_K^\gamma|_{3,\gamma}^2 + h^2\|f_K^\gamma\|_{0,\gamma}^2)\Big\}^{1/2} \|\boldsymbol{v}_h\|_h. \tag{4.20}$$

Using the identity (4.9), the estimates (4.19)-(4.20), and noting the fact that $J_1=J_4=0$ shown before, we find

$$\begin{aligned} |E_c(\boldsymbol{u})| \lesssim & h\Big\{ \sum_{\beta\in\Omega^2} (|u_3^\beta|_{3,\beta}^2 + h^2\|f_3^\beta\|_{0,\beta}^2) \\ & + \sum_{\gamma\in\Omega_2^1} \Big(\sum_{K=2}^{3} (|u_K^\gamma|_{3,\gamma}^2 + h^2\|f_K^\gamma\|_{0,\gamma}^2) + |u_4^\gamma|_{2,\gamma}^2 + \|f_4^\gamma\|_{0,\gamma}^2 \Big) \Big\}^{1/2}, \end{aligned}$$

and with (4.6)-(4.8) we get the error estimate (4.7). The proof of Theorem 4.1 is completed. □

5 Numerical examples

In this section, we want to solve an elastic plate-plate problem by means of the finite element method (2.14). Consider two elastic plate members $\beta_1 := \{0\}\times(-1,1)\times(0,1)$ and $\beta_2 := (0,1)\times(-1,1)\times\{0\}$, which are coupled together along the common edge $\gamma_1 := \{0\}\times(-1,1)\times\{0\}$ to form an elastic multi-structure Ω (see Figure 1). We choose the local coordinate systems as

$$(x_1^{\beta_1}, x_2^{\beta_1}, x_3^{\beta_1}) := (x_2, x_3, x_1),\ (x_1^{\beta_2}, x_2^{\beta_2}, x_3^{\beta_2}) := (x_1, x_2, x_3),$$

where (x_1, x_2, x_3) is the global coordinate system in R^3.

The displacement fields $\boldsymbol{u}$ on β_1 and β_2 are given respectively by

$$\begin{cases} u_1^{\beta_1} := (1-x_2^2)^2(1-x_3^2)^2, \\ u_2^{\beta_1} := (1-x_2^2)^2(1-x_3^2)^2, \\ u_3^{\beta_1} := (1-x_2^2)^2(1-x_3^2)^2, \end{cases}$$

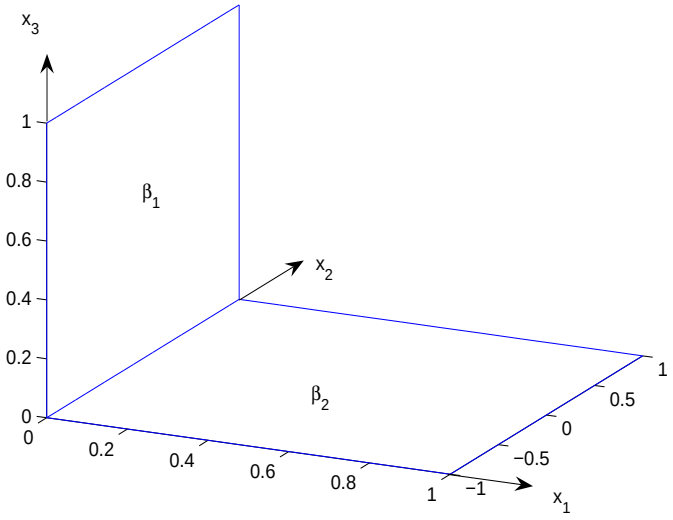


Figure 1: The geometric domain of the plate-plate problem.

with $-1 \le x_2 \le 1$, $0 \le x_3 \le 1$, and

$$\begin{cases} u_1^{\beta_2} := (1-x_1^2)^2(1-x_2^2)^2, \\ u_2^{\beta_2} := (1-x_1^2)^2(1-x_2^2)^2, \\ u_3^{\beta_2} := (1-x_1^2)^2(1-x_2^2)^2, \end{cases}$$

with $0 \le x_1 \le 1$, $-1 \le x_2 \le 1$. The corresponding material parameters are given respectively as $h_{\beta_1} = h_{\beta_2} = 0.02$, $\nu_{\beta_1} = \nu_{\beta_2} = 0.3$, and $E_{\beta_1} = E_{\beta_2} = \frac{12(1-\nu_{\beta_1}^2)}{h_{\beta_1}}$. It is obvious that $\boldsymbol{u}^{\beta_1}$ and $\boldsymbol{u}^{\beta_2}$ satisfy condition (2.2). Hence, $\boldsymbol{u} := (\boldsymbol{u}^{\beta_1}, \boldsymbol{u}^{\beta_2})$ is the unique function characterized by the variational formulation

$$D(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V},$$

where

$$\begin{aligned} \boldsymbol{V} := \{ & \boldsymbol{v} = (\boldsymbol{v}^{\beta_1}, \boldsymbol{v}^{\beta_2});\ \boldsymbol{v}^{\beta_I} \in H^2(\beta_I))^2 \times H^3(\beta_I), \\ & v_i^{\beta_1} = v_i^{\beta_2},\ \varepsilon(\beta_1,\gamma)\partial_{\boldsymbol{n}^{\beta_1}} v_3^{\beta_1} = \varepsilon(\beta_2,\gamma)\partial_{\boldsymbol{n}^{\beta_2}} v_3^{\beta_2} \text{ on } \gamma_1,\ 1 \le i \le 3\}, \end{aligned}$$

$$D(\boldsymbol{v}, \boldsymbol{w}) := \sum_I D^{\beta_I}(\boldsymbol{v}^{\beta_I}, \boldsymbol{w}^{\beta_I}) \qquad \forall\, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V},$$

and the functions required to form the right term $F(\boldsymbol{v})$ can be obtained by the equilibrium equations given in [12], since the exact solution $\boldsymbol{u}$ is available.

Next, we use the MATLAB commands: *initmesh* and *refinemesh* to form a family of quasi-uniform finite element triangulations $\mathcal{T}_h^{\Omega} := \{\mathcal{T}_h^{\beta_1}, \mathcal{T}_h^{\beta_2}\}$ of Ω, whose mesh sizes are denoted by h. For instance, see the Figure 2 for the generated meshes over β_1. Based on the triangulations obtained, we can solve the last problem via the finite element method (2.14), to get the approximate solutions $\boldsymbol{u}_h = (\boldsymbol{u}_h^{\beta_1}, \boldsymbol{u}_h^{\beta_2})$.

h	Number of unknowns	Error_Plate_1	Error_Plate_2	R_h
0.1118	1267	2.0387	2.5449	0.2515
0.055902	4767	0.37743	0.37591	0.1190
0.027951	18487	0.049017	0.03688	0.0298
0.013975	70807	0.00063498	0.0083052	0.0294

Table 1. The computational results with different mesh sizes h.

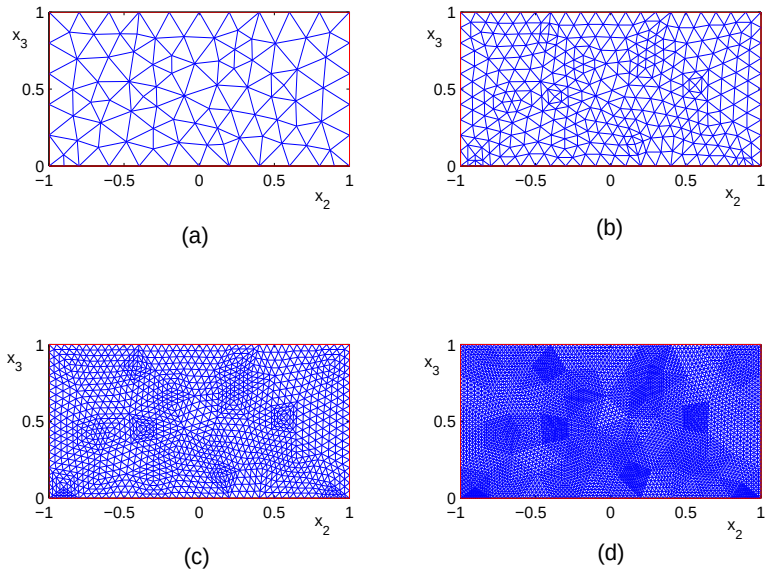


Figure 2: The generated meshes over plate β_1: (a) initial mesh, (b) 1st refinement, (c) 2nd refinement, (d) 3rd refinement.

To test the computation performance of our finite element method, we introduce some quantities as follows.

$$\text{Error_Plate_1} := \max_{1\le i\le 3} \frac{|u_{h,i}^{\beta_1}(\beta_{c,1}) - u_i^{\beta_1}(\beta_{c,1})|}{|u_i^{\beta_1}(\beta_{c,1})|}, \quad \text{Error_Plate_2} := \max_{1\le i\le 3} \frac{|u_{h,i}^{\beta_2}(\beta_{c,2}) - u_i^{\beta_2}(\beta_{c,2})|}{|u_i^{\beta_2}(\beta_{c,2})|},$$

$$R_h := \frac{\|\boldsymbol{I}_h\boldsymbol{u} - \boldsymbol{u}_h\|_h}{h\left\{\sum_{J=1}^{2}\left(\sum_{I=1}^{2}|u_I^{\beta_J}|_{2,\beta_J}^2 + \|u_3^{\beta_J}\|_{3,\beta_J}^2 + h^2\|f_3^{\beta_J}\|_{0,\beta_J}^2\right)\right\}^{1/2}},$$

where $\boldsymbol{I}_h$ denotes the usual nodal interpolation operator given in Section 4, and $\beta_{c,I}$ are the centers of plates β_I $(I = 1, 2)$, respectively.

The computational results with different mesh sizes h are given in Table 1 and the comparison between the numerical solution $\boldsymbol{u}_h$ after the first refinement and the exact solution $\boldsymbol{u}$ is shown in Figure 3, from which we may conclude that our finite element method is efficient in solving the plate-plate problem mentioned above. Moreover, since R_h has an absolute upper bound independent of h, by virtue of the usual estimate for $\boldsymbol{I}_h$ (see Lemma 4.1), we find that the numerical results support the theoretical estimate given in Theorem 4.1.

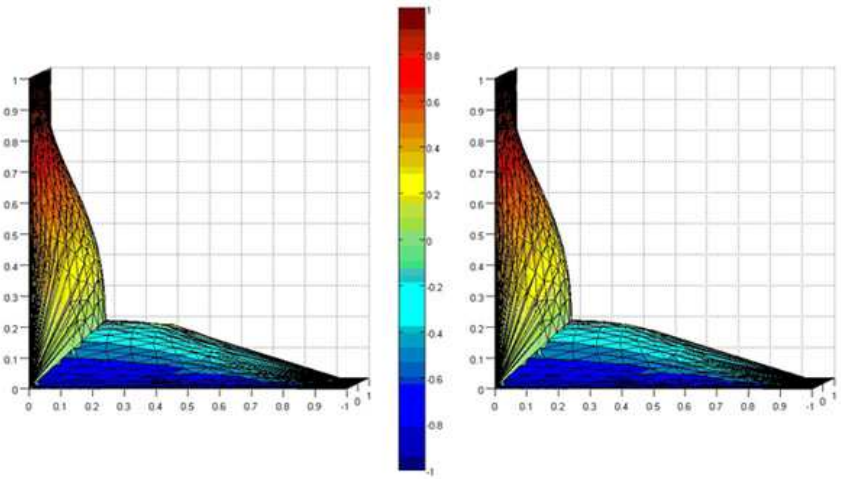


Figure 3: Left: the graph of the numerical solution after the 1st mesh-refinement, Right: the graph of the exact solution.

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A note on factored Fourier series

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Abstract

In this paper a main theorem on $\mid \bar{N}, p_n; \delta \mid_k$ summability factors of Fourier series has been proved. Also some new results have been obtained.

1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [6]).

The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{3}$$

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Keywords and Phrases: Absolute summability, Fourier series.

and it is said to be summable $\mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{4}$$

In the special case $p_n = 1$ for all values of n (resp. $\delta = 0$) $\mid \bar{N}, p_n; \delta \mid_k$ summability is the same as $\mid C, 1; \delta \mid_k$ (resp. $\mid \bar{N}, p_n \mid_k$) summability. Also if we take $k = 1$ and $p_n = \frac{1}{n+1}$ (resp. $k = 1$ and $\delta = 0$), then $\mid \bar{N}, p_n; \delta \mid_k$ summability becomes $\mid R, logn, 1 \mid$ (resp. $\mid \bar{N}, p_n \mid$) summability.

A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer n, where $\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of f(t) is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{5}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} A_n(t). \tag{6}$$

2. Known result. Bor [3] has proved the following theorem concerning the $\mid \bar{N}, p_n \mid_k$ summability factors for Fourier series.

Theorem A. If (λ_n) is a convex sequence such that $\sum p_n \lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty \quad as \quad n \to \infty$, and $\sum_{v=1}^{n} P_v A_v(t) = O(P_n)$, then the series $\sum A_n(t) P_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

Also quite recently Bor [4] has proved the following interesting and more general theorem.

Theorem B. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty \quad as \quad n \to \infty$, and $\sum_{v=1}^{n} P_v A_v(t) = O(P_n)$, then the series $\sum A_n(t) P_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

It should be noted that the conditions on the sequence (λ_n) in Theorem B, are somewhat more general than in Theorem A.

Main result. The aim of this paper is to generalize Theorem B for $\mid \bar{N}, p_n; \delta \mid_k$ summability in the following form.

Theorem. Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (λ_n) be a non-negative and non-increasing sequence such that $\sum p_n\lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty \quad as \quad \text{n}\to \infty$, and $\sum_{v=1}^{n} P_v A_v(t) = O(P_n)$. Furthermore if the conditions

$$\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k} P_v \Delta\lambda_v = O(1) \quad as \quad m \to \infty, \tag{7}$$

$$\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k} p_v\lambda_v = O(1) \quad as \quad m \to \infty, \tag{8}$$

$$\sum_{n=v+1}^{\infty}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}} = O\{(\frac{P_v}{p_v})^{\delta k}\frac{1}{P_v}\}, \tag{9}$$

are satisfied, then the series $\sum A_n(t)\lambda_n P_n$ is summable $\mid \bar{N}, p_n; \delta \mid_k$.

We need the following lemma for the proof of our theorem.

Lemma ([5]). If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty \quad as \quad \text{n}\to \infty$, then $P_n\lambda_n = O(1) \quad \text{as} \quad n \to \infty \quad and \quad \sum P_n\Delta\lambda_n < \infty$.

Proof of the Theorem. Let $T_n(t)$ denotes the $(\bar{N}, p_n)$ means of the series $\sum A_n(t)P_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} A_r(t)P_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})A_v(t)\lambda_v P_v.$$

Then, for $n \geq 1$, we have

$$T_n(t) - T_{n-1}(t) \quad = \quad \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}P_vA_v(t)\lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned} T_n(t) - T_{n-1}(t) &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta(P_{v-1}\lambda_v)\sum_{r=1}^{v} P_rA_r(t) + \frac{p_n}{P_n}\lambda_n\sum_{v=1}^{n} P_vA_v(t) \\ &= O(1)\{\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v - p_v\lambda_v - P_v\lambda_{v+1})P_v\} + O(1)p_n\lambda_n \\ &= O(1)\{\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vP_v\Delta\lambda_v - \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vp_v\lambda_v + p_n\lambda_n\} \\ &= O(1)\{T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t)\}, \quad say. \end{aligned}$$

Since

$$| T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t) |^k \leq 3^k \{| T_{n,1}(t) |^k + | T_{n,2}(t) |^k + | T_{n,3}(t) |^k\},$$

to complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1} | T_{n,r}(t) |^k < \infty, \quad for \quad r = 1, 2, 3. \tag{10}$$

Since

$$\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v \leq P_{n-1} \sum_{v=1}^{n-1} P_v \Delta\lambda_v$$

it follows by the Lemma that

$$\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v \leq \sum_{v=1}^{n-1} P_v \Delta\lambda_v = O(1) \quad as \quad m \to \infty. \tag{11}$$

Hence, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} | T_{n,1}(t) |^k &\leq \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \{\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v\} \times \{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v\}^{k-1}. \\
&= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v \\
&= O(1) \sum_{v=1}^{m} P_v P_v \Delta\lambda_v \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^{\delta k} P_v \Delta\lambda_v = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma. Again

$$\begin{aligned}
\sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k+k-1} | T_{n,2}(t) |^k &\leq \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \{\sum_{v=1}^{n-1} (P_v \lambda_v)^k p_v\} \times \{\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\}^{k-1} \\
&= O(1) \sum_{v=2}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} (P_v \lambda_v)^k p_v \\
&= O(1) \sum_{v=1}^{m} (P_v \lambda_v)^k p_v \sum_{n=v+1}^{m+1} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} (P_v \lambda_v)^k p_v (\frac{P_n}{p_n})^{\delta k} \frac{1}{P_v}
\end{aligned}$$

$$= O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}(P_v\lambda_v)^{k-1}p_v\lambda_v$$
$$= O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}p_v\lambda_v = O(1) \quad as \quad m\to\infty,$$

by virtue of the hypotheses of the Theorem and the Lemma. Finally as in $T_{n,1}(t)$, we have that

$$\sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k+k-1} \mid T_{n,3}(t) \mid^k = \sum_{n=1}^{m}(P_n\lambda_n)^{k-1}(\frac{P_n}{p_n})^{\delta k}p_n\lambda_n$$
$$= O(1)\sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k-1}p_n\lambda_n = O(1) \quad as \quad m\to\infty.$$

Therefore, we get that

$$\sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k+k-1} \mid T_{n,r}(t) \mid^k = O(1) \quad as \quad m\to\infty, for \quad r=1,2,3.$$

This completes the proof of the Theorem.

As special cases of this Theorem, one can obtain the following results.

1. If we take $p_n = 1$ for all values of n and $\delta = 0$, then we get a result concerning the $\mid C,1 \mid_k$ summability factors for Fourier series.

2. If we take $p_n = 1$ for all values of n, then we have a new result concerning the $\mid C,1;\delta \mid_k$ summability factors for Fourier series.

3. If we take $k = 1$, $p_n = 1/(n+1)$ and $\delta = 0$, then we get another new result related to $\mid R, logn, 1 \mid$ summability factors of Fourier series.

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On Pointwise Approximation Of Gauss-Weierstrass Operators

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Abstract The metric form $\Omega_x(f,\lambda)$, which was introduced by Zeng and Cheng in [8], plays an important role in the estimate of pointwise approximation of functions which have left limit $f(x-)$ and right limit $f(x+)$ at given point x. In this paper , Using the metric form $\Omega_x(f,\lambda)$ and the Bojanic - Khan - Cheng's method combining with analysis techniques , we obtain the asymptotic estimates on the rates of pointwise approximation of Gauss-Weierstrass operators on two classes of functions with certain growth condition.

Keywords: Gauss-Weierstrass operators, Metric form $\Omega_x(f,\lambda)$, Functions with certain growth condition, Lebesgue-Stieltjes integration.

Classification(MSC 2000): 41A30, 41A35, 41A36, 41A60

1 Introduction

Let $\{U_{n,x}\}$ be a sequence of normal distribution random variables with parameters $(n,x,\sigma)(n\geq 1,\sigma>0)$, and the probability density of $U_{n,x}$ is

$$w_{n,x}(t)=\frac{1}{\sqrt{2\pi n}}exp\{-\frac{1}{2n\sigma^2}(u-nx)^2\}$$

, then for a Baire function $f(x)$ on $(-\infty,+\infty)$, Gauss-Weierstrass operators W_n is given by

$$\begin{aligned}W_n(f;x)&=E(\tfrac{U_{n,x}}{n})=\int_{-\infty}^{+\infty}f(\tfrac{t}{n})dF_{U_{n,x}}(t)=\int_{-\infty}^{+\infty}f(t)dF_{n,x}(t)\\&=\tfrac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}f(t)exp\{-\tfrac{n}{2\sigma^2}(t-x)^2\}dt\end{aligned}\tag{1}$$

where $E(\xi)$ is the expectation of random variable ξ, $F_{U_{n,x}}(t)$ and $F_{n,x}(t)$ are the distribution functions of $U_{n,x}$ and $\frac{U_{n,x}}{n}$, respectively. And $dF_{U_{n,x}}(t)$ and $dF_{n,x}(t)$ are the Lebesgue-Stieltjes measures respectively deduced from $F_{U_{n,x}}(t)$ and $F_{n,x}(t)$.

Let $\sigma = \sqrt{2}$, then Z. Ditzian[1] proved that, if $f(x) \leq M(x^2+1)exp\{x^2/4\}$ $(-\infty < x < +\infty)$ and $f(x) \in C(R)$, then for $[a,b] \subset R$,

$$\|W_n[f(t);x] - f(x)\|_{C[a,b]} \to 0 \quad (n \to \infty).$$

For $\sigma = 1$, S. Guo and M.K. Khan[2] showed that

$$\left|W_n[f(t);x] - \frac{f(x+)+f(x-)}{2}\right| \leq \frac{3}{n}\sum_{i=0}^{n} V_{I_k}(g_x), \tag{2}$$

where $f(x) \in BV(-\infty,+\infty)$, $V_{I_k}(g_x)$ denotes the total variation of $g_x(t)$ on I_k ($I_k = [x-1/\sqrt{k}, x+1/\sqrt{k}\,]$, $k = 1,2,...,n$; $I_0 = (-\infty,+\infty)$), and

$$g_x(t) = \begin{cases} f(t)-f(x+) & t > x, \\ 0 & t = x, \\ f(t)-f(x-) & t < x. \end{cases} \tag{3}$$

And for $\sigma = 1$ and $f \in DBV(R) = \{f(x)|f(x) = c + \int\limits_{-\infty}^{x} h(u)du, \quad h(u) \in BV(R)\}$, R. Bojanic and M.K. Khan[3] established that

$$\left|W_n(f(t);x) - f(x) - \frac{[h(x+)-h(x-)]}{\sqrt{2\pi n}}\right| \leq \frac{2}{n}\int_0^{\sqrt{n}} \bigvee_{x-1/t}^{x+1/t}(\varphi_x)dt \tag{4}$$

where

$$\varphi_x(t) = \begin{cases} h(t)-h(x+) & t > x, \\ 0 & t = x, \\ h(t)-h(x-) & t < x. \end{cases} \tag{5}$$

In the present paper we investigate the pointwise approximation of Gauss-Weierstrass operators on a more general class of functions whose left limit $f(x-)$ and right limit $f(x+)$ at a fixed point x exist and which satisfy some growth conditions. Furthermore, an asymptotic formula of Gauss-Weierstrass operators is given on a more general class of absolutely continuous functions whose derivative $h(x)$ have left limit $h(x-)$ and right limit $h(x+)$ at a fixed point x, assuming that $h(x)$ exists almost everywhere over R. To be convenient for our discussing , two classes of functions Φ_B and Φ_{AB} are given as were introduced by X.M. Zeng in [6].

Let $f(x)$ be any Baire function on R, then a class of functions Φ_B is given as follows

$$\Phi_B = \{f(x)||f(x)| \leq Mexp\{\beta|x|\}\}$$

where both $M > 0$ and $\beta \geq 0$ are constants related to $f(x)$. And let $h(x)$ be any measurable and integrable function on every finite subinterval of $(-\infty,+\infty)$, then another class of functions Φ_A is given as follows

$$\Phi_A = \{f(x)|f(x) = \int_a^x h(t)dt + f(a)\}$$

where a is a fixed number. By definition, if $f(x) \in \Phi_A$, then the derivative function of $f(x)$ is $h(x)$ in the sense of "almost everywhere". We should point out that $BV(R) \subset \Phi_B$ and $DBV(R) \subset \Phi_A$. Moreover, let Φ_{AB} denote the intersection of Φ_B and Φ_A, i.e. $\Phi_{AB} = \Phi_B \cap \Phi_A$, then $DBV(R) \subset \Phi_{AB}$.

Furthermore, for $f(t) \in \Phi_B$ we give the metric form $\Omega_x(f, \lambda)$ as follows (see [6,7,8])

$$\Omega_x(f, \lambda) = \sup_{t \in [x-\lambda, x+\lambda]} |f(t) - f(x)|$$

where $x \in R$ is fixed and $\lambda \geq 0$. It is easily verified that
(1) $\Omega_x(f, \lambda)$ is monotone non-decreasing with respect to λ.
(2) $\lim_{\lambda \to 0} \Omega_x(f, \lambda) = 0$, if $f(x)$ is continuous at the point x.
(3) If $f(x)$ is bounded variation on $[a, b]$, and $\bigvee_a^b(f)$ denotes the total variation of $f(x)$ on$[a, b]$, then $\Omega_x(f, \lambda) \leq \bigvee_{x-\lambda}^{x+\lambda}(f)$.

By the Bojanic - Khan - Cheng's method (see [3, 4, 5]), using the metric form $\Omega_x(f, \lambda)$ and analysis techniques , we obtain the results given in the following section.

2 Main Results

Theorem 1 *Let $f \in \Phi_B$, if $f(x-)$ and $f(x+)$ exist at the fixed point x, then for $n > \frac{2\sigma^2\beta}{\rho}$, we have*

$$\begin{aligned} \left| W_n(f(t); x) - \frac{f(x+) + f(x-)}{2} \right| &\leq M_x[1 + \frac{\rho^2 n}{8\sigma^2}] exp\{-\frac{\rho^2 n}{8\sigma^2}\} \\ &+ \frac{2}{n} \max\{1, \frac{\sigma^2}{\rho^2}\} \sum_{i=1}^{n} \Omega_x(g_x, \frac{\rho}{\sqrt{i}}) \end{aligned} \tag{6}$$

where $\rho > 0$ is fixed after chosen, $M_x = \{Mexp\{\beta|x|\} + |f(x+)| + |f(x-)|\}exp\{\sigma^2\beta^2\}$, and $g_x(t)$ is given by (3).

According to Theorem 1, we note that, Gauss-Weierstrass operators approach $\frac{f(x+)+f(x-)}{2}$ as $n \to \infty$ and the rate of convergence depends on the second part on the right-hand side of inequality (6) since the first part has so much high rate of convergence. Moreover, it means more for Gauss-Weierstrass operators that the $\rho > 0$ in inequality (6) can be chosen.

From Theorem 1 and Property (3) of $\Omega_x(f, \lambda)$, it easily follows that

Corollary 1 *Let $f(x) \in \Phi_B$ be a function of bounded variation on every subinter-*

val of R,then for $n > \frac{2\sigma^2\beta}{\rho}$ *, we have*

$$\left|W_n(f(t);x) - \frac{f(x+)+f(x-)}{2}\right| \le M_x[1+\frac{\rho^2 n}{8\sigma^2}]exp\{-\frac{\rho^2 n}{8\sigma^2}\} + \frac{2}{n}\max\{1,\frac{\sigma^2}{\rho^2}\}\sum_{i=1}^{n}\bigvee_{x-\rho/\sqrt{i}}^{x+\rho/\sqrt{i}}(g_x). \tag{7}$$

We should point out that, let $\sigma = 1$, $\rho = 1$, and $n > 2\beta$, then we have

$$\left|W_n(f(t);x) - \frac{f(x+)+f(x-)}{2}\right| \le M_x[1+\frac{n}{8}]exp\{-\frac{n}{8}\} + \frac{2}{n}\sum_{i=1}^{n}\bigvee_{x-1/\sqrt{i}}^{x+1/\sqrt{i}}(g_x)$$

which is better than (2) that S. Guo and M.K. Khan[2] obtained .

Corollary 2 *Under the conditions of Theorem 1, if* $\Omega_x(g_x,\lambda) = o(\lambda)$*, then*

$$W_n(f(t);x) = \frac{f(x+)+f(x-)}{2} + o(\frac{1}{\sqrt{n}}). \tag{8}$$

Theorem 2 *Let* $f \in \Phi_{AB}$*,* $f(t) = \int_a^t h(u)du + f(a)$ *, if* $h(x-)$ *and* $h(x+)$ *exist at the fixed point* x*, then for* $n > \max\{\frac{2\sigma^2(1+\beta)}{\rho}, 4\}$ *, we have*

$$\left|W_n(f(t);x) - f(x) - \frac{[h(x+)-h(x-)]\sigma}{\sqrt{2\pi n}}\right| \le M_x[1+\frac{\rho^2 n}{8\sigma^2}]exp\{-\frac{\rho^2 n}{8\sigma^2}\} + \frac{4\max\{\rho^2,\sigma^2\}}{n\rho}\sum_{i=1}^{[\sqrt{n}]}\Omega_x(\varphi_x,\frac{\rho}{i}) \tag{9}$$

where $M_x = \{2Mexp\{\beta|x|\} + |h(x+)| + |h(x-)|\}exp\{\sigma^2(1+\beta)^2\}$*,* $\rho > 0$ *is fixed after chosen, and* $\varphi_x(t)$ *is given by (5).*

By Theorem 2 and Property (3) of $\Omega_x(f,\lambda)$ we have

Corollary 3 *Let* $f \in \Phi_{AB}$*,* $f(t) = \int_a^t h(u)du + f(a)$ *, where* $h(x)$ *is a function of bounded variation on every finite subinterval of* $(-\infty,+\infty)$*, then for* $n > \max\{\frac{2\sigma^2(1+\beta)}{\rho}, 4\}$ *, we have*

$$\left|W_n(f(t);x) - f(x) - \frac{[h(x+)-h(x-)]\sigma}{\sqrt{2\pi n}}\right| \le M_x[1+\frac{\rho^2 n}{8\sigma^2}]exp\{-\frac{\rho^2 n}{8\sigma^2}\} + \frac{4\max\{\rho^2,\sigma^2\}}{n\rho}\sum_{i=1}^{[\sqrt{n}]}\bigvee_{x-\rho/i}^{x+\rho/i}(\varphi_x). \tag{10}$$

Let $\sigma = 1$, $\rho = 1$ and $h(x) \in BV(R)$, then by Corollary 3 and for $n > 4$ we obtain

$$\left| W_n(f(t);x) - f(x) - \frac{[h(x+) - h(x-)]}{\sqrt{2\pi n}} \right| \leq M_x[1 + \frac{n}{8}]exp\{-\frac{n}{8}\} + \frac{4}{n}\sum_{i=1}^{[\sqrt{n}]} \bigvee_{x-1/i}^{x+1/i}(\varphi_x), \quad (11)$$

where the rate of convergence is not lower than that in (4) which was obtained by R. Bojanic and M.K. Khan[3] since

$$\frac{1}{n}\sum_{i=1}^{[\sqrt{n}]} \bigvee_{x-1/i}^{x+1/i}(\varphi_x) \leq \frac{1}{n}\int_0^{\sqrt{n}} \bigvee_{x-1/t}^{x+1/t}(\varphi_x)dt$$

and the first part on the right-hand side of inequality (11) has so much high rate of convergence.

Next we give the proofs of Theorem 1 and 2.

3 Proofs of Theorems

To complete the proofs of Theorem 1 and 2, the following lemmas are needed.
Lemma 1 For $\alpha > 0$, we have

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{\alpha\sigma}^{+\infty} exp\{-\frac{x^2}{2\sigma^2}\}dx \leq exp\{-\frac{\alpha^2}{2}\}[\frac{1}{2} - \frac{\alpha}{\sqrt{2\pi}} + \frac{\alpha^2}{4}]. \quad (12)$$

Proof:

$$I := \frac{1}{\sqrt{2\pi}\sigma}\int_{\alpha\sigma}^{+\infty} exp\{-\frac{x^2}{2\sigma^2}\}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_0^{+\infty} exp\{-\frac{1}{2\sigma^2}(t+\alpha\sigma)^2\}dt$$
$$= exp\{-\frac{\alpha^2}{2}\}\cdot\frac{1}{\sqrt{2\pi}\sigma}\int_0^{+\infty} exp\{-\frac{t^2}{2\sigma^2}\}\cdot exp\{-\frac{\alpha}{\sigma}t\}dt.$$

Since $exp\{x\} < 1 + x + x^2/2$ whenever $x < 0$, we have

$$exp\{-\frac{\alpha}{\sigma}t\} < 1 - \frac{\alpha}{\sigma}t + \frac{\alpha^2}{2\sigma^2}t^2.$$

then,

$$I \leq exp\{-\frac{\alpha^2}{2}\}\cdot\frac{1}{\sqrt{2\pi}\sigma}\int_0^{+\infty} exp\{-\frac{t^2}{2\sigma^2}\}(1 - \frac{\alpha}{\sigma}t + \frac{\alpha^2}{2\sigma^2}t^2)dt$$
$$= exp\{-\frac{\alpha^2}{2}\}[\frac{1}{2} - \frac{\alpha}{\sqrt{2\pi}} + \frac{\alpha^2}{4}].$$

Lemma 2 Let $F_n(t) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{-\infty}^{t} exp\{-\frac{n}{2\sigma^2}u^2\}du$, then for $t > 0$, we have

$$1 - F_n(t) \leq \frac{\sigma^2}{2nt^2}. \quad (13)$$

Note that $F_n(t) = F_{n,\,0}(t)$ when $x = 0$.

Proof: $F_n(t)$ is $N(0, \sigma/\sqrt{n})$ normal distribution function. Let its corresponding random variable be ξ_n, then

$$1 - F_n(t) = P(\xi_n > t) = \int_t^{+\infty} dF_n(u)$$

. As the proof of Chebyshev's inequality, we obtain

$$\int_0^{+\infty} u^2 dF_n(u) \geq \int_t^{+\infty} u^2 dF_n(u) \geq t^2 \int_t^{+\infty} dF_n(u) = t^2[1 - F_n(t)].$$

since

$$\int_0^{+\infty} u^2 dF_n(u) = \frac{\sigma^2}{2n}$$

, we establish (13).

Firstly, the proof of Theorem 1 is given.

Proof of Theorem 1: Let $f(t) \in \Phi_B$ and assume that $f(x+)$ and $f(x-)$ exist at the fixed point x, then we can write $f(t)$ as follows

$$f(t) = \frac{f(x+) + f(x-)}{2} + g_x(t) + \frac{f(x+) - f(x-)}{2} sign(t-x) + \delta_x(t)[f(x) - \frac{f(x+) + f(x-)}{2}]$$

where $g_x(t)$ is defined in (3), $sign(t)$ is sign function and

$$\delta_x(t) = \begin{cases} 1 & t = x, \\ 0 & t \neq 0. \end{cases}$$

Then,

$$\begin{aligned} W_n(f;x) &= \int_{-\infty}^{+\infty} f(t) dF_{n,x}(t) = \frac{f(x+) + f(x-)}{2} + W_n(g_x(t);x) \\ &+ \frac{f(x+) - f(x-)}{2} W_n(sign(t-x);x) + [f(x) - \frac{f(x+) + f(x-)}{2}] W_n(\delta_x(t);x). \end{aligned} \tag{14}$$

Obviously, $W_n(\delta_x(t);x) = 0$. And by direct calculation , it is easily verified that

$$W_n(sign(t-x);x) = 0$$

. Therefore, identity (14) can be simplified as

$$W_n(f;x) = \frac{f(x+) + f(x-)}{2} + W_n(g_x(t);x). \tag{15}$$

Now we give the estimate of $|W_n(g_x(t);x)|$. Let $\rho > 0$ be fixed after chosen, then $W_n(g_x(t);x)$ is split up into three parts as follows

$$W_n(g_x(t);x) = \int_{-\infty}^{+\infty} g_x(t) dF_{n,x}(t) = \Delta_{n,1} + \Delta_{n,2} + \Delta_{n,3} \tag{16}$$

where

$$\Delta_{n,1}=\int_{-\infty}^{x-\rho} g_x(t)dF_{n,x}(t),\quad \Delta_{n,2}=\int_{x-\rho}^{x+\rho} g_x(t)dF_{n,x}(t),\quad \Delta_{n,3}=\int_{x+\rho}^{+\infty} g_x(t)dF_{n,x}(t).$$

The first step is to estimate $|\Delta_{n,1}|+|\Delta_{n,3}|$. By the substitution rule of integrals, we have

$$|\Delta_{n,1}|\leq \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{\rho}^{+\infty}|g_x(x-u)|\,exp\{-\frac{n}{2\sigma^2}u^2\}du. \tag{17}$$

Observe the definition of $g_x(t)$ in (3), we obtain

$$|g_x(x-u)|\leq \overline{M}exp\{\beta u\}$$

where $\overline{M}=Mexp\{\beta|x|\}+|f(x+)|+|f(x-)|$. Therefore, it follows from (17) that

$$|\Delta_{n,1}|\leq \overline{M}\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{\rho}^{+\infty}exp\{\beta u-\frac{n}{2\sigma^2}u^2\}du$$

$$=\overline{M}exp\{\frac{\sigma^2\beta^2}{2n}\}\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{\rho-\frac{\sigma^2\beta}{n}}^{+\infty}exp\{-\frac{n}{2\sigma^2}u^2\}du.$$

Let $n>\frac{2\sigma^2\beta}{\rho}$, then by the above inequality, we establish

$$|\Delta_{n,1}|\leq \overline{M}exp\{\sigma^2\beta^2\}\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{\rho/2}^{+\infty}exp\{-\frac{n}{2\sigma^2}u^2\}du. \tag{18}$$

By Lemma 1, we show that

$$\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{\rho/2}^{+\infty}exp\{-\frac{n}{2\sigma^2}u^2\}du\leq[\frac{1}{2}+\frac{\rho^2n}{16\sigma^2}]exp\{-\frac{\rho^2n}{8\sigma^2}\}.$$

Hence from (18), it follows that

$$|\Delta_{n,1}|\leq M_x[\frac{1}{2}+\frac{\rho^2n}{16\sigma^2}]exp\{-\frac{\rho^2n}{8\sigma^2}\}$$

where $M_x=[Mexp\{\beta|x|\}+|f(x+)|+|f(x-)|]exp\{\sigma^2\beta^2\}$.

Similarly, we have

$$|\Delta_{n,3}|\leq M_x[\frac{1}{2}+\frac{\rho^2n}{16\sigma^2}]exp\{-\frac{\rho^2n}{8\sigma^2}\}.$$

Consequently, we obtain

$$|\Delta_{n,1}|+|\Delta_{n,3}|\leq M_x[1+\frac{\rho^2n}{8\sigma^2}]exp\{-\frac{\rho^2n}{8\sigma^2}\} \tag{19}$$

where $M_x=[Mexp\{\beta|x|\}+|f(x+)|+|f(x-)|]exp\{\sigma^2\beta^2\}$, and $n>\frac{2\sigma^2\beta}{\rho}$.

The second step is to estimate $|\Delta_{n,2}|$. Applying the substitution rule of integrals, we get

$$\begin{aligned} |\Delta_{n,2}| &= \left| \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{x-\rho}^{x+\rho} g_x(t) exp\{-\frac{n}{2\sigma^2}(t-x)^2\} dt \right| \\ &\leq \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_0^{\rho} [|g_x(x-u)| + |g_x(x+u)|] exp\{-\frac{n}{2\sigma^2}u^2\} du. \end{aligned} \tag{20}$$

Since $g_x(x) = 0$, it follows from (20) that

$$|\Delta_{n,2}| \leq \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_0^{\rho} 2\Omega_x(g_x, u) exp\{-\frac{n}{2\sigma^2}u^2\} du. \tag{21}$$

We split up the integral of the right-hand side of (21) into tow parts denoted by I_1 and I_2, respectively, i.e.

$$\frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_0^{\rho} 2\Omega_x(g_x, u) exp\{-\frac{n}{2\sigma^2}u^2\} du = I_1 + I_2 \tag{22}$$

where

$$\begin{aligned} I_1 &= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_0^{\rho/\sqrt{n}} 2\Omega_x(g_x, u) exp\{-\frac{n}{2\sigma^2}u^2\} du, \\ I_2 &= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\rho/\sqrt{n}}^{\rho} 2\Omega_x(g_x, u) exp\{-\frac{n}{2\sigma^2}u^2\} du. \end{aligned}$$

For I_1 , we obtain

$$I_1 \leq 2\Omega_x(g_x, \frac{\rho}{\sqrt{n}}) \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_0^{\rho/\sqrt{n}} exp\{-\frac{n}{2\sigma^2}u^2\} du \leq \Omega_x(g_x, \frac{\rho}{\sqrt{n}}). \tag{23}$$

For I_2, we rewrite it as follows

$$I_2 = \int_{\rho/\sqrt{n}}^{\rho} 2\Omega_x(g_x, u) dF_n(u)$$

where $F_n(u)$ is given in Lemma 2. Note that $F_n(u)$ is continuous and $\Omega_x(g_x, u)$ is monotone non-decreasing with respect to u, then applying the Lebesgue-Stieltjes integral by parts, we have

$$\begin{aligned} I_2 &= [2\Omega_x(g_x, u)F_n(u)]_{\rho/\sqrt{n}}^{\rho} - \int_{\rho/\sqrt{n}}^{\rho} F_n(u) d[2\Omega_x(g_x, u)] \\ &= 2\Omega_x(g_x, \frac{\rho}{\sqrt{n}})[1 - F_n(\frac{\rho}{\sqrt{n}})] - 2\Omega_x(g_x, \rho)[1 - F_n(\rho)] + \int_{\rho/\sqrt{n}}^{\rho} [1 - F_n(u)] d[2\Omega_x(g_x, u)]. \end{aligned}$$

By Lemma 2, it follows from the above equality that

$$\begin{aligned} I_2 &\leq \frac{\sigma^2}{\rho^2}\Omega_x(g_x, \frac{\rho}{\sqrt{n}}) + \frac{\sigma^2}{2n} \int_{\rho/\sqrt{n}}^{\rho} \frac{1}{u^2} d[2\Omega_x(g_x, u)] \\ &= \frac{\sigma^2}{\rho^2}\Omega_x(g_x, \frac{\rho}{\sqrt{n}}) + \frac{\sigma^2}{n} \left\{ \left[\frac{1}{u^2}\Omega_x(g_x, u) \right]_{\rho/\sqrt{n}}^{\rho} - \int_{\rho/\sqrt{n}}^{\rho} \Omega_x(g_x, u) d[\frac{1}{u^2}] \right\} \\ &= \frac{\sigma^2}{n\rho^2}\Omega_x(g_x, \rho) + \frac{\sigma^2}{n\rho^2} \int_1^n \Omega_x(g_x, \frac{\rho}{\sqrt{u}}) du \\ &\leq \frac{\sigma^2}{n\rho^2}\Omega_x(g_x, \rho) + \frac{\sigma^2}{n\rho^2} \sum_{i=1}^{n-1} \Omega_x(g_x, \frac{\rho}{\sqrt{i}}). \end{aligned} \tag{24}$$

Hence by (21)-(24), we establish

$$|\Delta_{n,2}| \leq \frac{2}{n}\max\{1, \frac{\sigma^2}{\rho^2}\}\sum_{i=1}^{n}\Omega_x(g_x, \frac{\rho}{\sqrt{i}}). \tag{25}$$

The final is to combine (15),(16),(19)and (25) that completes the proof of Theorem 1.

Secondly, we present the proof of Theorem 2

Proof of Theorem 2: Let $f(t) \in \Phi_{AB}$, $f(t) = \int_a^t h(u)du + f(a)$, and assume that $h(x+)$ and $h(x-)$ exist at the fixed point x, then we can write $h(t)$ as follows

$$h(t) = \frac{h(x+)+h(x-)}{2} + \varphi_x(t) + \frac{h(x+)-h(x-)}{2}sign(t-x) + \delta_x(t)[h(x) - \frac{h(x+)+h(x-)}{2}]$$

where $\varphi_x(t)$ is defined in (5), $sign(t)$ is sign function and $\delta_x(t) = \begin{cases} 1 & t = x \\ 0 & t \neq 0 \end{cases}$.
Integrating both sides of the above equality from x to t, and noting that

$$\int_x^t sign(u-x)du = |t-x|, \qquad \int_x^t \delta_x(u)du = 0$$

, we have

$$f(t) - f(x) = \frac{h(x+)+h(x-)}{2}(t-x) + \int_x^t \varphi_x(u)du + \frac{h(x+)-h(x-)}{2}|t-x|.$$

Hence,

$$W_n(f(t);x) - f(x) = W_n(\int_x^t \varphi_x(u)du; x) + \frac{h(x+)-h(x-)}{2}W_n(|t-x|;x). \tag{26}$$

By direct calculation , we get $W_n(|t-x|;x) = \sqrt{2}\sigma/\sqrt{\pi n}$. Therefore from (26), it follows that

$$W_n(f(t);x) - f(x) - \frac{[h(x+)-h(x-)]\sigma}{\sqrt{2\pi n}} = W_n(\int_x^t \varphi_x(u)du; x) \tag{27}$$

For a fixed $\rho > 0$, We split up the right-hand side of equality (27) into three parts as follows

$$W_n(\int_x^t \varphi_x(u)du; x) = \int_{-\infty}^{+\infty}(\int_x^t \varphi_x(u)du)dF_{n,x}(t) = \Theta_{n,1} + \Theta_{n,2} + \Theta_{n,3} \tag{28}$$

where

$$\Theta_{n,1} = \int_{-\infty}^{x-\rho}(\int_x^t \varphi_x(u)du)dF_{n,x}(t), \quad \Theta_{n,2} = \int_{x-\rho}^{x+\rho}(\int_x^t \varphi_x(u)du)dF_{n,x}(t),$$
$$\Theta_{n,3} = \int_{x+\rho}^{+\infty}(\int_x^t \varphi_x(u)du)dF_{n,x}(t).$$

Next we will give the estimates of $|\Theta_{n,1}|$, $|\Theta_{n,2}|$ and $|\Theta_{n,3}|$.

Applying the substitution rule of integrals, we have

$$|\Theta_{n,1}| \leq \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\rho}^{+\infty} \left| \int_{x}^{x-t} \varphi_x(u)du \right| exp\{-\frac{n}{2\sigma^2}t^2\}dt. \tag{29}$$

Observe that , for $t > 0$,

$$\left| \int_{x}^{x-t} \varphi_x(u)du \right| = |f(x-t) - f(x) + h(x-)t| \leq \overline{M} exp\{(1+\beta)t\}$$

where $\overline{M} = 2Mexp\{\beta|x|\} + |h(x+)| + |h(x-)|$, then by (29), we obtain

$$|\Theta_{n,1}| \leq \overline{M}\frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{\rho}^{+\infty} exp\{(1+\beta)t - \frac{n}{2\sigma^2}t^2\}dt.$$

Using the same techniques as in the estimate of $|\Delta_{n,1}|$ in Proof of Theorem 1, for $n > \frac{2\sigma^2(1+\beta)}{\rho}$, we establish

$$|\Theta_{n,1}| \leq M_x[\frac{1}{2} + \frac{\rho^2 n}{16\sigma^2}]exp\{-\frac{\rho^2 n}{8\sigma^2}\}$$

where $M_x = [2Mexp\{\beta|x|\} + |h(x+)| + |h(x-)|]exp\{\sigma^2(1+\beta)^2\}$.

Similarly, we get

$$|\Theta_{n,3}| \leq M_x[\frac{1}{2} + \frac{\rho^2 n}{16\sigma^2}]exp\{-\frac{\rho^2 n}{8\sigma^2}\}.$$

Therefore, we obtain

$$|\Theta_{n,1}| + |\Theta_{n,3}| \leq M_x[1 + \frac{\rho^2 n}{8\sigma^2}]exp\{-\frac{\rho^2 n}{8\sigma^2}\} \tag{30}$$

where $M_x = [2Mexp\{\beta|x|\} + |h(x+)| + |h(x-)|]exp\{\sigma^2(1+\beta)^2\}$ and $n > \frac{2\sigma^2(1+\beta)}{\rho}$.

On the other hand, using the substitution rule of integrals, we obtain

$$\begin{aligned} |\Theta_{n,2}| &= \left| \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{x-\rho}^{x+\rho} (\int_{x}^{t} \varphi_x(u)du) exp\{-\frac{n}{2\sigma^2}(t-x)^2\}dt \right| \\ &= \left| \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{0}^{\rho} \left[\int_{x}^{x+t} \varphi_x(u)du + \int_{x}^{x-t} \varphi_x(u)du \right] exp\{-\frac{n}{2\sigma^2}t^2\}dt \right| \\ &\leq \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{0}^{\rho} \left[\int_{0}^{t} (|\varphi_x(x+u)| + |\varphi_x(x-u)|)du \right] exp\{-\frac{n}{2\sigma^2}t^2\}dt. \end{aligned} \tag{31}$$

Since $\varphi_x(x) = 0$, it follows from (31) that

$$|\Theta_{n,2}| \leq \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{0}^{\rho} \left[\int_{0}^{t} 2\Omega_x(\varphi_x, u)du \right] exp\{-\frac{n}{2\sigma^2}t^2\}dt. \tag{32}$$

We decompose the integral of the right-hand side of (32) into tow parts denoted by J_1 and J_2, respectively, i.e.

$$\frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \int_{0}^{\rho} \left[\int_{0}^{t} 2\Omega_x(\varphi_x, u)du \right] exp\{-\frac{n}{2\sigma^2}t^2\}dt = J_1 + J_2 \tag{33}$$

where

$$J_1 = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_0^{\rho/\sqrt{n}}\left[\int_0^t 2\Omega_x(\varphi_x,u)du\right]exp\{-\frac{n}{2\sigma^2}t^2\}dt,$$
$$J_2 = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{\rho/\sqrt{n}}^{\rho}\left[\int_0^t 2\Omega_x(\varphi_x,u)du\right]exp\{-\frac{n}{2\sigma^2}t^2\}dt.$$

For J_1 , we obtain

$$J_1 \le 2\int_0^{\rho/\sqrt{n}}\Omega_x(\varphi_x,u)du\cdot\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_0^{\rho/\sqrt{n}}exp\{-\frac{n}{2\sigma^2}t^2\}dt \le \frac{\rho}{\sqrt{n}}\Omega_x(\varphi_x,\frac{\rho}{\sqrt{n}}). \tag{34}$$

For J_2, we rewrite it as follows

$$J_2 = \int_{\rho/\sqrt{n}}^{\rho}\left(\int_0^t 2\Omega_x(\varphi_x,u)du\right)dF_n(t)$$

where $F_n(t)$ is given in Lemma 2. Note that $F_n(t)$ is continuous and $\int_0^t 2\Omega_x(\varphi_x,u)du$ is monotone non-decreasing with respect to t, then applying the Lebesgue-Stieltjes integral by parts, we get

$$\begin{aligned}J_2 &= \left[\left(\int_0^t 2\Omega_x(\varphi_x,u)du\right)F_n(t)\right]_{\rho/\sqrt{n}}^{\rho} - \int_{\rho/\sqrt{n}}^{\rho}F_n(t)\cdot 2\Omega_x(\varphi_x,t)dt\\ &= \int_0^{\rho/\sqrt{n}}2\Omega_x(\varphi_x,u)du\cdot[1-F_n(\frac{\rho}{\sqrt{n}})] - \int_0^{\rho}2\Omega_x(\varphi_x,u)du\cdot[1-F_n(\rho)]\\ &+ \int_{\rho/\sqrt{n}}^{\rho}[1-F_n(t)]2\Omega_x(\varphi_x,t)dt.\end{aligned}$$

By Lemma 2, it follows from the above equality that

$$\begin{aligned}J_2 &\le \frac{\sigma^2}{\rho\sqrt{n}}\Omega_x(\varphi_x,\frac{\rho}{\sqrt{n}}) + \frac{\sigma^2}{n}\int_{\rho/\sqrt{n}}^{\rho}\frac{1}{t^2}\Omega_x(\varphi_x,t)dt\\ &= \frac{\sigma^2}{\rho\sqrt{n}}\Omega_x(\varphi_x,\frac{\rho}{\sqrt{n}}) + \frac{\sigma^2}{n\rho}\int_1^{\sqrt{n}}\Omega_x(\varphi_x,\frac{\rho}{u})du\\ &\le \frac{\sigma^2}{\rho\sqrt{n}}\Omega_x(\varphi_x,\frac{\rho}{\sqrt{n}}) + \frac{\sigma^2}{n\rho}\sum_{i=1}^{[\sqrt{n}]}\Omega_x(\varphi_x,\frac{\rho}{i}).\end{aligned} \tag{35}$$

Therefore by (32)-(35), we have

$$|\Theta_{n,2}| \le \frac{4\max\{\rho^2,\sigma^2\}}{n\rho}\sum_{i=1}^{[\sqrt{n}]}\Omega_x(\varphi_x,\frac{\rho}{i}), \tag{36}$$

where $n \ge 4$. It completes the proof of Theorem 2 to Combine (27),(28),(30) and (36).

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3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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